# Dissipative Non-linear Schrodinger equation with variable coefficient in a stenosed elastic tube filled with a viscous fluid

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## Abstract

In the present work, by considering the artery as a prestressed thin-walled elastic tube with a symmetrical stenosis and the blood as an incompressible viscous fluid, we have studied the amplitude modulation of nonlinear waves in such a composite medium through the use of the reductive perturbation method [23]. The governing evolutions can be reduced to the dissipative non-linear Schrodinger (NLS) equation with variable coefficient. The progressive wave solution to the above non-linear evolution equation is then sought.

**Keywords:** Elastic tube with stenosis, viscous fluid, nonlinear Schrodinger equation.

## 1. Introduction

Due to its applications in arterial mechanics, the propagation of pressure pulses in fluid-filled distensible tubes has been studied by several researchers (Pedley [1] and Fung [2]). As far as the biological applications are concerned, most of the works on wave propagation in compliant tubes have considered small amplitude waves ignoring the nonlinear effects and focused on the dispersive character of waves (see, Atabek and Lew [3], Rachev [4] and Demiray [5]). However, when the nonlinear terms arising from the constitutive equations and kinematical relations are introduced, one has to consider either finite amplitude, or small-but-finite amplitude waves, depending on the order of nonlinearity.

Rudinger [6], Ling and Atabek [7], Anliker et al [8] and Tait and Moodie [9] observed the propagation of finite amplitude waves in fluid-filled elastic or viscoelastic tube by using the characteristics method to study the formation of shock. On the other hand, the propagation of small-but-finite amplitude waves in distensible tubes has been investigated by Johnson [10], Hashizume [11], and Yomosa [12] by employing various asymptotic methods.

Later, in a series of works of Demiray and Antar [13]-[15], they treated the artery as an incompressible, prestressed, thin and isotropic elastic tube and the blood as an incompressible inviscid, viscous or layered fluid. Then, by using the reductive perturbation method in the long-wave approximation, they obtained KdV, Burgers' and KdV-Burgers' type equations, respectively.

Recently, Tay and co-workers [16]-[18] studied the non-linear waves propagation in a prestressed thin elastic tube with a symmetrical stenonis filled with inviscid, viscous and Newtonian fluid with variable viscosity, they showed that the governing equations can be reduced to forced Kortewed-de Vries, forced perturbed Kortewed-de Vries and forced Kortewed-de Vries-Burgers equations, respectively.

The modulation of small-but-finite amplitude pressure waves in a fluid-filled distensible, linear elastic tube has been examined by Ravindran and Prasad [19]. They obtained the non-linear Schrodinger (NLS) equation equation. The work of of non-linear waves modulation in a prestressed thin elastic tube filled with inviscid or viscous fluid has been carried out by Demiray and co-worker [20]-[22]. They showed that the governing equations can be reduced to NLS and dissipative NLS equations, respectively. The NLS equation is the simplest representative equation describing the self-modulation of one-dimensional monochromatic plane waves in dispersive media. It has a balance between the nonlinearity and dispersion.

In the present work, considering the artery as an incompressible, prestressed, thin-walled elastic tube with a symmetrical stenosis and the blood as an averaged viscous fluid, we have studied the amplitude modulation of non-linear waves in such a composite medium by using the reductive perturbation method [23]. We obtained the dissipative NLS equation with variable coefficient. We then sought the progressive wave solution to the non-linear evolution equation obtained.

## 2. Basic equations and theoretical preliminaries

In this section, we shall give the derivation of the field equations of an elastic tube, which is considered to be a model for an artery, and a viscous fluid, which is assumed to be a model for blood.

### *2.1 Equations of tube*

In this sub-section, we shall derive the governing equations of an elastic tube filled with a viscous fluid. Such a combination of a solid and a fluid is considered to be a model for blood flow in arteries.

For a healthy human being, the systolic pressure is about 120 mm Hg, and the diastolic pressure is around 80 mm Hg. This means that the arteries are subjected to a mean pressure $P_0 = 100$ mm Hg, and in the course of blood flow, a dynamical pressure increment $\Delta P = \pm$ 20 mm Hg is added on this initial field. Moreover, experimental studies (Fung [2]) revealed that the arteries are also subjected to an initial axial stretch $\lambda_z$, which is about $\lambda_z = 1.6$. These observations show that the arteries are initially subjected to static deformation both in the radial and the axial directions, and a dynamical pressure (or a radial displacement $u^*$ ) is superimposed on this initial deformation. Due to the external tethering in the axial direction, the effect of axial displacement is neglected.

Now, we consider a thin and long tube of circular cross-section with initial reference radius $R_0$ in the cylindrical polar coordinates $(R^*, \Theta, Z^*)$. Then, the position vector of a point on the tube can be described by

$$\mathbf{R} = R_0 \mathbf{e_r} + Z^* \mathbf{e_z}, \tag{1}$$

where $\mathbf{e_r}$, $\mathbf{e}_\theta$ and $\mathbf{e_z}$ are the unit base vectors in the cylindrical polar coordinates and $Z^*$ is the axial coordinates of a material point in the natural state.

The arc lengths along the meridional and circumferential curves are given by

$$dS_Z = dZ^*, \quad dS_\Theta = R_0 d\Theta. \tag{2}$$

Motivated with the experimental observations (Fung [2]), we shall assume that the elastic tube is subjected to an axial stretch ratio $\lambda_z$, and the static pressure $P_0^*(Z^*)$. Then, the deformation may be described by

$$\mathbf{r_0} = [r_0 - f^*(z^*)]\, \mathbf{e_r} + z^* \mathbf{e_z}, \quad z^* = \lambda_z Z^*, \tag{3}$$

where $z^*$ is the axial coordinate at the intermediate configuration, $r_0$ is the radius of the origin after finite static deformation, and $f(z)$ is the stenosis functions after the deformation. Thus the arc lengths after static deformation along the meridional and circumferential directions are given by

$$ds_z^0 = [1 + (-f^{*'})^2]^{1/2} dz^*, \quad ds_\theta = [r_0 - f^*(z^*)]\, d\theta, \tag{4}$$

where a prime denotes the differentiation of the corresponding field variable with respect to $z^*$.

Upon this initial static deformation, we shall superimpose a finite dynamical radial displacement $u^*(z^*, t^*)$, where $t^*$ is the time parameter, but, in view of the external tethering in the axial direction, the axial displacement is assumed to be negligible. Then, the position vector $\mathbf{r}$ of a generic point on the tube can be described by

$$\mathbf{r} = [r_0 - f^*(z^*) + u^*(z^*, t^*)]\mathbf{e_r} + z^*\mathbf{e_z}. \tag{5}$$

The arc lengths along the deformed meridional and circumferential curves are respectively given by

$$ds_z = \left[1 + \left(-f^{*'} + \frac{\partial u^*}{\partial z^*}\right)^2\right]^{1/2} dz^*, \quad ds_\theta = [r_0 - f^*(z^*) + u^*(z^*, t^*)]d\theta. \tag{6}$$

Then, the stretch ratios along the meridional and circumferential curves in the final configuration read, respectively, by

$$\lambda_1 = \lambda_z \left[1 + (-f^{*'} + \frac{\partial u^*}{\partial z^*})^2\right]^{1/2}, \lambda_2 = \frac{1}{R_0}[r_0 - f^*(z^*) + u^*(z^*, t^*)]. \tag{7}$$

The unit tangent vector $\mathbf{t}$ along the deformed meridional curve and the unit exterior normal vector $\mathbf{n}$ to the deformed membrane are given by

$$\mathbf{t} = \frac{(-f^{*'} + \frac{\partial u^*}{\partial z^*})\mathbf{e_r} + \mathbf{e_z}}{\Lambda}, \quad \mathbf{n} = \frac{\mathbf{e_r} - (-f^{*'} + \frac{\partial u^*}{\partial z^*})\mathbf{e_z}}{\Lambda}, \tag{8}$$

where the function $\Lambda$ is defined by

$$\Lambda = \left[1 + \left(-f^{*'} + \frac{\partial u^*}{\partial z^*}\right)^2\right]^{1/2}.$$

The material that we shall consider is assumed to be incompressible. This condition imposes the following restriction on the thickness $H$, and $h$, before and after final deformation respectively

$$h = \frac{H}{\lambda_1 \lambda_2}. \tag{9}$$

Let $T_1$ and $T_2$ be the membrane forces along the meridional and circumferential curves, respectively. Then, the equation of the radial motion of a small tube element placed between the planes $z^* = \text{const}$, $z^* + dz^* = \text{const}$, $\theta = \text{const}$ and $\theta + d\theta = \text{const}$ may be given by

$$\frac{\partial}{\partial z^*}\left[\frac{T_1}{\Lambda}(r_0 - f^* + u^*)\left(-f^{*'} + \frac{\partial u^*}{\partial z^*}\right)\right] - T_2\Lambda + \Lambda(r_0 - f^* + u^*)P_r^* = \rho_0 \frac{H}{\lambda_z} R_0 \frac{\partial^2 u^*}{\partial t^{*2}}, \tag{10}$$

where $\rho_0$ is the mass density of the membrane material, and $P_r^*$ is the radial fluid reaction force on the inner surface of the tube.

Let $\mu\Sigma$ be the strain energy density function of the tube material, where $\mu$ is the shear modulus. Then, the membrane forces $T_1$ and $T_2$ may be expressed in terms of the stretch ratios as

$$T_1 = \frac{\mu H}{\lambda_2}\frac{\partial \Sigma}{\partial \lambda_1}, \quad T_2 = \frac{\mu H}{\lambda_1}\frac{\partial \Sigma}{\partial \lambda_2}. \tag{11}$$

Introducing equation (11) into equation (10), the equation of motion of the tube in the radial direction takes the following form

$$\mu R_0 \frac{\partial}{\partial z^*}\left\{\frac{\left(-f^{*'} + \frac{\partial u^*}{\partial z^*}\right)}{\Lambda}\frac{\partial \Sigma}{\partial \lambda_1}\right\} - \frac{\mu}{\lambda_z}\frac{\partial \Sigma}{\partial \lambda_2} + \frac{\Lambda P_r^*}{H}(r_0 - f^* + u^*) - \rho_0 \frac{R_0}{\lambda_z}\frac{\partial^2 u^*}{\partial t^{*2}} = 0. \tag{12}$$

*2.2. Equations of fluid*

In general, blood is known to be an incompressible non-Newtonian fluid. However, in the course of flow in large arteries, the red blood cells in the vicinity of arterial wall move to the central region of the artery so that hematocrit ratio becomes quite low near the arterial wall, where the shear rate is quite high, as can be seen from Poiseuille flow. Experimental studies indicate when the hematocrit ratio is low and the shear rate is high, blood behaves like a Newtonian fluid (see [2, 6]). Therefore, for flow problems in large blood vessels, blood may be treated as incompressible Newtonian fluid whose axially symmetric motion in the cylindrical polar coordinates may be given by

$$\frac{\partial V_r^*}{\partial t^*} + V_r^* \frac{\partial V_r^*}{\partial r} + V_z^* \frac{\partial V_r^*}{\partial z^*} + \frac{1}{\rho_f} \frac{\partial \bar{P}}{\partial r} - \frac{\mu_v}{\rho_f} \left( \frac{\partial^2 V_r^*}{\partial r^2} + \frac{1}{r} \frac{\partial V_r^*}{\partial r} - \frac{V_r^*}{r^2} + \frac{\partial^2 V_r^*}{\partial z^{*2}} \right) = 0, \quad (13)$$

$$\frac{\partial V_z^*}{\partial t^*} + V_r^* \frac{\partial V_z^*}{\partial r} + V_z^* \frac{\partial V_z^*}{\partial z^*} + \frac{1}{\rho_f} \frac{\partial \bar{P}}{\partial z^*} - \frac{\mu_v}{\rho_f} \left( \frac{\partial^2 V_z^*}{\partial r^2} + \frac{1}{r} \frac{\partial V_z^*}{\partial r} + \frac{\partial^2 V_z^*}{\partial z^{*2}} \right) = 0, \quad (14)$$

$$\frac{\partial V_r^*}{\partial r} + \frac{V_r^*}{r} + \frac{\partial V_z^*}{\partial z^*} = 0 \quad \text{(incompressibility)}, \quad (15)$$

where $V_r^*$, $V_z^*$ denote the radial and the axial velocity components, $\rho_f$ is the mass density, $\bar{P}$ is the pressure function, $\mu_v$ is the viscosity of the fluid and $r_f = r - f^* + u^*$.

In general, it is quite difficult to deal with these exact equations of motion of a viscous fluid. Therefore, we shall make some simplifying assumptions so called "hydraulic approximations". In this approximation, it is assumed that the axial velocity is much larger than the radial one and an averaging procedure with respect to the cross-sectional area is permissible. Applying the averaging procedure to the equations (13)-(15), we have

$$\frac{\partial A^*}{\partial t^*} + \frac{\partial}{\partial z^*}(A w^*) = 0, \quad (16)$$

$$\frac{\partial w^*}{\partial t^*} + w^* \frac{\partial w^*}{\partial z^*} + \frac{1}{\rho_f} \frac{\partial P^*}{\partial z^*} - \frac{\mu_v}{\rho_f} \left( \frac{\partial^2 w^*}{\partial z^{*2}} - \frac{8 w^*}{r_f^2} \right) = 0, \quad (17)$$

where $A$ denotes the inner cross-sectional area, i.e., $A = \pi r_f^2$, $r_f = r - f^* + u^*$ is the final radius of the tube after deformation and other quantities are defined by

$$A w^* = 2\pi \int_0^{r_f} r V_z^* dr, \quad A P^* = 2\pi \int_0^{r_f} r \bar{P} dr. \quad (18)$$

Here $w^*$ is the averaged axial velocity and $P^*$ is the averaged pressure of the fluid. In obtaining (18), we have made use of the following assumption [24]:

$$A (w^*)^2 = 2\pi \int_0^{r_f} r V_z^{*2} dr, \quad \frac{2\mu_v}{\rho_f r_f} \frac{\partial V_z^*}{\partial r}|_{r=r_f} = -\frac{8 \mu_v w^*}{\rho_f r_f^2}. \quad (19)$$

Introducing the expression of $A$ into the equation (16) yields

$$2 \frac{\partial u^*}{\partial t^*} + 2 w^* \left[ -f'^* + \frac{\partial u^*}{\partial z^*} \right] + [r_0 - f^*(z^*) + u^*] \frac{\partial w^*}{\partial z^*} = 0. \quad (20)$$

For the present problem, the fluid reaction force $P_r^*$ takes the following form:

$$P_r^* = \frac{1}{\Lambda} \left[ P^* - \frac{4 \mu_v (-f^{*'} + \partial u^* / \partial z^*) w^*}{r_0 - f^*(z^*) + u^*} \right]. \quad (21)$$

At this stage it is convenient to introduce the following non-dimensional quantities

$$t^* = \left(\frac{R_0}{c_0}\right) t, \qquad z^* = R_0 z, \qquad u^* = R_0 u, \qquad f^* = R_0 f,$$
$$w^* = c_0 w, \qquad \mu_v = c_0 R_0 \rho_f \bar{\nu}, \qquad P^* = \rho_f c_0^2 p, \qquad r_0 = R_0 \lambda_\theta,$$
$$c_0^2 = \frac{\mu H}{\rho_f R_0}, \qquad m = \frac{\rho_0 H}{\rho_f R_0}, \tag{22}$$

where $\lambda_\theta = r_0/R_0$ is the initial stretch ratio, and $c_0$ is the Moens-Korteweg wave speed.

Introducing (22) into the equations (12), (20) and (17), the following non-dimensional equations are obtained

$$\begin{aligned} p \;=\; & \frac{m}{\lambda_z(\lambda_\theta - f(z)+u)}\frac{\partial^2 u}{\partial t^2} + \frac{1}{\lambda_z(\lambda_\theta - f(z)+u)}\frac{\partial \Sigma}{\partial \lambda_2} \\ & -\frac{1}{(\lambda_\theta - f(z)+u)}\frac{\partial}{\partial z}\left\{\frac{(-f' + \partial u/\partial z)}{[1+(-f'+\partial u/\partial z)^2]^{1/2}}\frac{\partial \Sigma}{\partial \lambda_1}\right\} \\ & + \frac{4\bar{\nu}(-f'+\partial u/\partial z)w}{(\lambda_\theta - f(z)+u)}, \end{aligned} \tag{23}$$

$$2\frac{\partial u}{\partial t} + 2w\left[-f' + \frac{\partial u}{\partial z}\right] + [\lambda_\theta - f(z) + u]\frac{\partial w}{\partial z} = 0, \tag{24}$$

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} - \bar{\nu}\left[\frac{\partial^2 w}{\partial z^2} - \frac{8w}{(\lambda_\theta - f(z)+u)^2}\right] = 0. \tag{25}$$

The equations (23)-(25) give sufficient relations to determine the field quantities $u$, $w$, and $p$ completely.

### 3.0 Non-Linear Wave Modulation

In this section, we will examine the amplitude modulation of weakly non-linear waves in a fluid-filled thin elastic with a stenosis whose non-dimensional governing equations are given in equations (23)-(25). Considering the dispersion relation of the linearized field equations and the nature of the problem of concern, which is a boundary-value problem, the following stretched coordinates is introduced:

$$\xi = \epsilon(z - \lambda t), \quad \tau = \epsilon^2 z, \tag{26}$$

where $\epsilon$ is a small parameter measuring the weakness of nonlinearity and $\lambda$ is a constant to be determined from the solution. Solving $z$ in terms of $\tau$, we get

$$z = \epsilon^{-2}\tau. \tag{27}$$

Introducing (27) into the expression of $f(z)$, we obtain

$$f(\epsilon^{-2}\tau) = \hat{h}(\tau). \tag{28}$$

In order to take the effect of stenosis into account, $f(z)$ must be of order of $\epsilon^4$. For the present work, we shall assume that $\hat{h}(\tau)$ have the following form

$$\hat{h}(\tau) = \epsilon^2 h(\tau). \tag{29}$$

Introducing the following differential relations

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} - \epsilon\lambda\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z} + \epsilon\frac{\partial}{\partial \xi} + \epsilon^2\frac{\partial}{\partial \tau}, \tag{30}$$

into the equations (23)-(25), we obtain

$$\begin{aligned} p &= \frac{m}{\lambda_z[\lambda_\theta - \epsilon^2 h(\tau) + u]}\left[\frac{\partial^2 u}{\partial t^2} - 2\epsilon\lambda\frac{\partial^2 u}{\partial t \partial \xi} - \epsilon^2\lambda^2\frac{\partial^2 u}{\partial \xi^2}\right] \\ &+ \frac{1}{\lambda_z[\lambda_\theta - \epsilon^2 h(\tau) + u]}\frac{\partial \Sigma}{\partial \lambda_2} \\ &- \frac{1}{\lambda_\theta - \epsilon^2 h(\tau) + u}\left[\frac{\partial}{\partial z} + \epsilon\frac{\partial}{\partial \xi} + \epsilon^2\frac{\partial}{\partial \tau}\right] \times \\ &\left\{\frac{\left(-\epsilon^2\frac{dh}{d\tau} + \frac{\partial u}{\partial z} + \epsilon\frac{\partial u}{\partial \xi} + \epsilon^2\frac{\partial u}{\partial \tau}\right)}{\left[1 + \left(-\epsilon^2\frac{dh}{d\tau} + \frac{\partial u}{\partial z} + \epsilon\frac{\partial u}{\partial \xi} + \epsilon^2\frac{\partial u}{\partial \tau}\right)^2\right]^{1/2}}\frac{\partial \Sigma}{\partial \lambda_1}\right\}, \end{aligned} \tag{31}$$

$$\begin{aligned} 2\left[\frac{\partial u}{\partial t} - \epsilon\lambda\frac{\partial u}{\partial \xi}\right] + 2w\left[-\epsilon^2\frac{\partial h}{\partial \tau} + \left(\frac{\partial u}{\partial z} + \epsilon\frac{\partial u}{\partial \xi} + \epsilon^2\frac{\partial u}{\partial \tau}\right)\right] & \\ +[\lambda_\theta - \epsilon^2 h(\tau) + u]\left[\frac{\partial w}{\partial z} + \epsilon\frac{\partial w}{\partial \xi} + \epsilon^2\frac{\partial w}{\partial \tau}\right] &= 0, \end{aligned} \tag{32}$$

$$\begin{aligned} \frac{\partial w}{\partial t} - \epsilon\lambda\frac{\partial w}{\partial \xi} + w\left(\frac{\partial w}{\partial z} + \epsilon\frac{\partial w}{\partial \xi} + \epsilon^2\frac{\partial w}{\partial \tau}\right) + \frac{\partial p}{\partial z} + \epsilon\frac{\partial p}{\partial \xi} + \epsilon^2\frac{\partial p}{\partial \tau} & \\ -\epsilon^2\nu\left[\frac{\partial^2 w}{\partial z^2} + 2\epsilon\frac{\partial^2 w}{\partial \xi \partial z} + 2\epsilon^2\frac{\partial^2 w}{\partial z \partial \tau} + \epsilon^2\frac{\partial^2 w}{\partial \xi^2}\right. & \\ \left. +2\epsilon^3\frac{\partial^2 w}{\partial \xi \partial \tau} + \epsilon^4\frac{\partial^2 w}{\partial \tau^2} - \frac{8w}{[\lambda_\theta - \epsilon^2 h(\tau) + u]^2}\right] &= 0. \end{aligned} \tag{33}$$

Here, in order to take the effect of viscosity into account, the order of viscosity is assumed to be $O(\epsilon^2)$, i.e. $\bar{\nu} = \epsilon^2\nu$. For the long wave limit, it is assumed that the field quantities may be expanded into asymptotic series of $\epsilon$ as

$$\begin{aligned} u &= \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + ..., \\ w &= \epsilon w_1 + \epsilon^2 w_2 + \epsilon^3 w_3 + ..., \\ p &= p_0 + \epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3 + ..., \\ h(\tau) &= \epsilon^2 h_1(\tau) + \epsilon^3 h_2(\tau) + ... \end{aligned} \tag{34}$$

Introducing the expansions (34) into the equations (31)-(33), the following sets of differential equations are obtained

*$O(\epsilon)$ equations*

$$\begin{aligned} p_1 = \frac{m}{\lambda_\theta\lambda_z}\frac{\partial^2 u_1}{\partial t^2} - \alpha_0\frac{\partial^2 u_1}{\partial z^2} + \beta_1 u_1, & \\ 2\frac{\partial u_1}{\partial t} + \lambda_\theta\frac{\partial w_1}{\partial z} = 0, \quad \frac{\partial w_1}{\partial t} + \frac{\partial p_1}{\partial z} = 0,. & \end{aligned} \tag{35}$$

*$O(\epsilon^2)$ equations*

$$
\begin{aligned}
p_2 \;=\;& \frac{m}{\lambda_\theta \lambda_z}\frac{\partial^2 u_2}{\partial t^2} - \alpha_0 \frac{\partial^2 u_2}{\partial z^2} + \beta_1 (u_2 - h_1) \\
& - \frac{2m\lambda}{\lambda_\theta \lambda_z}\frac{\partial^2 u_1}{\partial \xi \partial t} - 2\alpha_0 \frac{\partial^2 u_1}{\partial \xi \partial z} - \frac{m}{\lambda_\theta^2 \lambda_z} u_1 \frac{\partial^2 u_1}{\partial t^2} \\
& - \alpha_1 \left(\frac{\partial u_1}{\partial z}\right)^2 - \left(2\alpha_1 - \frac{\alpha_0}{\lambda_\theta}\right) u_1 \frac{\partial^2 u_1}{\partial z^2} + \beta_2 u_1^2, \\
& 2\frac{\partial u_2}{\partial t} + \lambda_\theta \frac{\partial w_2}{\partial z} - 2\lambda \frac{\partial u_1}{\partial \xi} + \lambda_\theta \frac{\partial w_1}{\partial \xi} + u_1 \frac{\partial w_1}{\partial z} + 2 w_1 \frac{\partial u_1}{\partial z} = 0, \\
& \frac{\partial w_2}{\partial t} + \frac{\partial p_2}{\partial z} - \lambda \frac{\partial w_1}{\partial \xi} + \frac{\partial p_1}{\partial \xi} + w_1 \frac{\partial w_1}{\partial z} = 0.
\end{aligned} \tag{36}
$$

*$O(\epsilon^3)$ equations*

$$
\begin{aligned}
p_3 \;=\;& \frac{m}{\lambda_\theta \lambda_z}\frac{\partial^3 u_3}{\partial t^3} - \alpha_0 \frac{\partial^2 u_3}{\partial z^3} - \frac{2m\lambda}{\lambda_\theta \lambda_z}\frac{\partial^2 u_2}{\partial \xi \partial t} - 2\alpha_0 \frac{\partial^2 u_2}{\partial \xi \partial z} \\
& - \alpha_0 \left(\frac{\partial^2 u_1}{\partial \xi^2} + 2\frac{\partial^2 u_1}{\partial z \partial \tau}\right) + \frac{m\lambda^2}{\lambda_\theta^2 \lambda_z}\frac{\partial^2 u_1}{\partial \xi^2} + \beta_1 (u_3 - h_2) \\
& - \frac{m}{\lambda_\theta^2 \lambda_z} u_1 \left(\frac{\partial^2 u_2}{\partial t^1} - 2\lambda \frac{\partial^2 u_1}{\partial \xi \partial t}\right) - \frac{m}{\lambda_\theta^2 \lambda_z}(u_2 - h_1)\frac{\partial^2 u_1}{\partial t^2} \\
& - 2\alpha_1 \frac{\partial u_1}{\partial z}\left(\frac{\partial u_2}{\partial z} + \frac{\partial u_1}{\partial \xi}\right) - \left(2\alpha_1 - \frac{\alpha_0}{\lambda_\theta}\right) u_1 \left(\frac{\partial^2 u_2}{\partial z^2} + 2\frac{\partial^2 u_1}{\partial z \partial \xi}\right) \\
& - \left(2\alpha_1 - \frac{\alpha_0}{\lambda_\theta}\right)(u_2 - h_1)\frac{\partial^2 u_1}{\partial z^2} + 2\beta_2 u_1 (u_2 - h_1) \\
& + \frac{m}{\lambda_\theta^3 \lambda_z} u_1^2 \frac{\partial^2 u_1}{\partial t^2} - \left(\alpha_2 - \frac{\alpha_1}{\lambda_\theta}\right) u_1 \left(\frac{\partial u_1}{\partial z}\right)^2 \\
& - \left(\alpha_2 - \frac{2\alpha_1}{\lambda_\theta} + \frac{\alpha_0}{\lambda_\theta^2}\right) u_1^2 \frac{\partial^2 u_1}{\partial z^2} - 3\left(\gamma_1 - \frac{\alpha_0}{2}\right)\left(\frac{\partial u_1}{\partial z}\right)^2 \frac{\partial^2 u_1}{\partial z^2} + \beta_3 u_1^3, \\
& 2\frac{\partial u_3}{\partial t} + \lambda_\theta \frac{\partial w_3}{\partial z} - 2\lambda \frac{\partial u_2}{\partial \xi} + \lambda_\theta \frac{\partial w_2}{\partial \xi} + 2 w_1 \left(\frac{\partial u_1}{\partial \xi} + \frac{\partial u_2}{\partial z}\right) \\
& + 2 w_2 \frac{\partial u_1}{\partial z} + \lambda_\theta \frac{\partial w_1}{\partial \tau} + u_1 \left(\frac{\partial w_2}{\partial z} + \frac{\partial w_1}{\partial \xi}\right) + (u_2 - h_1)\frac{\partial w_1}{\partial z} = 0, \\
& \frac{\partial w_3}{\partial t} + \frac{\partial p_3}{\partial z} - \lambda \frac{\partial w_2}{\partial \xi} + \frac{\partial p_2}{\partial \xi} + \frac{\partial p_1}{\partial \tau} + w_1 \frac{\partial w_2}{\partial z} + w_2 \frac{\partial w_1}{\partial z} + w_1 \frac{\partial w_1}{\partial \xi} \\
& - \nu \left(\frac{\partial^2 w_1}{\partial z^2} - \frac{8 w_1}{\lambda_\theta^2}\right) = 0.
\end{aligned} \tag{37}
$$

Here the coefficients of $\alpha_0$, $\alpha_1$, $\alpha_2$, $\beta_0$, $\beta_1$, $\beta_2$, $\beta_3$ and $\gamma_1$are defined by

$$
\begin{aligned}
&\alpha_0 = \frac{1}{\lambda_\theta}\frac{\partial \Sigma}{\partial \lambda_z}, && \alpha_1 = \frac{1}{2\lambda_\theta}\frac{\partial^2 \Sigma}{\partial \lambda_\theta \lambda_z}, && \alpha_2 = \frac{1}{2\lambda_\theta}\frac{\partial^3 \Sigma}{\partial \lambda_\theta^2 \lambda_z}, \\
&\beta_0 = \frac{1}{\lambda_\theta \lambda_z}\frac{\partial \Sigma}{\partial \lambda_\theta}, && \beta_1 = \frac{1}{\lambda_\theta \lambda_z}\frac{\partial^2 \Sigma}{\partial \lambda_\theta^2} - \frac{\beta_0}{\lambda_\theta}, && \beta_2 = \frac{1}{2\lambda_\theta \lambda_z}\frac{\partial^3 \Sigma}{\partial \lambda_\theta^3} - \frac{\beta_1}{\lambda_\theta}, \\
&\beta_3 = \frac{1}{6\lambda_\theta \lambda_z}\frac{\partial^4 \Sigma}{\partial \lambda^4} - \frac{\beta_2}{\lambda_\theta}, && \gamma_1 = \frac{\lambda_z}{2\lambda_\theta}\frac{\partial^2 \Sigma}{\partial \lambda_z^2}.
\end{aligned} \tag{38}
$$

Equation (38) are defined through series expansion of the stretch ratios $\lambda_1$ and $\lambda_2$, which read

$$\begin{aligned}\lambda_1 &= \lambda_z\left[1+\left(-\epsilon^4 h_1'(\tau)-\epsilon^5 h_2'(\tau)+\frac{\partial u}{\partial z}+\epsilon\frac{\partial u}{\partial \xi}+\epsilon^2\frac{\partial u}{\partial \tau}\right)^2\right]^{1/2},\\ \lambda_2 &= \lambda_\theta+\epsilon u_1+\epsilon^2[u_2-h_1(\tau)]+\epsilon^3[u_3-h_2(\tau)].\end{aligned} \tag{39}$$

### 3.1 Solution of the field equations

*3.1.1 The Solution of $O(\epsilon)$ equations*

Seeking the following type of solution to the differential equations (35):

$$\begin{aligned}u_1 &= (U_1 e^{i\theta}+c.c),\\ w_1 &= (W_1 e^{i\theta}+c.c),\\ p_1 &= \left(-\frac{m\omega^2}{\lambda_\theta\lambda_z}+\alpha_0 k^2+\beta_1\right)U_1 e^{i\theta}+c.c,\end{aligned} \tag{40}$$

where $U_1$ and $W_1$ are unknown functions of the slow variables $(\xi,\tau)$, $\theta=\omega t-kz$ is the phasor and $c.c$ is the complex conjugate of the corresponding expressions, $\omega$ is the angular frequency, $k$ is the wave number, we obtain

$$U_1=U(\xi,\tau),\quad W_1=\frac{2\omega}{\lambda_\theta k}U, \tag{41}$$

provided that the following dispersion relation holds true:

$$\omega^2=\frac{\lambda_\theta\lambda_z k^2(\alpha_0 k^2+\beta_1)}{2\lambda_z+mk^2}. \tag{42}$$

Here $U(\xi,\tau)$ is an unknown function whose governing equation will be obtained later.

*3.1.2 The Solution of $O(\epsilon^2)$ equations*

Introducing the solutions (40)-41) into (37) gives

$$\begin{aligned}p_2 &= \frac{m}{\lambda_\theta\lambda_z}\frac{\partial^2 u_2}{\partial t^2}-\alpha_0\frac{\partial^2 u_2}{\partial z^2}+\beta_1(u_2-h_1)+2\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z}+\alpha_1 k^2-\frac{\alpha_0 k^2}{\lambda_\theta}+\beta_2\right)|U|^2\\ &\quad+2i\left(\alpha_0 k-\frac{m\omega\lambda}{\lambda_\theta\lambda_z}\right)\frac{\partial U}{\partial \xi}e^{i\theta}+\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z}+3\alpha_1 k^2-\frac{\alpha_0 k^2}{\lambda_\theta}+\beta_2\right)U^2 e^{2i\theta}+c.c,\\ &2\frac{\partial u_2}{\partial t}+\lambda_\theta\frac{\partial w_2}{\partial z}+2\left(\frac{\omega}{k}-\lambda\right)\frac{\partial U}{\partial \xi}e^{i\theta}-6i\frac{\omega}{\lambda_\theta}U^2 e^{2i\theta}+c.c=0,\\ &\frac{\partial w_2}{\partial t}+\frac{\partial p_2}{\partial z}+\left(-2\frac{\lambda\omega}{\lambda_\theta k}-\frac{m\omega^2}{\lambda_\theta\lambda_z}+\alpha_0 k^2+\beta_1\right)\frac{\partial U}{\partial \xi}e^{i\theta}-4i\frac{\omega^2}{\lambda_\theta^2 k}U^2 e^{2i\theta}+c.c=0,\end{aligned} \tag{43}$$

where $|U|^2=UU*, U*$ is the complex conjugate of $U$.

Seeking the following type of solutions

$$\begin{aligned}u_2 &= U_2^{(0)}+\left(\sum_{l=1}^{2}U_2^{(l)}e^{il\theta}+c.c\right),\\ w_2 &= W_2^{(0)}+\sum_{l=1}^{2}W_2^{(l)}e^{il\theta}+c.c,\\ p_2 &= P_2^{(0)}+\sum_{l=1}^{2}P_2^{(l)}e^{il\theta}+c.c,\end{aligned} \tag{44}$$

to (43) yields:

$$P_2^0 = \beta_1(U_2^{(0)} - h_1) + 2\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z} + \alpha_1 k^2 - \frac{\alpha_0}{\lambda_\theta}k^2 + \beta_2\right)|U|^2, \tag{45}$$

$$2\omega U_2^{(1)} - \lambda_\theta k W_2^{(1)} = 2i\left(\frac{\omega}{k} - \lambda\right)\frac{\partial U}{\partial \xi}, \tag{46}$$

$$\omega W_2^{(1)} - k\left(-\frac{m\omega^2}{\lambda_\theta\lambda_z} + \alpha_0 k^2 + \beta_1\right)U_2^{(1)}$$
$$= i\left(-\frac{2\lambda\omega}{\lambda_\theta k} - \frac{2m\omega\lambda k}{\lambda_\theta\lambda_z} - \frac{m\omega^2}{\lambda_\theta\lambda_z} + 3\alpha_0 k^2 + \beta_1\right)\frac{\partial U}{\partial \xi}, \tag{47}$$

$$P_2^{(1)} = -\frac{m\omega^2}{\lambda_\theta\lambda_z}U_2^{(1)} + \alpha_0 k^2 U_2^{(1)} + \beta_1 U_2^{(1)} + 2i\left(\alpha_0 k - \frac{m\omega\lambda}{\lambda_\theta\lambda_z}\right)\frac{\partial U}{\partial \xi}, \tag{48}$$

$$2\omega U_2^{(2)} - \lambda_\theta k W_2^{(2)} = 3\frac{\omega}{\lambda_\theta}U^2, \tag{49}$$

$$\omega W_2^{(2)} - k\left(-\frac{4m\omega^2}{\lambda_\theta\lambda_z} + 4\alpha_0 k^2 + \beta_1\right)U_2^{(2)}$$
$$= \left(\frac{2\omega^2}{\lambda_\theta^2 k} + \frac{m\omega^2 k}{\lambda_\theta^2\lambda_z} + 3\alpha_1 k^3 - \frac{\alpha_0}{\lambda_\theta}k^3 + \beta_2 k\right)U^2. \tag{50}$$

Taking $U_2^{(1)} = 0$ and solving Eq.(46), we get

$$W_2^{(1)} = i\frac{2}{\lambda_\theta k}\lambda k\left(\lambda - \frac{\omega}{k}\right)\frac{\partial U}{\partial \xi}. \tag{51}$$

Introducing Eq. (51) into Eq.(47) leads to

$$\left[\lambda\omega k\left(2 + \frac{mk^2}{\lambda_z}\right) - (2\omega^2 + \lambda_\theta\alpha_0 k^4)\right]\frac{\partial U}{\partial \xi} = 0. \tag{52}$$

In order to have nonzero solution for $U$, the coefficient of $\frac{\partial U}{\partial \xi}$ in (52) must vanish, that is

$$\lambda\omega k\left(2 + \frac{mk^2}{\lambda_z}\right) - (2\omega^2 + \lambda_\theta\alpha_0 k^4) = 0. \tag{53}$$

or

$$\lambda = \frac{\lambda_z(2\omega^2 + \lambda_\theta\alpha_0 k^4)}{\omega k(2\lambda_z + mk^2)}(\text{group velocity}). \tag{54}$$

Solving Eqs (49)- (50) leads to

$$U_2^{(2)} = \Phi_0 U^2, \quad W_2^{(2)} = \frac{2\omega}{\lambda_\theta k}U_2^{(2)} - \frac{3\omega}{\lambda_\theta^2 k}U^2,$$
$$\Phi_0 = \frac{\frac{3\omega^2}{\lambda_\theta} + k^2\beta_1 + 3\alpha_1\lambda_\theta k^4 + \lambda_\theta\beta_2 k^2}{3(\beta_1\lambda_\theta k^2 - 2\omega^2)}. \tag{55}$$

*3.1.3 The Solution of $O(\epsilon^3)$ equations*

Introducing the following type of solutions

$$u_3 = U_3^{(0)} + \left(\sum_{l=1}^{3} U_3^{(l)} e^{il\theta} + c.c\right),$$
$$w_3 = W_3^{(0)} + \sum_{l=1}^{3} W_3^{(l)} e^{il\theta} + c.c,$$
$$p_3 = P_3^{(0)} + \sum_{l=1}^{3} P_3^{(l)} e^{il\theta} + c.c, \tag{56}$$

to into O($\epsilon^3$) equations (37), we obtain the zeroth- and first-order equations below:

$$\begin{aligned} -2\lambda\frac{\partial U_2^{(0)}}{\partial\xi}+\lambda_\theta\frac{\partial W_2^{(0)}}{\partial\xi}+\frac{6\omega}{\lambda_\theta k}\frac{\partial}{\partial\xi}|U|^2 &= 0,\\ -\lambda\frac{\partial W_2^{(0)}}{\partial\xi}+\frac{\partial P_2^{(0)}}{\partial\xi}+\frac{4\omega^2}{\lambda_\theta^2k^2}\frac{\partial}{\partial\xi}|U|^2 &= 0, \end{aligned} \tag{57}$$

$$\begin{aligned} P_3^{(1)} =& \left(\alpha_0k^2-\frac{m\omega^2}{\lambda_\theta\lambda_z}+\beta_1\right)U_3^{(1)}+\left(\frac{m\lambda^2}{\lambda_\theta\lambda_z}-\alpha_0\right)\frac{\partial^2U}{\partial\xi^2}+2i\alpha_0k\frac{\partial U}{\partial\tau}\\ &+\left(\frac{5m\omega^2}{\lambda_\theta^2\lambda_z}+6\alpha_1k^2-\frac{5\alpha_0k^2}{\lambda_\theta}+2\beta_2\right)U_2^{(2)}U*\\ &\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z}+2\alpha_1k^2-\frac{\alpha_0k^2}{\lambda_\theta}+2\beta_2\right)(U_2^{(0)}-h_1)U\\ &+\left[-\frac{3m\omega^2}{\lambda_\theta^3\lambda_z}+2\alpha_2k^2-\frac{5\alpha_1k^2}{\lambda_\theta}+\frac{3\alpha_0k^2}{\lambda_\theta^2}+3\left(\gamma_1-\frac{\alpha_0}{2}\right)k^4+3\beta_3\right]|U|^2U,\\ &2i\omega U_3^{(1)}-ik\lambda_\theta W_3^{(1)}+\lambda_\theta\frac{\partial W_2^{(1)}}{\partial\xi}+\frac{2\omega}{k}\frac{\partial U}{\partial\tau}-\frac{6i\omega}{\lambda_\theta}U_2^{(2)}U^*\\ &-2i\left(kW_2^{(0)}+\frac{\omega}{\lambda_\theta}U_2^{(0)}-\frac{\omega}{\lambda_\theta}h_1\right)U=0,\\ &i\omega W_3^{(1)}-ikP_3^{(1)}-\lambda\frac{\partial W_2^{(1)}}{\partial\xi}+\frac{\partial P_2^{(1)}}{\partial\xi}+\left(-\frac{m\omega^2}{\lambda_\theta\lambda_z}+\alpha_0k^2+\beta_1\right)\frac{\partial U}{\partial\tau}\\ &-\frac{2i\omega}{\lambda_\theta}W_2^{(2)}U^*-\frac{2i\omega}{\lambda_\theta}W_2^{(0)}U+\frac{2\nu\omega k}{\lambda_\theta}U+\frac{16\nu\omega}{\lambda_\theta^3k}U=0. \end{aligned} \tag{58}$$

From the solution of the equations (57) and (45), results in

$$\begin{aligned} U_2^{(0)} &= \Phi_1|U|^2-\Phi_2h_1,\quad W_2^{(0)}=\frac{2\lambda}{\lambda_\theta}U_2^{(0)}-\frac{6\omega}{\lambda_\theta^2k}|U|^2,\\ \Phi_1 &= \frac{\frac{3\lambda\omega}{\lambda_\theta k}+\frac{2\omega^2}{\lambda_\theta k^2}+\frac{m\omega^2}{\lambda_\theta\lambda_z}+\alpha_1\lambda_\theta k^2-\alpha_0k^2+\lambda_\theta\beta_2}{\lambda^2-\frac{\lambda_\theta\beta_1}{2}},\\ \Phi_2 &= \frac{\lambda_\theta\beta_1}{2\lambda^2-\lambda_\theta\beta 1}. \end{aligned} \tag{59}$$

Finally, eliminating $U_3^{(1)}$,$W_3^{(1)}$ and $P_3^{(1)}$ between Eq.(58) through the use of dispersion relation (42), Eqs (48), (51), (55) and (59), we obtain the following dissipative NLS equation with variable coefficient:

$$i\frac{\partial U}{\partial\tau}+\mu_1\frac{\partial^2U}{\partial\xi^2}+\mu_2|U|^2U-\mu_3h_1(\tau)U+i\mu_4U=0, \tag{60}$$

where the coefficients $\mu_1$, $\mu_2$, $\mu_3$ and $\mu_4$ are defined by

$$\begin{aligned}
\mu &= \frac{2\omega^2}{k} + 3\alpha_0\lambda_\theta k^3 - \frac{mk\omega^2}{\lambda_z} + \lambda_\theta\beta_1 k,\\
\mu_1 &= \mu^{(-1)}\left[-\frac{4\lambda\omega}{k} + \frac{2\omega^2}{k} + 2\lambda^2 + \frac{m\lambda^2k^2}{\lambda_z} - 3\alpha_0\lambda_\theta k^2 + \frac{2m\omega\lambda k}{\lambda_z}\right],\\
\mu_2 &= \mu^{(-1)}\left\{\left[\frac{10\omega^2}{\lambda_\theta} + \lambda_\theta k^2\left(\frac{5m\omega^2}{\lambda_\theta^2\lambda_z} + 6\alpha_1 k^2 - \frac{5\alpha_0 k^2}{\lambda_\theta} + 2\beta_2\right)\right]\Phi_0\right.\\
&\quad + \left[\frac{8\omega\lambda k}{\lambda_\theta} + \frac{2\omega^2}{\lambda_\theta} + \lambda_\theta k^2\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z} + 2\alpha_1 k^2 - \frac{\alpha_0 k}{\lambda_\theta} + 2\beta_2\right)\right]\Phi_1\\
&\quad \left. - \frac{30\omega^2}{\lambda_\theta} + \lambda_\theta k^2\left[-\frac{3m\omega^2}{\lambda_\theta^3\lambda_z} + 2\alpha_2 k^2 - \frac{5\alpha_1 k^2}{\lambda_\theta} + \frac{3\alpha_0 k^2}{\lambda_\theta^2} + 3\left(\gamma_1 - \frac{\alpha_0}{2}\right)k^4 + 3\beta_3\right]\right\}\\
\mu_3 &= \mu^{(-1)}\left\{\left[\frac{2\omega^2}{\lambda_\theta} + \frac{8\omega\lambda k}{\lambda_\theta} + k^2\lambda_\theta\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z} + 2\alpha_1 k^2 - \frac{\alpha_0 k^2}{\lambda_\theta} + 2\beta_2\right)\right]\Phi_2\right.\\
&\quad \left.\left[\frac{2\omega^2}{\lambda_\theta} + \lambda_\theta k^2\left(\frac{m\omega^2}{\lambda_\theta^2\lambda_z} + 2\alpha_1 k^2 - \frac{\alpha_0 k^2}{\lambda_\theta} + 2\beta_2\right)\right]\right\}\\
\mu_4 &= \mu^{(-1)}\left[2\nu\omega\left(k^2 + \frac{8}{\lambda_\theta^2}\right)\right].
\end{aligned} \tag{61}$$

Introducing the following change of variable:

$$U = V(\xi,\tau)\exp\left[-i\mu_3\int_0^\tau h_1(s)ds - \mu_4\tau\right], \tag{62}$$

equation (60) reduces to the following conventional NLS equations:

$$i\frac{\partial V}{\partial\tau} + \mu_1\frac{\partial^2 V}{\partial\xi^2} + \mu_2|V|^2V = 0. \tag{63}$$

## 4 Progressive Wave Solution

In this subsection, we will present the progressive wave solution to the evolution equation given in (63) of the following form :

$$V(\xi,\ \tau) = F(\zeta)\exp\left[i(K\xi - \Omega\tau)\right], \quad \zeta = \xi - c\tau, \tag{64}$$

where $\Omega$, $K$ and $c$ are some constants and $F(\zeta)$ is a real-valued unknown function to be determined from the solution. Introducing (64) into (63), we have

$$\mu_1\frac{\partial^2 F}{\partial\zeta^2} + i(2\mu_1 K - c)\frac{\partial F}{\partial\zeta} + (\Omega - \mu_1 K^2)F + \mu_2 F^3 = 0. \tag{65}$$

By letting $c = 2\mu_1 K$, the $\frac{\partial F}{\partial\zeta}$ term can be eliminated and choosing $\Omega = \mu_1 K^2 - \frac{\mu_2 a^2}{2}$, where $a$ is the amplitude of the wave, we obtain

$$\mu_1\frac{\partial^2 F}{\partial\zeta^2} - \frac{\mu_2 a^2}{2}F + \mu_2 F^3 = 0. \tag{66}$$

Multiplying the above equation (66) by $2\frac{\partial F}{\partial\zeta}$ and then integrate it yields

$$\mu_1\left(\frac{\partial F}{\partial\zeta}\right)^2 = A + \frac{\mu_2 a^2}{2}F^2 - \frac{\mu_2}{2}F^4, \tag{67}$$

where $A$ is the integration constant. A special case that gives the single soliton is where $F(\pm\infty) = 0$ and $A = 0$ yields

$$\mu_1(\frac{\partial F}{\partial \zeta})^2 = \frac{\mu_2 a^2}{2}F^2 - \frac{\mu_2}{2}F^4. \quad (68)$$

By solving the equation (68), the soliton modulated wave solution to NLS equation (63) is given by

$$V(\xi,\tau) = a\,sech\left[\sqrt{\frac{\mu_2}{2\mu_1}}(\xi - c\tau)\right]\exp[i(K\xi - \Omega\tau)], \quad (69)$$

where the modulus of $V(\xi,\tau)$ will be given by

$$V(\xi,\tau) = a\,sech\left[\sqrt{\frac{\mu_2}{2\mu_1}}(\xi - c\tau)\right]. \quad (70)$$

Substituting the solution of standard NLS equation (69) into equation (62), we obtain the solution of the dissipative NLS equation with variable coefficient (60) as

$$U(\xi,\tau) = a\,sech\left[\sqrt{\frac{\mu_2}{2\mu_1}}(\xi - c\tau)\right]\exp\left[i(K\xi - \Omega\tau - \mu_3\int_0^\tau h_1(s)ds) - \mu_4\tau\right], \quad (71)$$

where the modulus of $U(\xi,\tau)$ is given by

$$U(\xi,\tau) = a\,sech\left[\sqrt{\frac{\mu_2}{2\mu_1}}(\xi - c\tau)\right]\exp\left[-\mu_4\tau\right]. \quad (72)$$

The speed of the enveloping wave is constant and equal to $2\mu_1 K$. On the other hand, the speed of the harmonic wave is given by

$$v_p = \frac{K}{\Omega + \mu_3 h_1(\tau) - i\mu_4}. \quad (73)$$

## 5 Numerical Results

For numerical calculation, we need the values of the coefficients $\alpha_0$, $\alpha_1$, $\alpha_2$, $\beta_0$, $\beta_1$, $\beta_2$, $\beta_3$, $\gamma_1$,$\mu_1$, $\mu_2$, $\mu_3$, $\mu_4$, In order to do that, we must know the constitutive relation of the tube material. In this work, we will utilize the constitutive relation proposed by Demiray [25] for soft biological tissues. Following Demiray [25], the strain energy density function may be expressed as

$$\Sigma = \frac{1}{2\alpha}\{\exp\left[\alpha(\lambda_\theta^2 + \lambda_z^2 + \frac{1}{\lambda_\theta^2\lambda_z^2} - 3)\right] - 1\}, \quad (74)$$

where $\alpha$ is a material constant and $I_1$ is the first invariant of Finger deformation tensor defined by $I_1 = \lambda_\theta^2 + \lambda_z^2 + 1/(\lambda_\theta^2\lambda_z^2)$. Introducing (74) into equation (38), the explicit expressions of the coefficients $\alpha_0$, $\alpha_1$, $\alpha_2$, $\beta_0$, $\beta_1$, $\beta_2$, $\beta_3$ and $\gamma_1$ are obtained as :

$$
\begin{aligned}
\alpha_0 &= \frac{1}{\lambda_\theta}\left(\lambda_z - \frac{1}{\lambda_\theta^2\lambda_z^3}\right) G(\lambda_\theta, \lambda_z), \\
\alpha_1 &= \left[\frac{1}{\lambda_\theta^4\lambda_z^3} + \alpha\left(\lambda_z - \frac{1}{\lambda_\theta^2\lambda_z^3}\right)\left(1 - \frac{1}{\lambda_\theta^4\lambda_z^2}\right)\right] G(\lambda_\theta, \lambda_z), \\
\alpha_2 &= \left[\frac{2\alpha^2}{\lambda_\theta}\left(\lambda_z - \frac{1}{\lambda_\theta^2\lambda_z^3}\right)\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^2 + \frac{3\alpha}{\lambda_\theta^3\lambda_z^3}\left(1 - \frac{7}{3\lambda_\theta^4\lambda_z^2}\right)\right. \\
&\quad \left. + \frac{\alpha\lambda_z}{\lambda_\theta} + \frac{3}{\lambda_\theta^5\lambda_z}\left(\alpha - \frac{1}{\lambda_z^2}\right)\right] G(\lambda_\theta, \lambda_z), \\
\beta_0 &= \left[\frac{1}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)\right] G(\lambda_\theta, \lambda_z), \\
\beta_1 &= \left[\frac{2\alpha}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^2 + \frac{4}{\lambda_\theta^5\lambda_z^3}\right] G(\lambda_\theta, \lambda_z), \\
\beta_2 &= \left[\frac{2\alpha^2}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^3 - \frac{10}{\lambda_\theta^6\lambda_z^3}\right. \\
&\quad \left. + \frac{\alpha}{\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)\left(1 + \frac{11}{\lambda_\theta^4\lambda_z^2}\right)\right] G(\lambda_\theta, \lambda_z), \\
\beta_3 &= \left[\frac{20}{\lambda_\theta^7\lambda_z^3} + \frac{4\alpha^3}{3\lambda_\theta\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^4 + \frac{4\alpha^2}{\lambda_\theta\lambda_z}\left(1 + \frac{3}{\lambda_\theta^4\lambda_z^2}\right)\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^2\right. \\
&\quad - \frac{16\alpha}{\lambda_\theta^6\lambda_z^3}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right) + \frac{\alpha}{\lambda_\theta\lambda_z}\left(1 + \frac{3}{\lambda_\theta^4\lambda_z^2}\right)^2 - \frac{2\alpha^2}{\lambda_\theta^2\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)^3 \\
&\quad \left. - \frac{\alpha}{\lambda_\theta^2\lambda_z}\left(\lambda_\theta - \frac{1}{\lambda_\theta^3\lambda_z^2}\right)(1 + \frac{11}{\lambda_\theta^4\lambda_z^2})\right] G(\lambda_\theta, \lambda_z), \\
\gamma_1 &= \left[\frac{\alpha\lambda_z}{\lambda_\theta}\left(\lambda_z - \frac{1}{\lambda_\theta^2\lambda_z^3}\right)^2 + \frac{\lambda_z}{2\lambda_\theta} + \frac{3}{2\lambda_\theta^3\lambda_z^3}\right] G(\lambda_\theta, \lambda_z), \qquad (75)
\end{aligned}
$$

where the function $G$ is defined by

$$G(\lambda_\theta, \lambda_z) = \exp\left[\alpha\left(\lambda_\theta^2 + \lambda_z^2 + \frac{1}{\lambda_\theta^2\lambda_z^2} - 3\right)\right]. \qquad (76)$$

Right now, we need the value of the material constant $\alpha$. For the static case, the present model was compared by Demiray[25] with the experimental measurement by Simon et al [27] on canine abdominal artery with the characteristics $R_i = 0.31cm$, $R_0 = 0.38cm$ and $\lambda_z = 1.53$ and the value of the material constant $\alpha$ was found to be $\alpha = 1.948$. Using this numerical value of the coefficient $\alpha$, and for the initial deformation $\lambda_\theta = \lambda_z = 1.6$, we obtain $\alpha_0 = 78.6924$, $\alpha_1 = 233.7666$, $\alpha_2 = 1563.4837$, $\beta_0 = 49.1827$, $\beta_1 = 296.1049$, $\beta_2 = 991.4958$, $\beta_3$=2384.8778, $\gamma_1 = 418.3605$, $\omega = 41.6845$, $\lambda = 29.2660$, $\Phi_0 = -6.0631$, $\Phi_1 = 7.2986$, $\Phi_2 = 0.3823$, $\mu_1 = 0.003449$, $\mu_2 = 26.3303$, $\mu_3 = 7.3572$, $\mu_4 = 0.1082$, provided $m = 0.1$, $\nu = 1$, $k = 2$ and $K = 2$.

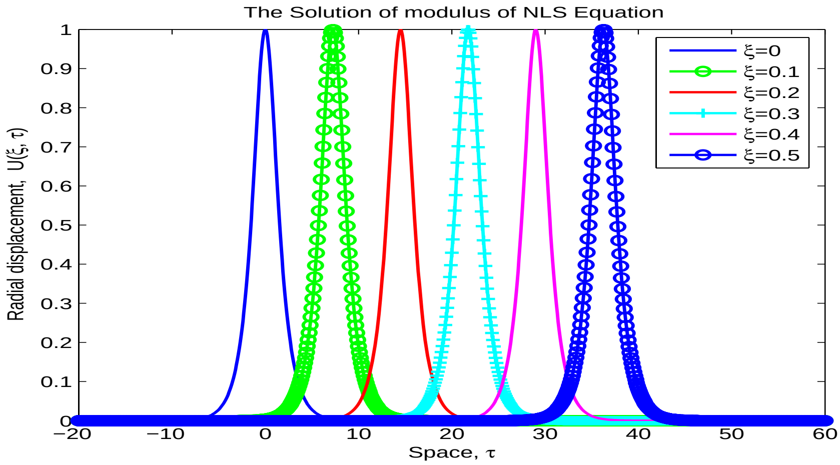


Figure 1: The solution of modulus of NLS equation versus space $\tau$

Figure (1) shows the solution of modulus of the NLS equation (63) versus space $\tau$ at different time $\xi$. It shows that the modulus of the NLS equation admits solitary wave solution and propagates to the right with same amplitude as time $\xi$ increases.

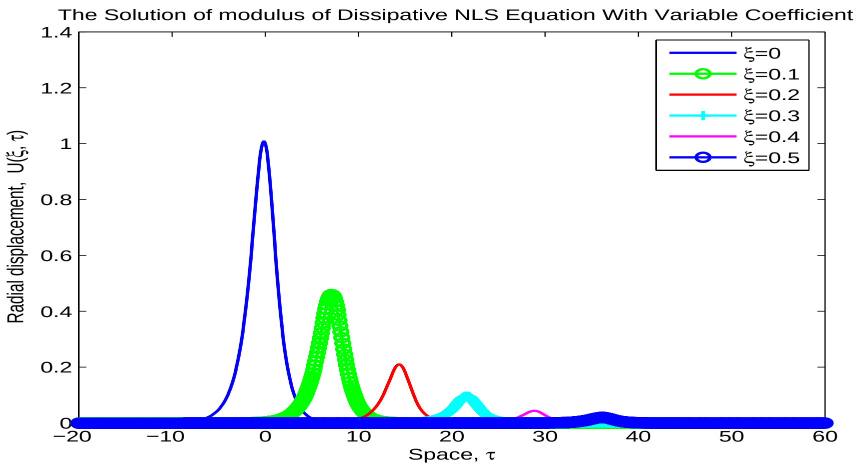


Figure 2: The solution of modulus of dissipative NLS equation with variable coefficient versus space $\tau$

The solution of modulus of the dissipative NLS equation with variable coefficient (60) versus space $\tau$ at different time $\xi$ is shown in Figure (2). It is shown that as time $\xi$ increases, the initial wave propagates to the right with decreasing amplitude due to the effect of the viscosity.

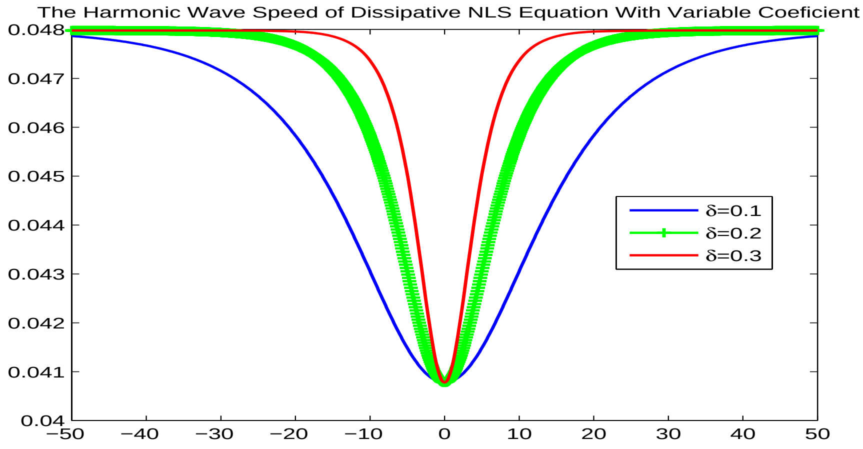


Figure 3: The speed of harmonic wave

Figure (3) illustrates the speed of harmonic wave of the dissipative NLS equation with variable coefficient versus space tau $\tau$ at different $\delta$, where $\delta$ specify the sharpness of stenosis function $f(\tau) = sech(\delta\tau)$. The graph shows the speed is minimum at the center of stenosis and increases to a constant value of 0.048 as it goes away from center of stenosis. If the shape of the stenosis is sharp, the wave speed increase rapidly.

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