

NUMERICAL CONFORMAL MAPPING
VIA THE BERGMAN KERNEL
USING FOURIER METHOD

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ABSTRACT

The Szegő kernel and the Bergman kernel of a simply connected region in the complex plane are kernel functions which are related to the Riemann mapping function. An efficient method based on the Kerzman-Stein-Trummer integral equation for computing the Szegő kernel has been known since 1986. In 1997, integral equation for the Bergman kernel which can be used effectively for numerical conformal mapping has also been established. Both of these integral equations have been solved by means of Nyström's method. Our subject of study is based on integral equation for Bergman kernel, where we had solved this integral equation by means of Fourier method. Since integral equation for Bergman kernel has not yet been solved using Fourier method, the numerical results can also be used to compare with those obtained from Nyström's method. As a result, Fourier method is capable to produce approximations of comparable accuracy to the Nyström's method; where these approximations are also suitable for numerical conformal mapping.

ABSTRAK

Inti Szegő dan inti Bergman terhadap rantau terkait ringkas dalam satah kompleks adalah fungsi inti yang berkaitan dengan fungsi pemetaan Riemann. Suatu kaedah berkesan yang berdasarkan persamaan kamiran Kerzman-Stein-Trummer untuk pengiraan inti Szegő sudah diketahui sejak tahun 1986. Pada tahun 1997, persamaan kamiran untuk inti Bergman yang dapat digunakan secara berkesan untuk pemetaan konformal berangka juga telah dibentuk. Kedua-dua persamaan kamiran ini telah diselesaikan dengan menggunakan kaedah Nyström. Kajian ini adalah untuk membicarakan persamaan kamiran untuk inti Bergman, yang telah kami selesaikan dengan menggunakan kaedah Fourier. Memandangkan persamaan kamiran untuk inti Bergman belum diselesaikan dengan kaedah Fourier, keputusan berangka yang dihasilkan boleh juga dibandingkan dengan keputusan yang dihasilkan oleh kaedah Nyström. Keputusannya, kaedah Fourier mampu menghasilkan anggaran yang ketepatannya adalah setanding dengan kaedah Nyström dan anggarannya juga sesuai untuk pemetaan konformal berangka.

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CHAPTER I

INTRODUCTION

1.1 Background of the Study

At the beginning of this section, we shall take a tour back in time to recognize a few developments and achievements of complex variable theory in the field of engineering in the past and in the present time. Secondly, we shall like to express the relationship between conformal mapping and physical problems, as well as the ideas and concepts lie behind the construction of conformal mapping through numerical approach. Then, we may as well give a brief discussion on the equivalence property between Fourier method and Nyström's method.

1.1.1 Applications of Conformal Mapping: Past and Present

Conformal mapping has been an important and indispensable tool of science and engineering since the development of complex analysis. The applications of conformal mapping to the solution of problems in electrostatics, fluid mechanics, and heat transfer must represent those of the great achievements of complex analysis (Wunsch, 2005, p. 555). Yet surprisingly little has been written on the history of this subject, perhaps because it is in the realm of applied mathematics, which often escapes the historian's interest. It is not clear if any one mathematician had a moment of saying "Eureka" upon realizing how useful mapping with analytic functions could be to the scientist or engineer. It is evident that conformal mapping was used increasingly throughout the 19th century to solve physical problems.

In Maxwell's famous *Treatise on Electricity and Magnetism*, published in 1873, the technique is used to great advantage to display electric field lines and equipotential surfaces surrounding charged conductors (Wunsch, 2005, p. 555). Two Germans, H. A. Schwarz and E.B. Christoffel, are credited, because of their work in the period 1869-1871, with greatly advancing the subject of mapping in a way that would help the engineers or scientists. A method of transformation bearing their names is sufficiently important to merit in the field of complex analysis (Wunsch, 2005, p. 555). Other names associated with the applications of conformal mapping are those of the German, Hermann von Helmholtz, who used it in the 1860s to describe fluid flow as well as the

Englishman, Lord Rayleigh (John William Strutt) who continued work on this field a generation later (Wunsch, 2005, p. 555). These are several well-known historical events in the past, and we are sure that there were others great achievements which contribute to the growth of complex analysis.

At present, all of the significant problems solvable with conformal mapping are probably done. Now and during the past generation, problems that once would have been attempted in an idealized or simplified form with conformal mapping have come to be solved more realistically with commercially available numerical software packages for the computer.

1.1.2 Numerical Conformal Mapping

In the last section, we have mentioned some applications of conformal mapping to the solution of several physical problems in electrostatics, fluid mechanics, and heat transfer. However, we still wonder how does conformal mapping technique is applied to these physical problems. In explaining this idea, we need to interpret these physical problems as complex boundary value problems.

Conformal mapping and complex boundary value problems are two major branches of complex variable theory. The former is the geometric theory of analytic functions and the latter is the analytic theory governing the close relationship between the abstract theory and many concrete problems. Actually,

conformal mapping uses functions of complex variables to transform a complicated boundary of a physical problem (or, boundary value problem) to a simpler one. In various applied problems, by means of conformal maps, problems for certain ‘physical regions’ are transplanted into problems on some standardized ‘model regions’ where they can be solved easily (Henrici, 1974, p. 337). By transplanting back we obtain the solutions of the original problems in the physical regions.

A physical illustration is the heat diffusion problem. Imagine that Ω is a thin plate of heat-conducting metal (see Figure 1.1).

Figure 1.1 Dirichlet problem

A function ϕ describes the temperature at each point (x, y) in Ω . It is a standard situation in engineering to consider heat sources that maintain fixed values of ϕ on the boundary Γ . One wants to find the steady state heat distribution on Ω which is determined by the given boundary conditions. If we

let ϕ_0 denote the temperature specified on the boundary Γ , then it turns out the temperature in the interior satisfies (Krantz, 1999, p. 164)

$$\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \text{ in } \Omega,$$

$$\phi = \phi_0 \text{ on } \Gamma.$$

We assume that there is a conformal map $w = f(z)$ of Ω onto the unit disc $|w| < 1$ and that f is such that it can be extended to a continuous map of Ω onto the closed disc $|w| \leq 1$. The transplanted solution then satisfies

$$\Delta\psi = \frac{\partial^2\psi}{\partial u^2} + \frac{\partial^2\psi}{\partial v^2} = 0, \quad |w| < 1,$$

and its value on the boundary are

$$\psi(w) = \psi_0(w) = \phi_0(f^{-1}(w)), \quad |w| = 1.$$

This simple heat problem can be consider as the Dirichlet problem for the region Ω with boundary data ϕ_0 .

A complex-valued function can be viewed as a mapping that transforms one region of the complex plane onto another region. Some mapping functions can be formulated using specific transformations method in complex variable theory, while the rest can be formulated through composition of various transformation formulas. For those who are familiar with complex analysis, linear fractional transformation and Schwarz-Christoffel transformation are the most common transformation methods. Linear fractional transformation is usually applicable to regions whose boundaries are straight lines and generalized circles, while Schwarz-Christoffel transformation usually maps half-plane to

polygon (Saff and Snider, 2003, p. 443). However, the practical use of these two transformations is severely limited to regions involving generalized circles and polygons. A more powerful function for mapping regions of general shape is described in the Riemann mapping theorem.

Riemann mapping theorem is a fundamental theorem which guarantees the existence and uniqueness of a conformal map of a bounded simply connected region of the complex plane onto the unit disc (Tutschke and Harkrishan, 2005, p. 321). The complex-valued function which satisfies the theorem is known as the Riemann mapping function, or often known as Riemann map. This function is less rigid compared to linear fractional transformation and Schwarz-Christoffel transformation. According to the theorem, any simply connected region can be map onto a unit disc if there exists a Riemann mapping function which does the job (Marsden and Hoffman, 1999, p. 321). Unfortunately, Riemann mapping theorem also has its own weakness. The theorem does not suggest any formulas which map the simply connected region onto the unit disc. So, it is a great challenge to discover various numerical methods of computing the Riemann mapping function. In Chapter II, we shall discuss the Riemann mapping theorem in more detail.

Currently, there are various methods to compute approximately the Riemann mapping function. Some of these methods have been frequently proposed in the literatures which draw our attention. They are: *expansion methods*, *integral equation methods*, *osculation methods*, *Cauchy-Riemann*

equations methods and methods of small parameter (Ali Hassan Mohamed Murid, 1997, p. 2). Of these methods, two of them are usually encountered in the literature and used by various researchers, namely, the expansion methods and the integral equation methods (Ali Hassan Mohamed Murid, 1997, p. 2). Common expansion methods are the Bergman and the Szegő kernels method, and the Ritz variational methods. For some perspectives of expansion method and integral equation method, see e.g. Henrici (1986) and Bergman (1970). Integral equation methods are sometimes more effective for numerical conformal mapping. Some well-known integral equation methods for computing the Riemann mapping function are the integral equations of Symm, Gerschgorin, and Kerzman-Stein-Trummer (see Henrici (1986) and Kerzman and Trummer (1986)). For the discussion on the rest of these methods (that is osculation method, Cauchy-Riemann equations method, and the method of small parameter), see e.g. Henrici (1986) and Razali (1983).

The Riemann mapping function can be considered as a conformal mapping of interior regions. There are also various integral equations and expansion methods for the numerical conformal mapping of exterior and multiply connected regions, (see Henrici (1986)). A region which is not simply connected is called multiply connected region. A multiply connected region is a region that contains ‘holes’ in it (Ahlfors, 1979, p. 146). The methods discussed so far involve computing a conformal mapping from a problem region onto a model region. Similarly, there also exist the problems of numerical conformal mapping where we need to compute conformal maps from a model region to a

problem region. For surveys and various perspectives on numerical conformal mapping see Henrici (1986), Kythe (1998), and Wegmann (2005). For theoretical aspects of conformal mappings see, Hille (1962), and Henrici(1986).

1.1.3 Fourier Method and Nyström's Method

It has been established that Fourier method and trapezoidal rule are suitable to integrate periodic functions numerically. A function $f(t)$ is said to be periodic, of period p , if $f(t+p) = f(t)$ (Franklin, 1949, p. 57). According to Henrici (1974, p. 489), discrete Fourier transforms in Fourier method are derived using trapezoidal rule, while Nyström's method for solving integral equations use trapezoidal rule as its quadrature formula (Razali *et al.*, 1997). Due to this similarity, Berrut and Trummer (1987) have shown that Fourier method is equivalent to Nyström's method for the numerical solution of Fredholm integral equation. We shall present this subject in Chapter III of this dissertation. However, Berrut and Trummer (1987) did not give any numerical examples to support their findings. Furthermore, no numerical comparison has also been given for the performance of the interpolation based on Fourier method and Nyström's method. So, it is necessary for us to provide relevant numerical examples to fill up this gap.

1.2 Statements of the Study

Our research problem aims to approximate a conformal map using integral equation method. This conformal map will correspond to the Riemann mapping function which maps a simply connected region onto a unit disc. Later, Fourier method will apply to solve the integral equation. Numerical results obtained by Fourier method also act as numerical examples to support the findings by Berrut and Trummer (1987). Finally, these numerical results will then be compared with the numerical results obtained by means of Nyström's method.

Our research problem is to compare numerically the performance of Fourier method and Nyström's method for solving numerically a Fredholm integral equation of the second kind related to conformal mapping. Performance of the interpolation formulas of both the Fourier method and Nyström's method will also be compared.

1.3 Objectives of the Study

The objectives of our study are listed as follows.

1. To study and understand the basic ideas and concepts of conformal mapping, in particular the interior mapping.

2. To study and understand the application of integral equation method by deriving an integral equation of the second kind for the Bergman kernel.
3. To apply Fourier method for solving the integral equation via the Bergman kernel.
4. To verify numerically that the Fourier method is equivalent to the Nyström's method with trapezoidal rule.

1.4 Scope of the Study

As we have stated in Section 1.2, we shall use integral equation method to determine the numerical approximation of the Riemann mapping function. In literature, the Szegő kernel and the Bergman kernel of a simply connected region are related to the Riemann mapping function which maps a simply connected region onto a unit disc (Henrici, 1986, p. 547, 553). Kerzman and Trummer (1986) have developed an effective numerical method for computing the Szegő kernel, by solving an integral equation now known as the Kerzman-Stein-Trummer integral equation (briefly, KST integral equation). Since there exists a relationship between the Szegő kernel and the Bergman kernel, it is natural to ask whether an integral equation may also be developed for the Bergman kernel. Eventually, in the more recent development, the integral equation of the second kind for the Bergman kernel has been derived by Razali *et al.* (1997).

In this dissertation, we just state the KST integral equation without deriving it. For the derivation of the KST integral equation, see Henrici (1986, p. 560-564) and Kerzman and Trummer (1986). Later, we shall take a closer look at the integral equation of the second kind for the Bergman kernel and rederive it according to the approach given by Razali *et al.* (1997). This helps us to study and understand the integral equation method even better. This method will be used to compute the values of the conformal map on the boundary. Once we have solved the integral equation, sup-norm error will be computed to evaluate the efficiency of the integral equation. Razali *et al.* (1997) have used Nyström's method with trapezoidal rule to solve the integral equation. In this dissertation, we use a different approach called the Fourier method to evaluate the integral equation. On the other hand, Berrut and Trummer (1987) have shown that Fourier method is equivalent to Nyström's method with trapezoidal rule. In view of this, we shall use Fourier method to evaluate the integral equation for two main reasons: firstly, we want to see whether Fourier method produces approximations of comparable accuracy to the Nyström's method with trapezoidal rule, and secondly, we want to provide numerical examples for the findings of Berrut and Trummer (1987). So, we shall compare the numerical results of Fourier method with the numerical results of Nyström's method. Also, this dissertation will focus only on the mapping of simply connected region onto the unit disc. Last but not least, all the numerical procedure and graphics related to the tested regions will be carried out by using MATHEMATICA 5.0.

1.5 Significance of the Study

The discovery of integral equation via the Bergman kernel is certainly an encouraging effort done by Razali *et al.* (1997). It provides us another new integral equation thereby enriching the integral equation methods for the numerical conformal mapping of the interior region. This dissertation is also an effort to conduct further research on the integral equation via the Bergman kernel. At present, the integral equation via the Bergman kernel has only been solved using the Nyström's method with trapezoidal rule. Due to this condition, more researches on various of method in solving the integral equation should be carried out so that the integral equation can approximate better results under these methods. So, we choose the Fourier method to solve the integral equation instead of using the Nyström's method with trapezoidal rule. Fourier method is chosen because of its equivalence property to the Nyström's method with trapezoidal rule (Berrut and Trummer, 1987). Moreover, Fourier method is not a direct method like the Nyström's method with trapezoidal rule. The computation procedure for Fourier method is rather tedious, where it involves computing the coefficient of the Fourier series before computing the integral equation in form of linear equation. Such procedures make the Fourier method less attractive numerically. However, we still hope that Fourier method gives approximation of comparable accuracy to the Nyström's method with trapezoidal rule. If it does, it will encourage researchers to implement Fourier method in their study. As a conclusion, this dissertation is to use Fourier method to solve integral equation via the Bergman kernel, and makes

comparison with the results by means of Nyström's method. Hopefully, we can spot at least one or more significance features of using Fourier method to solve integral equation via the Bergman kernel.

1.6 Outline of the Dissertation

In Chapter II, we review some basic facts about general mappings, conformal mappings, Riemann mapping with boundary correspondence function, and integral equations. Several papers which discuss the researches on numerical conformal mapping using integral equation method are also presented as the rationale for our study.

Chapter III is all about the discussion of Fourier method which we are going to use to solve the integral equation of the second kind. This chapter contains reviews on the basic facts of Fourier series, continuous Fourier transform, discrete Fourier transform (one-dimensional and two-dimensional transform), and trigonometric interpolation. The equivalence of Nyström's method and Fourier method for the numerical solution of Fredholm integral equations is also presented by using the approach in Berrut and Trummer (1987).

In Chapter IV, we first introduce some fundamental facts about the Bergman kernel, the Szegő and their relationships. The KST integral equation is also stated at the beginning of the chapter as motivation to our derivation of integral equation via the Bergman kernel. Next, we rederive the integral equation of the second kind for the Bergman kernel by adapting the method shown in Razali *et al.* (1997). The objective of deriving this integral equation is to provide an alternative numerical method for computing the Riemann mapping function.

After obtaining the integral equation of the second kind for the Bergman kernel in Chapter IV, we can parametrize this integral equation and solve it numerically. So, in Chapter V, we present how does Fourier method solves this integral equation for some test regions. Thus, some related numerical results are also presented.

Chapter VI contains a summary of the main results of the dissertation and several recommendations.

With the above summaries, conclusions and suggestions for further studies, we conclude this dissertation.

REFERENCES

- Ahlfors, L. V. (1979). *Complex Analysis*. 3rd. ed. New York: McGraw-Hill Book Company. 146.
- Ali Hassan Mohamed Murid (1997). *Boundary Integral Equation Approach for Numerical Conformal Mapping*. Universiti Teknologi Malaysia: Ph.D. Thesis. 2.
- Bergman, S. (1970). The Kernel Function and Conformal Mapping. 2nd (revised) ed. *Providence: American Mathematical Society*.
- Berrut, J. P. (1986). A Fredholm Integral Equation of the Second Kind for Conformal Mapping. *Journal of Computational and Applied Mathematics*. 14:99 - 110.
- Berrut, J. P. and Trummer, M. R. (1987). Equivalence of Nyström's Method and Fourier Methods for the Numerical Solution of Fredholm Integral Equations. *Mathematics of Computation*. 48(178):617 - 623.
- Brown, J. W. and Churchill, R. V. (2003). *Complex Variables and Applications*. 7th. ed. New York: McGraw-Hill. 36.

Don, E. (2001). *Schaum's Outline of Theory and Problem of MATHEMATICA*.
New York: McGraw-Hill.

Fausett, L.V. (1999). *Applied Numerical Analysis Using MATLAB*. New Jersey:
Prentice Hall, Inc. 332.

Franklin, P. (1949). *Fourier Methods*. London: McGraw-Hill Book Company.
57.

Henrici, P. (1974). *Applied and Computational Complex Analysis*. Volume 1.
New York: John Wiley & Sons. 186, 329, 337, 489.

Henrici, P. (1979). Fast Fourier Methods in Computational Complex Analysis.
SIAM Review. 21(4):481-527.

Henrici, P. (1986). *Applied and Computational Complex Analysis*. Volume 3.
New York: John Wiley & Sons. 43, 89, 282, 345, 358, 377 - 379, 380, 381 -
391, 394, 395 - 396, 398 - 401, 547, 553, 560 - 564.

Hille, E. (1962). *Analytic Function Theory*. Volume 2. Boston: Ginn and
Company. 176.

Kanwal, R. P. (1971). *Linear Integral Equations: Theory and Technique*. New
York: Academic Press. 1, 2, 3.

- Kammler, D. W. (2000). *A First Course in Fourier Analysis*. New Jersey: Prentice-Hall, Inc. 24.
- Kerzman, N. and Stein, E. M. (1978). The Cauchy Kernel, the Szegö Kernel, and the Riemann Mapping Function. *Math. Ann.* 236:85 - 93.
- Kerzman, N. and Trummer, M. R. (1986). Numerical Conformal Mapping via the Szegö Kernel. *Journal of Computational and Applied Mathematics.* 14:111 - 123.
- Krantz, S. G. (1999). *Handbook of Complex Variables*. Boston: Birkhäuser.
- Kreyszig, E. (1978). *Introductory Functional Analysis with Applications*. New York: John Wiley & Sons, Inc. 453.
- Kythe, P. K. (1998). *Computational Conformal Mapping*. Boston: Birkhäuser.
- Marsden, J. E. and Hoffman, M. J. (1999). *Basic Complex Analysis*. 3rd. ed. New York: W. H. Freeman and Company. 247, 320, 321, 323.
- Mitra, S. K. (2002). *Digital Signal Processing: A Computer-Based Approach*. 2nd. ed. New York: McGraw-Hill. 535, 538.

- Razali, M. R. M. (1983). *Singular Integral Equations and Mixed Boundary Value Problems for Harmonic Functions*. University of Southampton: Ph.D. Thesis.
- Razali, M. R. M., Nashed, M. Z. and Murid, A. H. M. (1997). Numerical Conformal Mapping via the Bergman Kernel. *Journal of Computational and Applied Mathematics*. 82:333 - 350.
- Ralston, A. & Rabinowitz, P. (1978). *A First Course in Numerical Analysis*. New York: McGraw Hill Book Company.
- Reichel, L. (1986). A Fast Method for Solving Certain Integral Equations of the First Kind with Application to Conformal Mapping." *Journal of Computational and Applied Mathematics*. 14:125 - 142.
- Saff, E. B. and Snider, A. D. (2003). *Fundamentals of Complex Analysis with Applications to Engineering and Science*. 3rd. ed. New Jersey: Pearson Education, Inc. 150, 369, 443.
- Spiegel, M. R. (1974). *Schaum's Outline of Theory and Problems of Fourier Analysis*. New York: McGraw-Hill, Inc. 21.
- Trefethen, L. N. (Ed.) (1986). *Numerical Conformal Mapping*. Amsterdam: North-Holland.

- Trummer, M. R. (1986). An Efficient Implementation of a Conformal Mapping Method based on the Szegö Kernel. *SIAM Journal of Numerical Analysis*. 23:853 - 872.
- Tutschke, W. and Harkrishan L. V. (2005). *An Introduction to Complex Analysis: Classical and Modern Approaches*. Boca Raton: Chapman & Hall/CRC. 321.
- Wegmann, R. (2005). Methods for Numerical Conformal Mapping. In: Kühnau, R. ed. *Handbook of Complex Analysis: Geometric Function Theory*. Volume 2. Amsterdam: Elsevier.
- Weinberger, H. F. (1965). *A First Course in Partial Differential Equations with Complex Variables and Transform Methods*. New York: John Wiley & Sons. 298.
- Wunsch, A.D. (2005). *Complex Variables with Applications*. New York: Pearson Education, Inc. 42, 517, 520, 530, 555.
- Zill, D. G. and Shanahan, P. D. (2003). *A First Course in Complex Analysis with Applications*. Massachusetts: Jones and Bartlett Publishers. 31, 58, 88, 391.

Zwillinger, D. ed. (1996). *CRC Standard Mathematical Tables and Formulae*.

Florida: CRC Press, Inc. 63, 426, 427.