MATHEMATICAL MODELLING OF BOUNDARY LAYER FLOW AND HEAT TRANSFER IN FORCED CONVECTION

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To my beloved family

R. Ismail & Siti Khodijah
R. Bazlin, R.M. Firdaus
& R. Syakireen Shahida

Thanks for all the sacrifices, support and hope which is given so far....

To all the UTM lecturers
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ABSTRACT

A mathematical model for the boundary layer flow and heat transfer in forced convection is developed. Boundary layer is a narrow region of thin layer that exists adjacent to the surface of a solid body where the effects of viscosity are obvious, manifested by large flow velocity and temperature gradient. The concept of boundary layer was first introduced by Ludwig Prandtl (1875-1953) in 1905. The derivation of both velocity and temperature boundary layer equations for flow past a horizontal flat plate and semi-infinite wedge are discussed. The velocity and temperature boundary layer equations are first transformed into ordinary differential equations via a similarity transformation. The differential equations corresponding to the flow past a horizontal flat plate and a semi-infinite wedge are nonlinear and known respectively as the Blasius and the Falkner-Skan equation. The approximate solutions of these equations are obtained analytically using a series expansion, namely the Blasius series and an improved perturbation series using the Shanks transformation. The solutions presented include the velocity and temperature profiles, the skin friction and the heat transfer coefficient.

ABSTRAK

Model matematik bagi aliran lapisan sempadan dan pemindahan haba dalam perolakan paksa telah dibina. Lapisan sempadan merupakan suatu kawasan nipis yang wujud pada suatu permukaan, di mana kesan kelikatan terhadap aliran bendalir adalah nyata yang mengakibatkan wujud kecerunan halaju dan suhu yang besar. Konsep lapisan sempadan buat pertama kalinya telah diperkenalkan oleh Ludwig Prandtl (1875-1953) pada tahun 1905. Penerbitan bagi persamaan-persamaan lapisan sempadan halaju dan suhu bagi aliran merentasi suatu plat rata yang mendatar dan merentasi bucu semi-infiniti telah dibincangkan. Kedua-dua persamaan lapisan sempadan halaju dan suhu terlebih dahulu dijelmakan kepada persamaan-persamaan pembezaan biasa menggunakan penjelmaan keserupaan. Persamaan pembezaan yang diperoleh bagi kes aliran merentasi plat rata dan bucu semi-infiniti masing-masing dikenali sebagai persamaan Blasius dan persamaan Falkner-Skan. Kemudian, persamaan Blasius diselesaikan menggunakan pengembangan siri yang dikenali sebagai siri Blasius dan persamaan Falkner-Skan diselesaikan menggunakan kaedah usikan yang dipertingkatkan dengan penjelmaan Shanks. Keputusan yang diperoleh adalah merangkumi profil halaju dan suhu, tegasan ricih dan pekali pengaliran haba.

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LIST OF SYMBOLS

C_f	-	Friction coefficient
C_{fx}	-	Local friction coefficient
Ec	-	Eckert number
h	-	Heat transfer coefficient
h_x	-	Local heat transfer coefficient
$\boldsymbol{i}, \boldsymbol{j}$	-	Unit vector in Cartesian system
k	-	Thermal conductivity
L	-	Length
Nu	-	Nusselt number
Nu_x	-	Local Nusselt number
p	-	Pressure
Pr	-	Prandtl number
q	-	Heat transfer rate
q'''	-	Heat flux
Re	-	Reynolds number
Re_x	-	Local Reynolds number
T	-	Temperature
T_w	-	Wall temperature
T_∞	-	Free stream temperature
u	-	Component-x of velocity
$\boldsymbol{u}$	-	Velocity vector
$U(x)$	-	Free stream velocity function
U_∞	-	Free stream velocity
v	-	Component-y of velocity
x, y, z	-	Space coordinate in Cartesian system
α	-	Thermal diffusivity
β	-	Coefficient of thermal expansion

β	-	Parameter of Falkner-Skan equation
δ	-	Velocity boundary layer thickness
δ_T	-	Temperature boundary layer thickness
η	-	Similarity variable
θ	-	Temperature difference
μ	-	Dynamic viscosity
ρ	-	Density
τ_w	-	Wall shear stress
υ	-	Kinematics viscosity
Φ_v	-	Dissipation function
ψ	-	Stream function

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Boundary layer is a narrow region of thin layer that exists adjacent to the surface of a solid body when a real fluid flows past the body. In this region, the effect of viscosity is obvious on the flow of the fluid which resulted in large velocity gradient and the presence of shear stress. The various transfer processes which take place in fluids and between solids and fluids are momentum, mass, and heat transfer. When formulating the conservation laws of mass, momentum, and energy, the laws of thermodynamics and gas dynamics have to be observed. This means that along with the boundary layer flow, there are also the thermal boundary layer and the mutual influence of these boundary layers upon one another to be accounted for. The concept of boundary layer plays an important role in many branches of engineering sciences, especially in hydrodynamics, aerodynamics, automobile and marine engineering (Kundu and Cohen, 2004).

This report contains the derivation of both velocity and thermal boundary layer equations. Both velocity and temperature boundary layer are modelled in view of flow past a horizontal flat plate and semi-infinite wedge cases. In each cases of flow, the velocity and the thermal boundary layer equations are transformed to a single nonlinear and a linear differential equation respectively via similarity transformation. The nonlinear equations are known as the Blasius equation and the Falkner-Skan equation; each corresponds to the cases of flow past a horizontal flat plate and semi-infinite wedge respectively. Then the Blasius equation is solved via series expansion namely the Blasius series while the Falkner-Skan equation is solved using perturbation method, i.e. perturbation series together with Shanks transformation. From the solution of velocity and temperature boundary layer equations, the analysis of results is made in consideration of the skin friction and heat transfer coefficient.

In this chapter, the objective, methodology and scope of this project are described. The historical background of the boundary layer is also included here.

1.2 Objectives and Scope of Research

The objectives of this research are:

1. To derive the velocity and temperature boundary layer equations in forced convection.
2. To find the solution of the velocity and temperature boundary layer equations past a horizontal flat plate and a semi-infinite wedge via similarity transformation.
3. To solve the Blasius equation using series expansion.
4. To solve the Falkner-Skan equation using the perturbation series which is improved further using Shanks transformation.

The scope of this project is to derive the existing models of velocity and thermal boundary layers in a more comprehensive manner. No new mathematical models will be developed. The immersed bodies considered are the horizontal flat plate and the semi-infinite wedge.

1.3 Historical Background

Until the beginning of the twentieth century, analytical solution of a steady fluid flows were generally known for two typical situations. One of these was that of parallel viscous flows and low Reynolds number flows, in which the nonlinear advective terms were zero and the balance of forces was that between the pressure and the viscous forces. The second type of solution was that of inviscid flows around bodies of various shapes, in which the balance of forces was that between the inertia and pressure forces. Although the equations of motion are nonlinear in this case, the velocity field can be determined by solving the linear Laplace equation. These irrotational solutions predicted pressure forces on a streamlined body that agreed surprisingly well with experimental data for flow of fluids of small viscosity. However these solutions also predicted a zero drag force and a nonzero tangential velocity at the surfaces, features that did not agree with experiments.

In 1905 Ludwig Prandtl, an engineer by profession and therefore motivated to find realistic fields near bodies of various shapes, first hypothesized that, for small viscosity, the various forces are negligible everywhere except close to the solid boundaries where the no-slip condition had to be satisfied. The thickness of these boundary layers approaches zero as the viscosity goes to zero. The hypothesis of Prandtl reconciled two rather contradicting facts. On one hand he supported our intuitive idea that the effects of viscosity are indeed negligible in most of the flow field if the kinematics viscosity is small. At the same time Prandtl was able to account for drag by insisting that the no-slip condition must be satisfied at the wall, no matter how small the viscosity is. Prandtl also showed how the equations of motion within the boundary layer can be simplified. Since the time of Prandtl, the concept of the boundary layer has been generalized, and the mathematical techniques involved have been formalized, extended, and applied to various other branches of physical science. The concept of boundary layer is considered one of the cornerstones in the history of fluid mechanics. Besides, just as the hydrodynamic boundary layer was defined as that region of the flow where viscous forces are felt, a thermal boundary layer may be defined as that region where temperature gradients would result from a heat exchange process between the fluid and the wall (Kundu and Cohen, 2004).

1.4 Introduction to Chapters

This report contains six chapters. In Chapter 2, we clarify the derivation of the velocity boundary layer equations, which is actually represented via approximation. This chapter starts with the visualization of the physical model of boundary layer flow. It follows with the derivation of the velocity boundary layer equations, which is the main objective in this chapter. Then the order of boundary layer thickness and the Reynolds Number will be discussed. The derivation of the dimensionless boundary layer equations and the selection of boundary conditions will also be discussed in this chapter.

The main objective in Chapter 3 is to derive the temperature boundary layer equation. This chapter contains an explanation of some basic principles of convection heat transfer. It follows with the derivation of the temperature boundary layer equation. Next, the concept of thermal boundary layer thickness and the Prandtl number, and the heat transfer coefficient and the Nusselt number will be discussed. This chapter ends with the description of the relation between fluid friction and heat transfer.

The next two chapters describe the models of velocity and thermal boundary layers past immersed bodies, namely the horizontal flat plate and semi-infinite wedge. Chapter 4 first illustrates the physical models of boundary layer flow past the bodies. Then the nondimensionalization of the boundary layer equations which have been obtained in Chapter 2 will be shown. Next, the equations will be transformed via similarity transformation for each case of flow. The transformation will result in an ordinary differential equation, namely the Blasius equation for flow past a horizontal flat plate. After that the solution of Blasius equation using series expansion will be described. On the other hand, the similarity transformation will result in the Falkner-Skan equation for flow past a semi-infinite wedge. The Falkner-Skan equation will be solved via perturbation method. Finally the result which provides the velocity profiles and the skin friction coefficient will be analyzed for each case of flow in this chapter.

Chapter 5 will explain the models of thermal boundary layer. In this chapter we will apply the thermal boundary layer equation obtained in Chapter 3 to the problem of steady laminar flow past a horizontal flat plate and a semi-infinite wedge. This chapter first describes the physical models of thermal boundary layer past the bodies, and then the derivation of dimensionless thermal boundary layer equation follows. Then the thermal boundary layer equation will be transformed to another equation using similarity transformation technique. Next, the solution of the transformed equations will be obtained. This chapter ends with the analysis of results which provides the temperature profiles and the heat transfer coefficient.

Finally, the conclusion of this project will be included in Chapter 6. This chapter also contains some suggestions for future studies.

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