BOUNDARY INTEGRAL EQUATION METHOD FOR CONFORMAL MAPPING OF UNBOUNDED MULTIPLY CONNECTED REGIONS

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A thesis submitted in fulfilment of the
requirements for the award of the degree of
Doctor of Philosophy (Mathemtics)

Faculty of Science
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AUGUST 2013

ABSTRACT

This work presents two methods for numerical conformal mappings of unbounded multiply connected regions onto several classes of canonical slit regions. The first method is only limited to conformal mapping of unbounded multiply connected regions onto the first category of Koebe's canonical regions. It is based on three boundary integral equations formed with the classical adjoint Neumann kernel, adjoint generalized Neumann kernel and modified Neumann kernel. These integral equations are constructed from a boundary relationship satisfied by an analytic function on the unbounded multiply connected regions. By adding some normalizing conditions, the integral equations are uniquely solvable. The second method is for numerical conformal mapping and its inverse of unbounded multiply connected regions onto the first, second, third and fourth category of Koebe's canonical regions. It is based on reformulating the conformal mapping problem as a Riemann-Hilbert problem and an adjoint Riemann-Hilbert problem. Two integral equations formed with the adjoint generalized Neumann kernel are constructed. With some normalizing conditions, the integral equations are uniquely solvable. For both methods, discretizing the integral equations with their normalizing conditions lead to systems of linear algebraic equations which are solved by Gaussion elimination method to obtain the boundary values of the mapping functions. The interior values of the mapping functions are then determined by using Cauchy's integral formula. Cauchy's integral formula is also used to approximate the interior values of the inverse mapping functions for the second method. Some numerical examples are presented to illustrate the effectiveness of both methods for computing the conformal mappings of unbounded multiply connected regions.

ABSTRAK

Kajian ini mempersembahkan dua kaedah bagi pemetaan konformal berangka rantau terkait berganda tak terbatas kepada beberapa kelas rantau belahan berkanun. Kaedah pertama adalah terhad kepada pemetaan konformal rantau terkait berganda tak terbatas ke kategori pertama rantau berkanun Koebe. Kaedah ini adalah berasaskan tiga persamaan kamiran yang dibentuk bersama inti Neumann dampingan klasik, inti Neumann teritlak, dan inti Neumann terubahsuai. Persamaan kamiran ini dibentuk daripada hubungan sempadan yang ditepati oleh fungsi analisis atas rantau terkait berganda tak terbatas. Dengan menambahkan beberapa syarat penormalan, persamaan kamiran yang dibentuk dapat diselesaikan dengan unik. Kaedah kedua adalah untuk pemetaan konformal berangka dan juga pemetaan sonsangan bagi rantau terkait berganda tak terbatas ke kategori pertama, kedua, ketiga dan keempat rantau berkanun Koebe. Kaedah ini berasaskan kepada merumus semula masalah pemetaan konformal sebagai masalah Riemann-Hilbert dan masalah Riemann-Hilbert dampingan. Dua persamaan kamiran bersama inti Neumann dampingan teritlak dibentuk. Bersama dengan beberapa syarat penormalan, persamaan kamiran dapat diselesaikan dengan unik. Bagi kedua-dua kaedah ini, persamaan kamiran yang dibentuk bersama syarat-syarat penormalan telah didiskretkan menghasilkan beberapa sistem persamaan algebra linear yang diselesaikan menggunakan kaedah penghapusan Gaussian untuk memperoleh nilai-nilai sempadan bagi fungsi pemetaan. Nilai-nilai pedalaman bagi fungsi pemetaan dikenal pasti dengan menggunakan formula kamiran Cauchy. Formula kamiran Cauchy juga digunakan bagi membuat anggaran nilai-nilai pedalaman fungsi pemetaan songsang untuk kaedah kedua. Beberapa contoh berangka dipersembahkan bagi memaparkan keberkesanan teknik yang dipersembahkan bagi mengira pemetaan konformal rantau terkait berganda tak terbatas.

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LIST OF SYMBOLS

$K(t,s)$	-	Modified Neumann kernel
$N(s,t)$	-	Generalized Neumann kernel
$N^*(t,s)$	-	Adjoint generalized Neumann kernel
λ	-	Eigenvalue
χ	-	Eigenfunction
Ω^-	-	Unbounded multiply connected region
Ω^+	-	Complement of Ω^-
Γ_j	-	Boundary of Ω_j^+
$T(\eta(t))$	-	Unit tangent at $\eta(t)$
i	-	$\sqrt{-1}$
R	-	Riemann mapping function
$\in$	-	Element of
$\cup$	-	Union
$\neq$	-	Not equal to
$<$	-	Less than
$>$	-	Greater than
H	-	Space of all real Hölder continuous 2π-periodic functions
L	-	Subspace of H
π	-	Pi($pi \approx 3.142\ldots$)
Im	-	Imaginary part
Re	-	Real part
$\sum$	-	Summation
$\int$	-	Integration

CHAPTER 1

INTRODUCTION

1.1 Introduction

Conformal mapping is known as a special mapping that uses function of complex variable to transform any given planar regions onto other planar regions with unique characteristic. Conformal mapping has the properties that the angles between curves are preserved in magnitude as well as in their direction. Conformal mapping (also known as conformal maps, conformal transformation, angle preserving transformation or biholomorphic map) is a mapping of analytic function $w = f(z)$ where the analytic function has nonzero derivative for any point z in the original region (Wunsch, 2005, p. 520).

Conformal mapping plays an important role in solving several problems in the fields of sciences and engineering. It has the advantage of transforming a complicated boundary to a simpler and more manageable configuration. By means of conformal maps, problem from the original region (physical region) can be transformed into some standardized region (canonical region) where they can be solved easily. By transplanting it back to the original region, the solution of the original problem can be obtained. This process has been applied in several applied problem such as aerofoil problem, fluid flow, electrostatics, heat transfer problem and image processing. For these physical problems, see Henrici (1974), Wunsch (2005), and Saff and Snider

(2003). Despite its advantages on solving physical problems of modern technology, it has a limitation that exact conformal mapping functions are known only for certain regions. Therefore, numerous researchers have applied numerical methods to overcome this limitation.

For the case of simply connected region, there exists a special class of conformal map which maps any given simply connected region onto a unit disk namely the Riemann map. The Riemann mapping theorem guarantees the existence and uniqueness of the Riemann map. However, this theorem is not applicable for multiply connected regions. For theoretical aspects on existence and uniqueness of conformal mappings of bounded and unbounded multiply connected regions, see Nehari (1952), Ahlfors (1979), Henrici (1986), Goluzin (1969), Koebe (1916), Wen (1992), and Andreev and McNicholl (2012).

1.2 Research Background

To overcome the major setback in conformal mapping that only for certain special regions are exact conformal maps known, numerous researchers have resort to numerical approach. Trefethen (1986) has discussed several methods for computing conformal mapping numerically. Generally, these methods are based on expansion methods, iterative methods or integral equation methods. A canonical region in conformal mapping is known as a set of finitely connected regions S such that each finitely connected non-degenerate region is conformally equivalent to a region in S. Basically, when dealing with problem related to conformal mapping, we are dealing with problem on how to find a mapping from problem region onto the canonical region or vice versa. There exist several classes of canonical region with regards to conformal mapping of multiply connected regions as listed in Nehari (1952), Henrici (1986), Wen (1992), Andreev et al. (2008), Andreev and McNicholl (2012) and Koebe (1916). Koebe (1916) have cataloged 39 types of canonical regions onto five categories, see

Koebe (1916) and Nasser (2013). Briefly, these 39 types of the canonical regions are the product of the combination between the entire w-plane, upper half plane, a disk or annulus with these two following types of slit:

(i) logarithmic spiral slit (the special classes of this category are the circular slit and the radial slit),

(ii) rectilinear slit (the special classes of this category are the finite linear slit, the parallel slit, semi infinite linear slit, infinite linear slit).

From these 39 types of canonical slit regions, Nehari (1952) have listed the five types of the canonical slit regions which are regarded as important canonical regions for conformal mapping of multiply connected regions, namely (see Figure 1.1)

(i) disc with concentric circular slits (Figure 1.1a),

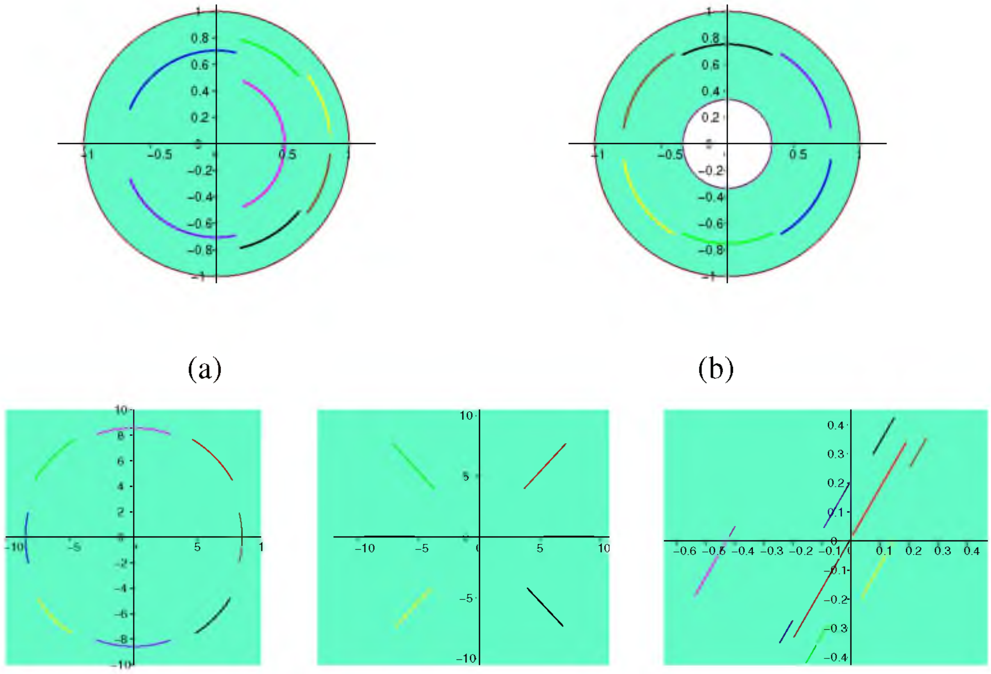
(ii) annulus with concentric circular slits (Figure 1.1b),

(iii) circular slit regions (Figure 1.1c),

(iv) radial slit regions (Figure 1.1d), and

(v) parallel slit regions (Figure 1.1e).

It is well known that any finite multiply connected regions can be mapped conformally onto these five canonical regions. These mapping functions are unknown except for some special regions. Numerical conformal mapping of multiply connected regions is presently still a subject of interest. Boundary integral equation related to a boundary relationship satisfied by a function which is analytic in a simply or doubly connected region bounded by closed smooth Jordan curves has been given by Murid (1997) and Murid and Razali (1999). Special realizations of this integral equation are the integral equations related to the Szegö kernel, Bergmann kernel, Riemann map, and



(a) (b)

(c) (d) (e)

Figure 1.1: An example of five types canonical slit regions of connectivity four.

Ahlfors map. The kernels arise in these integral equations are the Neumann kernel and the Kerzman-Stein kernel.

Murid and Hu (2009) managed to construct a boundary integral equation for numerical conformal mapping of a bounded multiply connected region onto a unit disk with slits. However, the integral equation involves unknown radii which lead to a system of nonlinear algebraic equations upon discretization of the integral equation. The resulting system of nonlinear algebraic equations is then solved by means of Lavemberg-Marquadt algorithm. The advantage of this method is that it calculates the boundary correspondence functions and unknown radii simultaneously and of the same accuracy. Nasser (2009a), Nasser (2009b), Nasser (2011), Nasser (2013) and Nasser and Al-Shihri (2013) produced another technique for numerical conformal mapping of bounded and unbounded multiply connected regions by expressing the mapping function in terms of the solution of a uniquely solvable Riemann-Hilbert problem. This uniquely solvable Riemann Hilbert problem can be solved by means of boundary integral equation with the generalized Neumann kernel.

Recently, Sangawi (2012), Sangawi et al. (2011), Sangawi et al. (2012a), Sangawi et al. (2012b), Sangawi et al. (2012d), and Sangawi et al. (2012c) have constructed new linear boundary integral equations for conformal mapping of bounded multiply region onto five class of the canonical slit regions, which significantly improves the work of Murid and Hu (2009) by eliminating the problem of nonlinearity. But these works have not considered conformal mapping of unbounded multiply connected regions and the inverse maps.

1.3 Problem Statements

The research problem is to formulate some boundary integral equations for the purpose of numerical conformal mapping and its inverse of unbounded multiply connected region with smooth and non-smooth boundaries onto the first, second, third and fourth categories of Koebe's canonical region via the classical Neumann kernel, modified Neumann kernel and the adjoint generalized Neumann kernel which are suitable for numerical purpose.

1.4 Research Objectives

The objectives of this research are:

(i) To formulate and extend the construction of boundary integral equation related to a boundary relationship satisfied by a function which is analytic in an unbounded simply connected region by Murid (1997) to an unbounded multiply connected region.

(ii) To derive new linear boundary integral equations for numerical conformal mapping of unbounded multiply connected region via the classical Neumann kernel, modified Neumann kernel and the adjoint generalized Neumann kernel.

(iii) To solve the resulting integral equations by means of numerical method to obtain the boundary and their interior values of the mapping function.

(iv) To approximate the inverse mapping function for each canonical region.

(v) To validate the numerical results by comparing the proposed method with some existing methods or analytic solution.

1.5 Scope of the Study

This research focuses on the boundary integral equation method to compute the conformal mapping function of unbounded multiply connected region onto canonical slit region. The theoretical development of the integral equations are based on the approaches given by Murid (1997), Nasser (2005) and Hu (2009).

In this research, some new boundary and linear integral equations will be constructed from boundary relationship satisfied by analytic functions on unbounded multiply connected region. Some new integral equations will be constructed for analytic functions satisfying the exterior Riemann-Hilbert problem. The kernels involve in this research are classical Neumann kernel, modified Neumann kernel and adjoint generalized Neumann kernel. These integral equations will be applied to compute the conformal mapping of unbounded multiply connected region onto the canonical region. These integral equations will be discretized by the Nyström method with the trapezoidal rule which will lead to a system of equations. Some normalizing are required to obtain the unique solutions.

This research will describe a numerical procedure based on Cauchy's integral formula for mapping of the exterior points. Cauchy's integral formula will also be used to find the inverse mapping function from the canonical region onto the original region.

1.6 Significance of Findings

Conformal mapping is important in solving several problems that arises from the field of science and engineering. Nowadays, many practical problems can be modeled as complex boundary value problem for analytic functions involving smooth or non smooth region. Basically, conformal mapping will transform boundary of complicated boundary to a much simpler one. Sakajo (2009) has stated that the study of numerical conformal mapping of multiply connected region may play important role in the study of dynamics of point vortices where it has wide applications in environmental flows, bio-fluids, etc. Amano and Okano (2010) have stated that conformal mapping of bounded domains onto circular and radial slits domains are important since the steady heat flow problem are usually solved as mixed boundary value in finite domains. Crowdy and Marshall (2006) mentioned some application of conformal mapping of multiply connected regions such as reductions of integrable hierarchy, forming the basis for an analytical study of the motion of point vortices through gaps in walls, and construction from circular domains to multiply connected polygonal regions.

The main contribution of this research will be the linear integral equation method to the conformal mapping of unbounded multiply connected regions and its inverse maps. This method will focuses on solving the conformal mapping by means of numerical computation from smooth and non smooth boundary to the first four category of Koebe's canonical regions (Koebe, 1916) and its inverse. Several computer algorithms will be written by using MATLAB software for the numerical experiments of the conformal mapping of unbounded multiply connected regions. Some of the results will be published in national and regional conference, journal indexed by Scopus or indexed in Thomson Reuters web of knowledge (formerly known as ISI). These findings will contribute to some new findings in the fields of complex analysis.

1.7 Thesis Organization

This thesis consists of six chapters including introductory and literature review chapters. The first chapter contains seven sections which are introduction, background of the problem, problem statement, objectives of the research, scope of the study, significant of findings and thesis organization.

Some fundamental ideas of univalent conformal mapping of unbounded multiply connected regions onto Koebe's canonical regions are presented in eight sections of Chapter 2 beginning with some explanations on concepts of conformal mapping. Introduction and theories related to Riemann mapping function, unbounded multiply connected region, canonical regions in conformal mapping are all presented in the second section followed by a section describing an integral equation for homogeneous exterior boundary relationship derived in Murid (1997) for conformal mapping of unbounded simply connected region. Some theories and definitions for the Riemann-Hilbert problem and its connection with boundary integral equation with the generalized Neumann kernel are presented and finally some modifications on integral equation derived in Kress (1990) is included.

Chapter 3 presents some new linear integral equations for unbounded multiply connected regions onto first category of Koebe's canonical regions. This chapter begins with a brief review on auxiliary material followed by formulation of a boundary integral equation for homogeneous boundary relationship for the case of unbounded multiply connected regions. The theoretical development is based on the boundary integral equation for exterior simply connected region derived in Murid (1997). By using the boundary relationship satisfied by the mapping function, a related system of integral equation with the modified Neumann kernel is constructed to find the values of the derivatives of the mapping function $\Phi(z)$. Then by using the Riemann-Hilbert approach with the adjoint generalized Neumann kernel (Wegmann and Nasser, 2008), the values of derivatives of unknown function $S(t)$ is computed. The values of piecewise real constant functions $R(t)$ are then calculated using the method presented in (Nasser et al., 2011). Discretizing all the integral equations leads to a system of linear equations. This system of linear equations will be solved by means of Gaussian elimination method. Next the mapping functions $\Phi(z)$ are then calculated using the relationship derived in this chapter. The mapping of interior points $z \in \Omega^-$ are then calculated by the Cauchy's integral formula. Some numerical examples are given to show the effectiveness of the present method.

A new method is proposed in Chapter 4 for the conformal mapping and its inverse for unbounded multiply connected regions onto first category of Koebe's canonical slits regions. In this chapter, the conformal mapping functions of $\Phi(z)$ are approximated by determining the values of unknown functions $S(t)$ and the conformal moduli $R(t)$. The values of $S'(t)$ and $R(t)$ will be determined by solving the same uniquely solvable boundary integral equations with the adjoint generalized Neumann kernel but with different terms in the right hand side of the integral equations. Discretizing these integral equations leads to a system of linear equations which will be solved by means of Gaussian elimination method. The value of unknown function $S(t)$ is computed by finding the anti-derivative of $S'(t)$ via Fourier series representation. Modification to the integral equations for regions with piecewise smooth boundaries is also presented. Next the mapping functions $\Phi(z)$ are then calculated using the relationship given in this chapter. The mapping of interior points $z \in \Omega^-$ and the inverse maps $\Phi^{-1}(w)$ are then calculated by the Cauchy's integral formula as proposed in (Henrici, 1986). Some numerical examples are given to show the effectiveness of the present method.

We extend the result in Chapter 4 to conformal mapping and its inverse of unbounded multiply connected regions onto second, third and fourth category of

Koebe's canonical slits regions in the Chapter 5. Here, for each class of the canonical region, two uniquely solvable integral equations with the adjoint generalized Neumann kernel are constructed. These boundary integral equations are constructed from a boundary relationship satisfied by an analytic function on an unbounded multiply connected region. The integral equations are all linear and by finding all the necessary parameters, we can use these information to find their corresponding inverse maps.

Finally, the concluding chapter, Chapter 6, contains a summary of this thesis and some recommendations for future research.

There are five appendices in this thesis. Appendix A presents the list of the papers that have been published or submitted during the authors candidature. Appendix B presents the derivation for the continuity of kernel $M(s,t)$ and $N(s,t)$. Appendix C shows the derivation of non-homogeneous boundary integral equation of unbounded multiply connected regions. Appendix D displays some samples of computer programs coded in MATLAB's R2011a. Finally, an image transformation which is conformal will be computed by using MATLAB's `imtransform` function is illustrated in Appendix E.

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