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Category of Fuzzy Graph and Its Relation to Morphism

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Abstract Category theory deals in an abstract way with mathematical structures and relationships between them. Fuzzy graph is a result of an application of fuzzy characteristics into a crisp graph. In this paper, we discuss the Category of Fuzzy Graph. In addition, the Subcategory of Fuzzy Graph Type-3 is revealed and some types of morphisms on Category of Fuzzy Graph were studied.

Keywords Fuzzy graph; Category theory

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1 Introduction

The study of categories is an attempt to axiomatically capture what is commonly found in various classes of related mathematical structures by relating them to the structure-preserving functions between them. Categories were first introduced by Eilenberg and MacLane in connection with algebraic topology [1]. The concept was initially used in homology theory and homological algebra which led to the definition by Pareigis [2]. However definition given by Higgins [3] is considered in this paper.

Definition 1 A category C is a structure comprising the following mathematical entities:

I. A class whose members A, B, C are called the objects of **C**.

II. For each pair of objects A, B, a set **C** (A, B) is called the set of morphism from A to B (in **C**). We write $f : A \to B$ to mean that $f \in$ **C** (A, B).

III. For each triple of objects A, B, C, law of composition : C (A, B) C (B, C) C (A, C);

That is for $f : A \to B$ and $g : B \to C$, we define a "composite" morphism $g \circ f$:$A \to C$. These mathematical entities are subject to the following axioms:

i. Associative law: If $f : A \to B$, $g : B \to C$ and $h : C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

ii. Identities: For every object A, there exist a morphism $e_A \in$ **C** (A, A) (an identity mapping of A onto itself) such that for any $f : A \to B$ and $g : C \to A$, one has $f \circ e_A = f$ and $e_A \circ g = g$.

2 Fuzzy Graph

Fuzzy graph is a result of an application of fuzzy theory into crisp graph. Rosenfeld [4] has defined fuzzy graph in which both vertices and edges are fuzzy sets.

Definition 2 Let S be a set. Fuzzy graph $\mathcal{G}= (\sigma, \mu)$ is a pair of function $\sigma : S \to [0, 1]$ and $\mu : S \times S \to [0, 1]$, we have $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$.

The mapping $\sigma : S \to [0, 1]$ assigns each $A \in S$ to membership value, $0 \leq \sigma_A \leq 1$. A fuzzy vertex of S is denoted as $(A,\ \sigma_A)$. Similarly, the mapping $\mu : S \times S \to [0, 1]$ is the fuzzy relation on σ which assigns each ordered pair of element (A, B); i.e. an edge with a degree of membership or value, $0 \leq \mu_{(A,B)} \leq 1$ denoted it as $((A, B),\ \mu_{(A,B)})$ such that $\mu_{(A,B)} \leq \sigma_A \wedge \sigma_B$. In other words, for μ to be a fuzzy relation on σ, we require that the degree of membership of a pair of elements never exceeds the degree of membership of either of the element.

Definition 3 Let μ and υ be fuzzy relations on σ , a composite of μ and υ is a fuzzy set $\mu \circ \upsilon$ defined by $(\mu \circ \upsilon)(A, C) = \bigvee_{B \in S} [\mu(A, B) \wedge \upsilon(B, C)]$ for $\forall\ A, C \in S$.

3 Category of Fuzzy Graph

In this section, we shall use the Definition 1 to prove the Category of Fuzzy Graph, G_F.

Theorem 1 *Fuzzy Graph, $\mathcal{G}_F$ is a category.*

Proof The objects of $\mathcal{G}_F$, $(A,\ \sigma_A)$, (B, σ_B), (C, σ_C) , are vertices with its own membership value, where we defined the class of objects of $\mathcal{G}_F$ as $Ob\ \mathcal{G}_F$ and morphisms f are arcs from $(A,\ \sigma_A)$ to (B, σ_B), i.e. $f = ((A,B), \mu_{(A,B)})$ or $f \in \hom(\mathcal{G}_F)$, where $hom(\mathcal{G}_F)$ is defined as a class of morphisms of $\mathcal{G}_F$.

Next, we look at the composition law. Suppose $f, g \in \hom(G_F)$ where $f= ((A, B),\ \mu_{(A,B)})$ i.e. $f : (A, \sigma_A) \to (B, \sigma_B)$ and $g = ((B, C),\ \mu_{(B,C)})$, then there exist a composite morphism $g \circ f = ((A, C),\ \mu_{(A,C)})$, where $\mu_{(A,C)} = \bigvee_{B \in Ob\ G_F} (\mu_{(A,B)} \wedge \mu_{(B,C)}) \leq \sigma_A \wedge \sigma_C$.

The above is true subject to the following axioms:

(i) *Associativity*:

It is well known that composition of fuzzy relation is associative, that is for every morphisms $f, g, h \in \hom(\mathcal{G}_F)$, where $f : (A, \sigma_A) \to (B, \sigma_B) f = ((A, B), \mu_{(A,B)})$, $g : (B, \sigma_B) \to (C, \sigma_C)\ g = ((B, C),\ \mu_{(B,C)})$ and $h : (C, \sigma_C) \to (D, \sigma_D)\ h = ((C, D),\ \mu_{(C,D)})$. We have $(h \circ g) \circ f = h \circ (g \circ f)$.

(ii) *Identity*

For every object A, there exists a morphism $e_A \in \mathcal{G}_F(A, A)$ such that for $f : A \to B$ and $g : C \to A$, one has $f \circ e_A = f$ and $e_A \circ g = g$. Let us consider the following cases:

1. Morphism $e_A \in \mathcal{G}_F(A, A)$ is a loop (Figure 1).

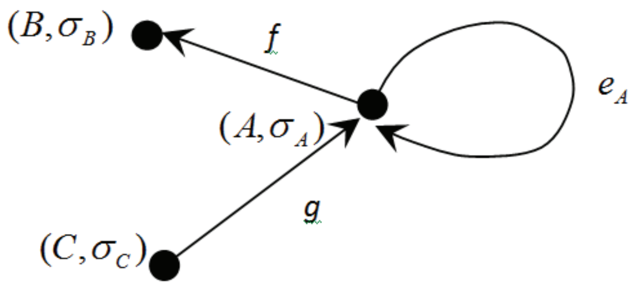

Figure 1: Identity Mapping for e_A as a Loop of Fuzzy Graph.

2. Morphism $e_A \in \mathcal{G}_F(A, A)$ is a cycle which go through other vertices.

   a. Morphism $f : A \to B$ and morphism $g : C \to A$ are not in the cycle of identity mapping e_A (Figure 2).

   b. Morphism $f : A \to B$ and morphism $g : C \to A$ are in the cycle of identity mapping e_A (Figure 3).

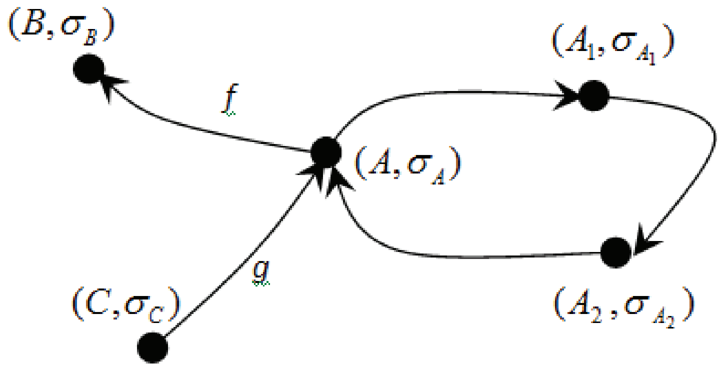

Figure 2: Identity Mapping with Cycle Identity which Goes through A_1 and A_2 of Fuzzy Graph.

To look at these two cases , we need the following definition.

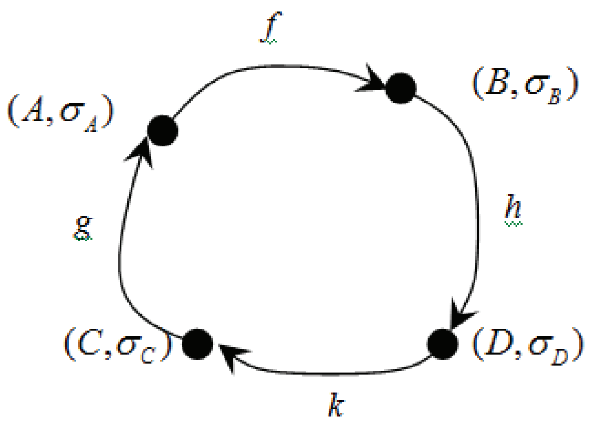

Figure 3: Identity Mapping with f and g are Arcs in Identity Cycle of Fuzzy Graph.

Definition 4 [4]: A path ρ in a fuzzy graph is a sequence of distinct nodes $x_0, x_1, ..., x_n \ni \mu(x_{i-1}, x_i) > 0$ for $1 \leq i \leq n$. A path in a fuzzy graph is called cycle if $x_0 = x_n$ for $n \geq 3$.

Case 1: Morphism $e_A \in \mathcal{G}_F(A, A)$ is a loop

An identity is an arc or morphisms $e_A = ((A, A),\ \mu_{(A,A)})$ such that if $f = ((A,B),\mu_{(A,B)})$ and $g = ((C, A),\ \mu_{(C,A)})$, then

$$f \circ e_A(A, \sigma_A) = f((A, A), \mu_{(A,A)}) = ((A, B), \mu_{(A,B)}) = f$$

$$e_A \circ g(C, \sigma_C) = e_A((C, A), \mu_{(C,A)}) = ((C, A), \mu_{(C,A)}) = g$$

Case 2: Morphism $e_A \in \mathcal{G}_F(A, A)$ is a cycle which goes through other vertices where morphism $f : A \to B$ and morphism $g : C \to A$ are not in the cycle.

An identity $e_A = ((A, A),\ \mu_{(A,A)})$ is a composite morphism $e_3 \circ e_2 \circ e_1$ go through (A_1, σ_{A_1}) and (A_2, σ_{A_2}) (Figure 2). One can see $e_1 = ((A, A_1), \mu_{(A,A_1)})$, $e_2 = ((A_1, A_2), \mu_{(A_1,A_2)})$ and $e_3 = ((A_2, A), \mu_{(A_2,A)})$ are fuzzy edges and $\mu_{(A,A)} = \vee[\mu_{(A,A_1)} \wedge \mu_{(A_1,A_2)} \wedge \mu_{(A_2,A)}]$ such that for $f = ((A,B),\mu_{(A,B)})$, then $f \circ e_A(A, \sigma_A) = f((A, A), \mu_{(A,A)}) = ((A, B), \mu_{(A,B)}) = f$ if and only if $\mu_{(A,B)} \leq \mu_{(A,A)}$.

Similarly for $g = ((C, A),\ \mu_{(C,A)})$, then $e_A \circ g(C, \sigma_C) = e_A((C, A), \mu_{(C,A)}) = ((C, A), \mu_{(C,A)}) = g$ if and only if $\mu_{(C,A)} \leq \mu_{(A,A)}$.

If the identity mapping e_A is a cycle which goes through arbitrary n-vertices, then for any $f : A \to B$ and $g : C \to A$, one has $f \circ e_A = f$ and $e_A \circ g = g$ if and only if the membership value of f and g are always less or equal to the weakest arc of the cycle of identity mapping e_A.

Case 2b: Morphism C (A, A) is a cycle which go through other vertices with morphism and morphism are in the cycle of identity mapping (Figure 3).

Consider the identity mapping $e_A = \{f,\ h,\ k,\ g\}$ with membership value, $\mu_{(A,A)} = \vee[\mu_{(A,B)} \wedge \mu_{(B,D)} \wedge \mu_{(D,C)} \wedge \mu_{(C,A)}]$ such that $f = ((A,B),\mu_{(A,B)})$,$h = ((B,D),\mu_{(B,D)})$,$k = ((D,C),\mu_{(D,C)})$ and $g = ((C,A),\mu_{(C,A)})$. For $f = ((A,B),\mu_{(A,B)})$, then $f \circ e_A(A,\sigma_A) = f((A,A),\mu_{(A,A)}) = ((A,B),\mu_{(A,B)}) = f$ if and only if morphism f is the weakest arc of the identity cycle. On the other hand if $g = ((C,A),\mu_{(C,A)})$, then $e_A \circ g(C,\sigma_C) = e_A((C,A),\mu_{(C,A)}) \neq ((C,A),\mu_{(C,A)}) = g$ since f is the weakest arc, i.e. $\mu_{(C,A)} \geq \mu_{(A,B)}$.

Therefore, morphisms f and g do not exist in the same cycle of identity mapping. This contradicts to the definition of identity with respect to the definition of category. We could then say that morphisms f and g do not exist in the cycle of identity mapping.

Thus, the axiom (ii) (Definition 1) is satisfied only when

1. the identity is a loop, or
2. the identity is a cycle with condition that for any morphisms f and g which are not arcs in the cycle, the membership value of f and g are always less or equal to the weakest arc of the cycle of identity mapping.

Consequently, fuzzy graph is a category. □

Furthermore, Tahir et. al. [5] have shown that

i. Every crisp graph is a fuzzy graph.

ii. Every Autocatalytic Set is a fuzzy graph.

iii. Every Fuzzy Autocatalytic Set (FACS) is also a fuzzy graph.

We can have the following corollaries immediately by invoking to Theorem 1:

Corollary 1:
Crisp graph is a category. □

Corollary 2:
Autocatalytic set is a category. □

Corollary 3:
FACS is a category. □

3.1 Subcategory of Fuzzy Graph Type-3

Tahir et. al. [5] defined the Fuzzy Graph Type 3, G_F^3 as a fuzzy graph where both the vertex and edge sets are crisp, but the edges have fuzzy heads and tails (Figure 4).
In order to complete the definition of Fuzzy Graph Type 3, they also introduced the following definition:

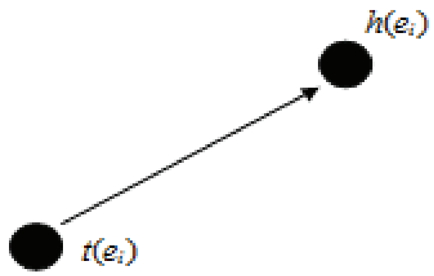

Figure 4: Fuzzy Head and Tail

Definition 5 Let $e_i \in E$. The fuzzy head of e_i denotes as $h(e_i)$ and the fuzzy tail $t(e_i)$ are functions of e_i such that $h : E \to [0,1]$ and $t : E \to [0,1]$ for $e_i \in E$. A fuzzy edge connectivity is a tuple $(t(e_i), h(e_i))$ and the set of all fuzzy edge connectivity is denoted as $C = \{(t(e_i), h(e_i)) : e_i \in E\}$. The membership value for each fuzzy edge connectivity is denoted as $\mu(e_i) = \min\{t(e_i), h(e_i)\}$.

Definition 6 [5]: Let denote the fuzzy edge connectivity between node i and node j, then

$$C_{F_{ij}} = \begin{cases} 0 & ,if\ i = j \qquad ,e_i \notin E \\ \mu(e_i) & ,if\ i \neq j \end{cases}$$

Although the above two definitions (Definition 5 and Definition 6) are designed to formulate a fuzzy graph for the clinical waste incineration process [5, 6] (Figure 5), we will use these definitions to prove the Subcategory of Fuzzy Graph Type 3 by using the Definition 7 defined by MacLane [7]:

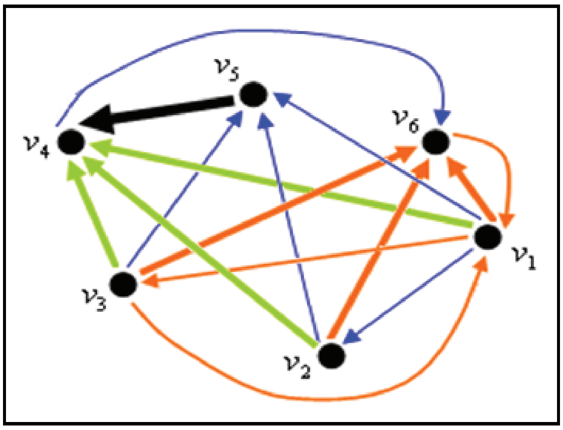

Figure 5: Fuzzy Graph of Type 3 G_{d_F} Built for a Clinical Incineration Process [6].

Definition 7 Let C be a category. A subcategory S of C is given by

- a subclass of objects of C, denoted as $Ob(\mathbf{S})$,
- a subclass of morphisms of C , denoted hom(S),

such that

i. for every X in Ob(S), the identity morphism idX is in hom(S),

ii. for every morphism is in hom(S), both the source X and the target Y are in Ob(S),

iii. for every pair of morphisms f and g in hom(S) the composite is in hom(S) whenver it is defined.

These conditions ensure that S is a category in its own right.

We already proved that Fuzzy Graph, $\mathcal{G}_F$ is a Category. In the following theorem, we will prove that Fuzzy Graph Type 3 is a subcategory.

Theorem 2 *Fuzzy Graph Type 3 is a subcategory.*

Proof We know that the Fuzzy Graph is a category, and Fuzzy Graph Type 3 is a Fuzzy Graph. Observe that $Ob(S) = Ob(\mathcal{G}_F^3) = \{V_{\mathcal{G}_F^3}\}$ is a class of crisp vertices and $\hom(S) = \hom(\mathcal{G}_F^3) = \{E_{\mathcal{G}_F^3}\}$ is a class of crisp edges with membership value of fuzzy edge connectivety,$\mu_c(e_i)$. These mathematical entities as term used in Definition 7 are subject to the following condition:

(i) Every $A \in Ob\ (\mathcal{G}_F^3)$, the identity morphism I_A is in $\hom(\mathcal{G}_F^3)$:

By the Definition 6, there exists no loop in Fuzzy Graph Type 3. We may consider our identity morphism as a cycle, which connected from one node to itself through other vertices or node. In other words, let the identity mapping I_A , be a cycle which goes through arbitrary n-crisp vertices i.e., $\rho = A_1, A_2, ..., A_{n-1}, A_n : A_1 = A_n$ where $e_{I_A} = \{I_{A_1}, I_{A_2}, ..., I_{A_{n-1}}\}$, for $I_{A_i} \in \hom(\mathcal{G}_F^3)$ with membership value of fuzzy edge connectivity given as $\mu_c(e_{I_A}) = \vee[\mu_c(e_{I_{A_1}}) \wedge \mu_c(e_{I_{A_2}}) \wedge ... \wedge \mu_c(e_{I_{A_n}})]$. Thus $I_A \in \hom(\mathcal{G}_F^3)$.

(ii) Every morphism $f : A \to B$ is in $\hom\mathcal{G}_F^3$, both the source A and the target B are in $Ob\ \mathcal{G}_F^3$:

This is obvious since our morphisms f are crisp arc or edge from A to B for every $A, B \in Ob\ \mathcal{G}_F^3$ with fuzzy edge connectivity.

(iii) Every pair of morphisms f and g in $\hom\mathcal{G}_F^3$ the composite $g \circ f$ is in $\hom\mathcal{G}_F^3$:

Suppose f and g are two crisp morphisms where $f = ((A, B), \mu_c(e_f))$ and $g = ((B, C), \mu_c(e_g))$, then there is a composite morphism $g \circ f = ((A, C), \mu_c(e_{g\circ f}))$ where $\mu_c(e_{g\circ f}) = \vee[\mu_c(e_f) \wedge$

$\mu_c(e_g)]$. Therefore $g \circ f \in \hom\mathcal{G}_F^3$ since $\hom\mathcal{G}_F^3$ is a set of crisp morphisms with fuzzy edge connectivity.

Consequently, the Fuzzy Graph Type 3, $\mathcal{G}_F^3$, is a subcategory of category of fuzzy graph, $\mathcal{G}_F$. $\square$

4 Types of Morphisms of Category of Fuzzy Graph

Category of Fuzzy Graph also yields some types of morphisms. We will use the definition given by Mitchell [8] to accomplish our task.

Theorem 3 *Every morphisms between two components of Category of Fuzzy Graphs is a monomorphism.*

Proof Let $f : (A, \sigma_A) \to (B, \sigma_B)$ such that $f = ((A, B), \mu_{(A,B)})$ and $g_1, g_2 : X \to A$ such that $g_1 = ((X, A), \mu_{g_1(X,A)})$ and $g_2 = ((X, A), \mu_{g_2(X,A)})$ for $A, B, X \in Ob\ \mathcal{G}_F$. Suppose $f \circ g_1 = f \circ g_2$, then $(f \circ g_1)(X, \sigma_X) = f((X, A), \mu_{(X,A)}) = ((X, B), \mu^1_{(X,B)})$ such that $\mu^1_{(X,B)} = \bigvee_A [\mu_{g_1(X,A)} \wedge \mu_{(A,B)}]$.

On the other hand $f \circ g_2(X, \sigma_X) = f((X, A), \mu_{(X,A)}) = ((X, B), \mu^2_{(X,B)})$ such that $\mu^2_{(X,B)} = \bigvee_A [\mu_{g_2(X,A)} \wedge \mu_{(A,B)}]$. It follows that $\mu_{g_1(X,A)} = \mu_{g_2(X,A)}$ which implies $g_1 = g_2$.

Hence $f : (A, \sigma_A) \to (B, \sigma_B)$ is a monomorphism. $\square$

Theorem 4 *Every morphism between two components of Category of Fuzzy Graphs is an epimorphism.*

Proof : Let $f : (A, \sigma_A) \to (B, \sigma_B)$ such that $f = ((A, B), \mu_{(A,B)})$ and $g_1, g_2 : B \to X$ such that $g_1 = ((B, X), \mu_{g_1(B,X)})$ and $g_2 = ((B, X), \mu_{g_2(B,X)})$ for $A, B, X \in Ob\ \mathcal{G}_F$. If $g_1 \circ f = g_2 \circ f$, then $(g_1 \circ f)(A, \sigma_A) = g_1((A, B), \mu_{(A,B)}) = ((A, X), \mu^1_{(A,X)})$ such that $\mu^1_{(A,X)} = \bigvee_B [\mu_{(A,B)} \wedge \mu_{g_1(B,X)}]$.

On the other hand $(g_2 \circ f)(A, \sigma_A) = g_2((A, B), \mu_{(A,B)}) = ((A, X), \mu^2_{(A,X)})$ such that $\mu^2_{(A,X)} = \bigvee_B [\mu_{(A,B)} \wedge \mu_{g_2(B,X)}]$. It follows that $\mu_{g_1(X,A)} = \mu_{g_2(X,A)}$ which implies $g_1 = g_2$. Therefore, $f : (A, \sigma_A) \to (B, \sigma_B)$ is an epimorphism. $\square$

Theorem 5 *Every morphism between two components of the Category of Fuzzy Graph is bimorphism.*

Proof By Theorem 3 and Theorem 4, every morphism between two components of the Category of Fuzzy Graph is bimorphism. $\square$

Theorem 6 *Every morphism between two components of the Category of Fuzzy Graph is an endomorphism.*

Proof By the definition of endomorphism, f is a mapping which map into itself, i.e. $f : a \to a$. However, by our definition of identity of Category of Fuzzy Graph, f is an identity mapping where $e_A = ((A, A),\ \mu_{(A,A)})$ which is a loop or a cycle which goes through other vertices.
Thus, f is an endomorphism. □

In the Category of Fuzzy Graph, the mappings are the morphisms $f : A \to B$ with grade of membership, $\mu_{(A,B)}$ for every $A, B \in Ob\ \mathcal{G}_F$. f is not an injective due to their membership value since there exists $a_1 \neq a_2 \in A$ and $\sigma_{a_1} \neq \sigma_{a_2}$ such that $f(a_1) = f(a_2) = b \in B$ where b has the grade of membership as below which defined by Zadeh [9]:

$$\sigma_b = \max_{a_i \in f^{-1}(b)} \sigma_{a_i}$$

and $f^{-1}(b)$ is the set of point in A which mapped into b by f. Therefore, the morphisms of Category of Fuzzy Graph are not injective which implies the morphisms of Category of Fuzzy Graph are neither retraction nor section. For that reason, we obtained the following results:

1. The morphisms of Category of Fuzzy Graph are not an isomorphism.
2. The morphisms of the Category of Fuzzy Graph are not an automorphism.

5 Conclusion

In this paper, we have proven that Fuzzy Graph is a Category. Then, we proved the Fuzzy Graph Type-3 is a subcategory. We also studied some types of morphisms of the category of fuzzy graph and from that study we can see that the morphism of fuzzy graph is monomorphism, epimorphism, bimorphism and also endomorphism.

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