

COMMUTATIVITY DEGREES AND RELATED INVARIANTS  
OF SOME FINITE NILPOTENT GROUPS

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*To My Mom and Dad*

*To My Beloved Husband*

*To My Family*

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## ABSTRACT

In this research, two-generator  $p$ -groups of nilpotency class two, which is referred to as  $G$  are considered. The commutativity degree of a finite group  $G$ , denoted as  $P(G)$ , is defined as the probability that a random element of the group  $G$  commutes with another random element in  $G$ . The main objective of this research is to derive the general formula for  $P(G)$  and its generalizations. This research starts by finding the formula for the number of conjugacy classes of  $G$ . Then the commutativity degree of each of these groups is determined by using the fact that the commutativity degree of a finite group  $G$  is equal to the number of conjugacy classes of  $G$  divided by the order of  $G$ . The commutativity degree can be generalized to the concepts of  $n$ -th commutativity degree,  $P_n(G)$ , which is defined as the probability that the  $n$ -th power of a random element commutes with another random element from the same group. Moreover,  $P_n(G)$  can be extended to the relative  $n$ -th commutativity degree,  $P_n(H, G)$ , which is the probability of commuting the  $n$ -th power of a random element of  $H$  with an element of  $G$ , where  $H$  is a subgroup of  $G$ . In this research, the explicit formulas for  $P_n(G)$  and  $P_n(H, G)$  are computed. Meanwhile, another generalization of the commutativity degree, which is called commutator degree and denoted by  $P_g(G)$ , is the probability that the commutator of two elements in  $G$  is equal to an element  $g$  in  $G$ . In this research, an effective character-free method is used for finding the exact formula for  $P_g(G)$ . Finally, the exterior degree of the wreath product of  $A$  and  $B$ ,  $P^\wedge(A \wr B)$ , is found where  $A$  and  $B$  are two finite abelian groups.

## ABSTRAK

Dalam penyelidikan ini, kumpulan- $p$  berpenjana-dua dengan kelas nilpoten dua, dirujuk sebagai  $G$ , dipertimbangkan. Darjah kekalisan tukar tertib bagi suatu kumpulan terhingga  $G$ , ditandakan sebagai  $P(G)$ , ditakrifkan sebagai kebarangkalian bahawa satu unsur dalam  $G$  yang dipilih secara rawak adalah kalis tukar tertib dengan unsur lain yang dipilih secara rawak dalam  $G$ . Objektif utama kajian ini adalah untuk mendapatkan rumus umum bagi  $P(G)$  dan pengitlakan bagi  $P(G)$ . Kajian ini dimulakan dengan mencari rumus untuk bilangan kelas kekonjugatan bagi  $G$ . Kemudian darjah kekalisan tukar tertib bagi kumpulan-kumpulan ini ditentukan dengan menggunakan fakta bahawa darjah kekalisan tukar tertib bagi suatu kumpulan terhingga  $G$  adalah sama dengan bilangan kelas kekonjugatan dibahagi dengan peringkat kumpulan. Darjah kekalisan tukar tertib boleh diitlakan kepada konsep darjah kekalisan tukar tertib kali ke- $n$ ,  $P_n(G)$ , yang ditakrifkan sebagai kebarangkalian bahawa kuasa ke- $n$  bagi suatu unsur yang dipilih secara rawak berkalis tukar tertib dengan unsur yang lain daripada kumpulan yang sama. Selain itu,  $P_n(G)$  boleh dilanjutkan kepada konsep darjah kekalisan tukar tertib secara relatif,  $P_n(H, G)$  yang ditakrifkan sebagai kebarangkalian berkalis tukar tertib kuasa ke- $n$  bagi suatu unsur dalam  $H$  yang dipilih secara rawak dengan suatu unsur dalam  $G$ , dengan  $H$  adalah sub-kumpulan bagi  $G$ . Dalam kajian ini, rumus eksplisit untuk  $P_n(G)$  dan  $P_n(H, G)$  dikira. Sementara itu, pengitlakan yang lain bagi darjah kekalisan tukar tertib, yang dinamai darjah pengalis tukar tertib dan dilambangi  $P_g(G)$ , merupakan kebarangkalian bahawa penukar tertib bagi dua unsur dalam kumpulan  $G$  adalah sama dengan unsur  $g$  yang diberikan dalam kumpulan tersebut. Dalam penyelidikan ini, satu kaedah bebas-aksara yang berkesan telah digunakan untuk mencari rumus tepat bagi  $P_g(G)$ . Akhir sekali, darjah peluaran bagi hasil darab kalungan  $A$  dan  $B$ ,  $P^{\wedge}(A \uplus B)$ , diperoleh bagi dua kumpulan abelian terhingga  $A$  dan  $B$ .

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## LIST OF SYMBOLS

$P(G)$	-	Commutativity degree of a group $G$ .
$P_n(G)$	-	$n$ -th commutativity degree of a group $G$ .
$P_n(H, G)$	-	Relative $n$ -th commutativity degree of a group $G$ .
$P_g(G)$	-	Commutator degree of a group $G$ .
$P^\wedge(G)$	-	Exterior degree of a group $G$ .
$G \wedge G$	-	Exterior square of a group $G$ .
$1$	-	The identity of a group $G$ .
$1_\wedge$	-	The identity of exterior square of a group $G$ .
$G \otimes G$	-	Nonabelian tensor square of a group $G$ .
$M(G)$	-	Schur multiplier.
$k(G)$	-	The number of conjugacy classes of a group $G$ .
$\Omega_i(G)$	-	The group generated by all elements $x \in G$ such that $x^{p^i} = 1$ , for every integer $i \geq 0$ .
$C_G^\wedge(x)$	-	Exterior centralizer.
$Z^\wedge(G)$	-	Exterior center of a group $G$ .
$A \wr B$	-	The wreath product of two finite abelian groups $A$ and $B$ .
$A \# A$	-	The factor group of $A \otimes A$ over the subgroup generated by the elements of the form $a \otimes b + b \otimes a$ .
$Inv(A)$	-	An element of order two in any group $A$ .
$G/H$	-	The factor group of $H$ in $G$ .

## CHAPTER 1

### INTRODUCTION

#### 1.1 Introduction

The probability that two elements of a group commute is called the commutativity degree of the group  $G$ , which is denoted by  $P(G)$ . This notion was first introduced by Miller [1] in 1944 and has been generalized in a number of ways.

Mohd Ali and Sarmin [2] in 2006 extended the definition of commutativity degree and defined a new generalization of this degree which is called the  $n$ -th commutativity degree,  $P_n(G)$ . This concept can be written as the probability that the  $n$ -th power of a random element commutes with another random element from the same group,  $G$ . A few years later, Erfanian *et al.* [3] gave the relative case of  $n$ -th commutativity degree. They identify the probability that the  $n$ -th power of a random element of a subgroup  $H$  of  $G$  commutes with another random element of  $G$ , denoted as  $P_n(H, G)$ .

The commutativity degree can also be defined as the probability that the commutator of two elements in a group,  $G$ , is equal to identity. Therefore, using this definition, Pournaki and Sobhani [4] provide a new generalization of commutativity degree, that is, the probability that the commutator of two elements in a group,  $G$ , is equal to a given element. In this research, this probability is defined as the commutator degree.

The exterior square of a group  $G$ , denoted as  $G \wedge G$ , is defined as  $G \wedge G = G \otimes G / \nabla(G)$  where  $G \otimes G$  is the nonabelian tensor square of  $G$  and  $\nabla(G)$  is the subgroup of  $G \otimes G$  generated by  $x \otimes x$  for  $x \in G$ . Recently, Niroomand and Rezaei [5] gave some relations between the concept of exterior square and commutativity degree by defining the exterior degree of a group  $G$ ,  $P^\wedge(G)$ , as the probability that two elements  $g$  and  $g'$  in  $G$  satisfy  $g \wedge g' = 1_\wedge$ , where  $1_\wedge$  is the identity of  $G \wedge G$ .

In this research, the commutativity degree,  $n$ -th commutativity degree, relative  $n$ -th commutativity degree and the commutator degree for two-generator  $p$ -groups of nilpotency class two are determined. Moreover, an upper and lower bound of the wreath product of two abelian groups are found.

## 1.2 Research Background

A group is called an abelian group if every pair of its elements commutes. This means that for a group  $G$ ,  $ab = ba$ , for all  $a, b \in G$ . However, not all groups are abelian, the ones that are not are called non-abelian groups. Can one measure in a certain sense how commutative a non-commutative group be?

### 1.2.1 Commutativity Degree and Its Generalizations

All groups considered in this research are assumed to be finite. The commutativity degree of a group  $G$ , which is denoted by  $P(G)$ , is the probability that two elements of the group  $G$ , chosen randomly with replacement, commute. This can be written as,

$$P(G) = \frac{\text{Number of ordered pairs } (x, y) \in G \times G \text{ such that } xy = yx}{\text{Total number of ordered pairs } (x, y) \in G \times G}.$$

In other words, the commutativity degree is a kind of measure for the abelianness of a group.

As mentioned before, there are other generalizations of the probability that two elements commute. One of them is the probability that the  $n$ -th power of a random element commutes with another random element from the same group, denoted by  $P_n(G)$ . The definition of  $P_n(G)$  can be written as the following ratio:

$$P_n(G) = \frac{\text{Number of ordered pairs } (x, y) \in G \times G \text{ such that } x^n y = y x^n}{\text{Total number of ordered pairs } (x, y) \in G \times G}.$$

In order to find  $P_n(G)$ , firstly, the power of each element is needed to be gradually raised until the power  $n$  is achieved. However, if the first copy of  $G$  is substituted with a subgroup  $H$  of  $G$ , then this probability is called a relative  $n$ -th commutativity degree and is denoted as  $P_n(H, G)$ . That is,  $P_n(H, G)$  is the probability that the  $n$ -th power of a random element of  $H$  commutes with an element of  $G$  and can be written as:

$$P_n(H, G) = \frac{|\{(h, g) \in H \times G : [h^n, g] = 1\}|}{|H||G|}.$$

The commutativity degree can be considered as a special case of a more general probability  $P_g(G)$ , which is defined for each  $g \in G'$  where  $G'$  is the commutator subgroup of  $G$ , denoted by

$$P_g(G) = \frac{|\{(x, y) \in G \times G : [x, y] = g\}|}{|G|^2},$$

that is the probability that the commutator of two randomly chosen elements of  $G$  is equal to the given element  $g$ . It is easy to see that  $P_g(G) = P(G)$  if  $g = 1$ , the identity element of  $G$ .

Recently, the concept of commutativity degree has been connected with the exterior square of a group. This connection is called the exterior degree of a group  $G$ , denoted as  $P^\wedge(G)$ , and is defined as the probability for two elements  $x$  and  $y$  in  $G$  to satisfy  $x \wedge y = 1_\wedge$ , where  $1_\wedge$  is the identity of  $G \wedge G$ . In mathematical symbols, this notion can be written as

$$P^\wedge(G) = \frac{|\{(x, y) \in G \times G \mid x \wedge y = 1_\wedge\}|}{|G|^2}.$$

### 1.2.2 Motivation

This research has its foundation with the classification of two-generator  $p$ -groups of nilpotency class two. Bacon and Kappe [6] and Kappe *et al.* [7] classified these groups for  $p$  is an odd prime and  $p = 2$ , respectively. The commutativity degree of a group  $G$  can be computed by dividing the number of conjugacy classes by the order of the group. The number of conjugacy classes of two-generator  $p$ -groups of nilpotency class two where  $p$  is an odd prime has been computed by Ahmad in [8]. Meanwhile, Ilangoan and Sarmin [9 – 12] computed the same for the case  $p = 2$ .

Since the commutativity degrees for two-generator  $p$ -groups of nilpotency class two have not been found yet, this research computes the exact formula for these degrees using the results by Ahmad [8]. Then this research also considers the other generalizations of commutativity degrees which are the  $n$ -th commutativity degrees, relative  $n$ -th commutativity degrees and the commutator degrees. Motivated by a paper written by Erenenko and Sury [13] in 2008, this research also found the exterior degree of the wreath product of two abelian groups.

### 1.3 Problem Statements

Students who study probability and algebra might well ask the following questions:

- (i) What is the probability that two elements of a group commute?
- (ii) Can one measure how commutative a non-commutative group can be?

This research is motivated by those questions. Hence, in this research, the following questions will be addressed and answered.

- (i) What is the probability that two elements of a group, chosen randomly with replacement commute?
- (ii) What is the probability that the  $n$ -th power of a random element of a group commutes with another random element from the same group?

- (iii) What is the probability that the  $n$ -th power of a random element of a subgroup of a group commutes with an element of the group?
- (iv) What is the probability that the commutator of two elements in a group is equal to a particular element of the group?
- (v) What is the probability that two elements  $g$  and  $g'$  in a group  $G$  satisfy  $g \wedge g' = 1_\wedge$ , where  $1_\wedge$  is the identity element of  $G \wedge G$ ?

#### 1.4 Research Objectives

The objectives of this research are:

- (i) to determine the commutativity degree of two-generator  $p$ -groups of nilpotency class two and give the generalization formula.
- (ii) to compute the  $n$ -th and relative  $n$ -th commutativity degree for two-generator  $p$ -groups of nilpotency class two and give the generalization formula.
- (iii) to find the commutator degree of a nilpotent  $p$ -group of class two.
- (iv) to obtain the exterior degree of the wreath product of two abelian groups.

#### 1.5 Scope of the Study

This thesis focuses on the two-generator  $p$ -groups of nilpotency class two for characterizing the commutativity degree,  $n$ -th commutativity degree, relative  $n$ -th commutativity degree and the commutator degree. Meanwhile, in computing the exterior degree of a group, the wreath product of two abelian groups is considered.

## 1.6 Significance of Findings

The aim of this research is to present new results in group theory in the forms of theorems. Using these theorems, the commutativity degree,  $n$ -th commutativity degree, relative  $n$ -th commutativity degree, commutator degree and the exterior degree are determined. The methods applied and the results obtained can be used for computing the commutativity degree,  $n$ -th commutativity degree, relative  $n$ -th commutativity degree, commutator degree and the exterior degree for other groups. Besides, the result of the commutativity degree can be transferred to a non-commuting graph where this kind of graph can be used to characterize the group theory properties of a group. For example, if a non-commuting graph of a group  $G$  is isomorphic to the non-commuting graph of the alternating group  $A_n$  where  $n \geq 4$ , then  $G \cong A_n$ . Research papers have been accepted and will be sent to be published in indexed local/international journals. Research papers also have been and will be presented in local and international seminars/conferences (see Appendix B).

## 1.7 Research Methodology

In the first step, by rewriting the results of [8 – 12], the exact formula for the number of conjugacy classes of two generator  $p$ -groups of nilpotency class two for any  $p$  are found. Then the commutativity degree of these groups is determined by using the equation which was introduced by Gustafson in [14]. In the second step, the  $n$ -th commutativity degree and the relative  $n$ -th commutativity degree are computed by using the definition of these degrees. The third step is focused on the commutator degree of two-generator  $p$ -groups of nilpotency class two and nilpotent  $p$ -groups with several conditions. By applying these results, some theorems are developed. Groups, Algorithms and Programming (GAP) software has been used in some calculations in Chapter 5. Lastly, bounds for the exterior degree of the wreath product of two abelian groups are found by computing the Schur multiplier and considering the bounds of exterior degree of a finite group.



## 1.8 Groups, Algorithms and Programming (GAP)

Groups, Algorithms and Programming (GAP) software is a system which provides a programming language, a library of thousands of functions implementing algebraic algorithms written in the GAP language as well as large data libraries of algebraic objects. This software is used in research and teaching for studying groups and their representations, rings, vector spaces, algebras, combinatorial structures and some others.

GAP also has many built in functions, operations, and algebraic structures. Thus GAP can be used to quickly provide numerous examples with more complexity than could be done by hand. In this research, GAP is used to calculate some of the commutator degrees.

## 1.9 Thesis Outline

This thesis is divided into seven chapters which includes the introduction, literature review, the commutativity degree of two-generator  $p$ -groups of nilpotency class two, the  $n$ -th and relative  $n$ -th commutativity degree of two-generator  $p$ -groups of nilpotency class two, the commutator degree of nilpotent  $p$ -groups of class two, the exterior degree of the wreath product of two abelian groups and conclusion.

In the first chapter, the introduction to the whole thesis is given including the research background, problem statement, research objectives, scope of the study, significance of findings and research methodology. This chapter also includes the concepts of the commutativity degree,  $n$ -th commutativity degree, relative  $n$ -th commutativity degree, the commutator degree and the exterior degree.

Chapter 2 presents the literature review of this research. Various works by different researchers regarding the commutativity degree and its generalizations are stated. Commutativity degree has been discovered for 68 years since 1944 where

Miller [1] was the first to introduce the concept of commutativity degree. The classifications of two-generator  $p$ -groups of nilpotency class two for any prime  $p$  are also given in this chapter.

Chapter 3 shows a general formula for finding the conjugacy classes of nilpotent  $p$ -groups of class two. Besides, this chapter gives the determination of commutativity degree of two-generator  $p$ -groups of nilpotency class two for any prime  $p$ . In addition, some concepts and basic results on the commutativity degree are also presented in this chapter.

Next, in Chapter 4, some preliminary results for  $n$ -th and relative  $n$ -th commutativity degree are included. The purpose of this chapter is to compute the explicit formula for  $P_n(G)$  and  $P_n(H, G)$  where  $G$  is a two-generator  $p$ -group of nilpotency class two and  $H$  is a subgroup of  $G$ . Furthermore, this chapter shows that if there are two pairs of relative isoclinic groups, then they will have equal relative  $n$ -th commutativity degree,  $P_n(H, G)$ .

In Chapter 5, the precise formulas of the commutator degree for two-generator  $p$ -groups of nilpotency class two are given. Some computations of the results of this degree have been obtained by using GAP software. Another result in this chapter is to determine the commutator degree for special cases of two-generator  $p$ -groups of nilpotency class two.

Chapter 6 focuses on the computation of the exterior degree of a group which is the wreath product of two abelian groups. Some basic concepts and results of exterior square, exterior degree and the wreath product of two abelian groups are also included in this chapter.

Finally, the last chapter presents the summary and conclusion of this research. Some suggestions for future research on the commutativity degree, the  $n$ -th commutativity degree, the relative  $n$ -th commutativity degree, the commutator degree and the exterior degree are also given in this chapter.

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