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The Nonabelian Tensor Squares and Homological Functors of Some 2-Engel Groups

Kuasa Dua Tensor Tak Abelan dan Fungtor Homologi beberapa kumpulan Engel-2

Hazzirah Izzati Mat Hassim[1], Nor Haniza Sarmin[1] and Nor Muhainiah Mohd Ali[2]
[1]Department of Mathematical Sciences, Faculty of Science
Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor
[2]Ibnu Sina Institute for Fundamental Science Studies
Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor
e-mail: [1]hazzirah_hassim@yahoo.com, [1]nhs@utm.my & [2]normuhainiah@utm.my

Abstract

Originated in homotopy theory, the nonabelian tensor square is a special case of the nonabelian tensor product. The nonabelian tensor square of a group G, denoted as $G \otimes G$ is generated by the symbols $g \otimes h$, for all $g, h \in G$ subject to the relations $gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h)$ and $g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$, for all $g, g', h, h' \in G$ where the action is taken to be conjugation. The homological functors of a group including $J(G)$, $\nabla(G)$, the exterior square, the Schur multiplier, $\nabla(G)$, the symmetric square and $\tilde{J}(G)$ d(G) are closely related to the nonabelian tensor square of the group. In this paper, the nonabelian tensor squares and homological functors of some 2-Engel groups will be presented.

Keywords nonabelian tensor square, homological functors, 2-Engel group

Abstrak

Berasal daripada teori homotopi, kuasa dua tensor tak abelan merupakan kes istimewa bagi hasil darab tensor tak abelan. Kuasa dua tensor tak abelan bagi suatu kumpulan *G,* dilabel sebagai $G \otimes G$ adalah dijanakan oleh simbol $g \otimes h$, bagi semua $g, h \in G$ tertakluk kepada hubungan $gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h)$ dan $g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$, bagi semua $g, g', h, h' \in G$ dengan tindakannya adalah kekonjugatan. Fungtor homologi bagi suatu kumpulan termasuk $J(G)$, $\nabla(G)$, kuasa dua peluaran, pekali Schur, $\nabla(G)$, kuasa dua simetrik dan $\tilde{J}(G)$ d(G) adalah berkait rapat dengan kuasa dua tensor tak abelan bagi kumpulan tersebut. Dalam artikel ini, kuasa dua tensor tak abelan dan fungtor homologi bagi beberapa kumpulan Engel-2 akan ditunjukkan.

Kata kunci kuasa dua tensor tak abelan, fungtor homologi, kumpulan Engel-2

Introduction

The nonabelian tensor square of a group was originated in algebraic K-theory as well as in homotopy theory. For any group G, its nonabelian tensor square, denoted as $G \otimes G$ is generated by the symbols $g \otimes h$, for all $g, h \in G$ subject to the relations $gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h)$ and $g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$ for all $g, g', h, h' \in G$ where ${}^{g}g' = gg'g^{-1}$. Note that the multiplication of two elements in the nonabelian tensor square of a group cannot be obtained as a usual multiplication of two elements in a group.

The homological functors of a group including $J(G)$, $\nabla(G)$, the exterior square, the Schur multiplier, ∇(G), the symmetric square and $\tilde{J}(G)$ are closely related to the nonabelian tensor square of the group. The group G acts naturally on the tensor products by ${}^{g}(g'\otimes h) = {}^{g}g'\otimes {}^{g}h$ and there exists a homomorphism mapping κ, where $\kappa : G\otimes G'$ defined by $\kappa(g\otimes h) = [g,h]$, and $[g,h] = ghg^{-1}h^{-1}$. Thus $J(G)$ is the kernel of κ and is a G-trivial subgroup of $G\otimes G$ contained in its center. The next homological functors $\nabla(G)$ and $\nabla(G)$ denote the subgroup of $J(G)$ generated by the elements $x\otimes x$ for $x\in G$ and the subgroup of $J(G)$ generated by the elements $(x\otimes y)\,(y\otimes x)$ for $x, y \in G$ respectively. The quotient group ${}^{G\otimes G}\!/\!_{\nabla(G)}$ is known as the exterior square of the group G and is denoted as $G\wedge G$. Meanwhile the symmetric square of G is defined as $G\tilde{\otimes} G = {}^{G\otimes G}\!/\!_{\nabla(G)}$ and the Schur multiplier of G is defined as $M(G) = {}^{JG}\!/\!_{\Delta(G)}$. Furthermore, $\tilde{J}(G)$ is defined as the factor group ${}^{JG}\!/\!_{\Delta(G)}$.

The study on the nonabelian tensor squares for various groups has been conducted by many researchers throughout the years. Brown *et al*. (1987) did explicit computation of the nonabelian tensor squares of all nonabelian groups up to order 30. A combination of reduction steps has been used to simplify the presentation resulting from the definition of the nonabelian tensor square. Tietze transformation is performed using a computer program to assist in simplifying the presentation. Then, the simplified presentation is examined in order to determine the isomorphism type of the nonabelian tensor square.

The nonabelian tensor squares of the 2-generator p-group of class 2, where p is an odd prime have been determined by Bacon and Kappe (1993). By considering the case for $p = 2$, Kappe *et al*. (1999) extended the research by Bacon and Kappe (1993) to determine the nonabelian tensor squares of 2-generator 2-groups of class 2. Then, Sarmin (2002) classified all infinite 2-generator groups of class two and determined their nonabelian tensor squares.

The homological functors were also originated in homotopy theory.This is the study of homotopy groups which are used in algebraic topology to classify topological spaces. Using their nonabelian tensor squares, Bacon and Kappe (2003) determined various homological functors for the 2-generator groups p-groups of class two.

A research on the homological functors of infinite nonabelian 2-generators groups of nilpotency class two has been conducted by Mohd Ali *et al*. (2007). The computations were done using the classification of the groups and their nonabelian tensor squares. In this research, Groups, Algorithm and Programming (GAP) algorithms are constructed for the computation of the homological functors of these groups. Besides, Beurle and Kappe (2008) classified all infinite metacyclic groups and determined their nonabelian tensor squares. The homological functors such as the exterior square and the Schur multiplier have also been computed in their research.

Ramachandran *et al*. (2008) showed the hand computations of the nonabelian tensor square and homological functors of the symmetric group of order six. Using the definition of the nonabelian tensor square of a group, the Cayley table of $S_3 \otimes S_3$ was found. Hence, based on the nonabelian tensor square of this groups, the homological functors were then computed. Then, the results found were verified using GAP.

In this paper, the nonabelian tensor squares and some homological functors are computed for the 2-Engel groups of order eight. A group G is called a 2-Engel group if $[g,{}_2h] = [g,h,h] = e$ for all elements g and h in G. All 2-Engel groups of order at most twenty have been found by Sarmin and Yusof (2006). Based on their research, two groups have been identified as the 2-Engel groups of order eight. They are the Dihedral group of order eight, D_4 and the quaternion group of order eight, Q_2.

Preliminaries

Some preparatory results that are vital in the computation of the nonabelian tensor squares and homological functors of the Dihedral group of order eight and quaternion group of order eight are included in this section.

Theorem 1(Kappe *et al*., 1999)

Let G be a finite nonabelian 2-generator 2-group of class 2. If $G \cong (\langle c\rangle \times \langle a\rangle) \times \langle b\rangle$ where $[a,b]=c, [a,c]=[b,c]=1, |a|=2^{\alpha}, |b|=2^{\beta}, |c|=2^{\gamma}$ with $\alpha=\beta=\gamma$, then

$$G \otimes G = \langle a\otimes a\rangle \times \langle b\otimes b\rangle \times \langle (a\otimes b)(b\otimes a)\rangle \times \langle a\otimes b\rangle \times \langle (a\otimes b)^2 (a\otimes c)\rangle \times \langle (a\otimes b)^2 (b\otimes c)\rangle$$
$$\cong C_{2^{\gamma}}^3 \times C_{2^{\gamma+1}} \times C_{2^{\gamma-1}}^2$$

Theorem 2(Kappe *et al*., 1999)

Let G be a finite nonabelian 2-generator 2-group of class 2. If $G \cong (\langle c\rangle \times \langle a\rangle)\ \langle b\rangle$,

where $|a|=|b|=2^{\gamma+1}$, $|[a,b]|=2^{\gamma}$, $|c|=2^{\gamma-1}$, $[a,b]=a^2c$,

$[c,b]=a^{-4}c^{-2}$, $a^{2^{\gamma}}=b^{2^{\gamma}}$, $\gamma \in \mathrm{N}$, then

$$G\otimes G \cong \begin{cases} C_2^2 \times C_4^4 & \text{for } \gamma=1, \\ C_{2^{\gamma}}^3 \times C_{2^{\gamma+1}} \times C_{2^{\gamma-1}}^2 & \text{for } \gamma \geq 2. \end{cases}$$

Proposition 1(Brown *et al*., 1987)

Let $m=4r+k$, where $k=0$ or 2. Then $J(Q_m)$ is isomorphic to $C_4 \times C_{k+2} \times C_2$, generated by $(x\otimes x)(x\otimes y)^m (y\otimes y)^{k/2}$, $y\otimes y$, and $(x\otimes y)(y\otimes x)$, respectively.

Lemma 2 (Dummit and Foote, 2004)

Let A and B be groups with $M \lhd A$ and $N \lhd B$. Then $(A\times B)/(M\times N) \cong A/M \times B/N$.

Lemma 3 (Dummit and Foote, 2004)

Let $A=\langle a\rangle$ and $B=\langle a^h\rangle$ be groups. If $\langle a\rangle \cong C_{\infty}$, then the quotient of cyclic group is cyclic, i.e $A/B \cong C_h$. If $\langle a \rangle \cong C_k$, then $\left| a^h \right| = \dfrac{k}{\gcd(h,k)}$. Furthermore, $\langle a^h \rangle \cong C_t$ where $t = \dfrac{k}{\gcd(h,k)}$. Thus, $A/B = \langle a \rangle / \langle a^h \rangle \cong C_{k/t} = C_{\gcd(h,k)}$.

Proposition 4 (Bacon and Kappe, 2003)

Given a group G of nilpotency class 2 and a generating set X for G, we have

$$G \otimes G = \langle u \otimes v, u \otimes [v,w] \mid u,v,w \in X \rangle, \tag{1}$$

$$\nabla(G) = \langle u \otimes u, (u \otimes v)(v \otimes u) \mid u,v \in X \rangle, \tag{2}$$

$$\Delta(G) = \langle (u \otimes u)^2, (u \otimes v)(v \otimes u) \mid u,v \in X \rangle. \tag{3}$$

The Computation of the Nonabelian Tensor Squares and Homological Functors of D_4 and Q_2

In this section, the computation of the nonabelian tensor square and some homological functors of the Dihedral group of order eight, D_4 and quaternion group of order eight, Q_2 are shown. The results are given in the following theorems.

Theorem 3

Let $G = D_4$ with the presentation $\langle a,b : a^4 = b^2 = e, ba = a^3 b \rangle$. Then

$$G \otimes G = \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (a \otimes b)(b \otimes a) \rangle \times \langle a \otimes b \rangle \cong C_2^3 \times C_4, \tag{4}$$

$$\nabla(G) = \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (a \otimes b)(b \otimes a) \rangle \cong C_2^3, \tag{5}$$

$$G \wedge G = \langle a \wedge b \rangle \cong C_4, \tag{6}$$

$$\Delta G = \langle \langle a \otimes a \rangle^2 \rangle \times \langle \langle b \otimes b \rangle^2 \rangle \times \langle \langle a \otimes b \rangle \langle b \otimes a \rangle \rangle \cong C_2 \tag{7}$$

$$G \tilde{\otimes} G = \langle a \tilde{\otimes} a \rangle \times \langle b \tilde{\otimes} b \rangle \times \langle a \tilde{\otimes} b \rangle \cong C_2^2 \times C_4. \tag{8}$$

Proof

Based on Theorem 1, by letting $\alpha = \beta = \gamma = 1$, the nonabelian tensor square of G is determined as follows:

$$G \otimes G = \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (a \otimes b)(b \otimes a) \rangle \times \langle a \otimes b \rangle \cong C_{2^1}^3 \times C_{2^{1+1}} \cong C_2^3 \times C_4.$$

This shows that $D_4 \otimes D_4$ is isomorphic to three copies of cyclic group of order two and a copy of cyclic group of order four. Note that $|a \otimes a| = |b \otimes b| = |(a \otimes b)(b \otimes a)| = 2$ and $|a \otimes b| = 4.$ By (2) in Proposition 4 and order restriction on the generators, it follows that $\nabla(G) = \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (a \otimes b)(b \otimes a) \rangle \cong C_2^3.$ Thus (5) holds.

Using Lemma 2, Lemma 3 and observing that $\nabla(G) \lhd (G \otimes G)$,

$$\begin{aligned}
G \wedge G &= {(G \otimes G)}/{\nabla(G)} \\
&= {\langle a \otimes a \rangle}/{\langle a \otimes a \rangle} \times {\langle b \otimes b \rangle}/{\langle b \otimes b \rangle} \times {\langle (a \otimes b)(b \otimes a) \rangle}/{\langle (a \otimes b)(b \otimes a) \rangle} \times {\langle a \otimes b \rangle}/{\langle 1_\otimes \rangle} \\
&= \langle (a \otimes b) \nabla(G) \rangle = \langle a \wedge b \rangle.
\end{aligned}$$

Then, by the order restrictions on the generators, $G \wedge G \cong C_4,$ the desired result. Similarly as proof of (5), using (3) in Proposition 4, we have

$$\Delta(G) = \langle (a \otimes a)^2 \rangle \times \langle (a \otimes b)(b \otimes a) \rangle \times \langle (b \otimes b)^2 \rangle.$$

From Lemma 3, $|(a \otimes a)^2| = |(b \otimes b)^2| = \dfrac{2}{\gcd(2,2)} = 1.$ By these and the order restrictions on the generators, we obtain $\Delta(G) \cong C_2$. Thus, (7) holds.

Note that $\langle (a \otimes a)^2 \rangle$ is a proper subgroup of index two in $\langle a \otimes a \rangle$ and $\langle (b \otimes b)^2 \rangle$ is a proper subgroup of index two in $\langle b \otimes b \rangle$. Thus, ${\langle a \otimes a \rangle}/{\langle (a \otimes a)^2 \rangle} \cong {\langle b \otimes b \rangle}/{\langle (b \otimes b)^2 \rangle}$ $\cong C_2$. Together with Lemma 2, Lemma 3 and observing that $\Delta(G) \lhd (G \otimes G),$ we obtain

$$\begin{aligned}
G \tilde{\otimes} G &= {(G \otimes G)}/{\Delta(G)} \\
&= {\langle a \otimes a \rangle}/{\langle (a \otimes a)^2 \rangle} \times {\langle b \otimes b \rangle}/{\langle (b \otimes b)^2 \rangle} \times {\langle (a \otimes b)(b \otimes a) \rangle}/{\langle (a \otimes b)(b \otimes a) \rangle} \\
&\quad \times {\langle a \otimes b \rangle}/{\langle 1_\otimes \rangle} \\
&= \langle (a \otimes a) \nabla(G) \rangle \times \langle (b \otimes b) \nabla(G) \rangle \times \langle (a \otimes b) \nabla(G) \rangle = \langle a \tilde{\otimes} a \rangle \times \langle b \tilde{\otimes} b \rangle \times \langle a \tilde{\otimes} b \rangle.
\end{aligned}$$

Using the order restrictions on the generators, this leads to the desired result (8).

Theorem 4

Let $G = Q_2$ with the presentation $\langle a,b : a^4 = b^4 = e, a^2 = b^2, ba = a^3b \rangle$. Then

$$G \otimes G \cong C_2^2 \times C_4^2, \tag{9}$$

$$J(G) \cong C_2 \times C_4^2. \tag{10}$$

Proof

Since the quaternion group of order eight is a finite nonabelian 2-generator 2-group of class 2 where $G \cong (\langle c \rangle \times \langle a \rangle) \langle b \rangle$, with $|a| = |b| = 2^{\gamma+1}$, $|[a,b]| = 2^{\gamma}$, $|c| = 2^{\gamma-1}$, $[a,b] = a^2c$, $[c,b] = a^{-4}c^{-2}$, $a^{2^{\gamma}} = b^{2^{\gamma}}$ for $\gamma = 1$, then based on Theorem 2, $G \otimes G \cong C_2^2 \times C_4^2$. This shows that $Q_2 \otimes Q_2$ is isomorphic to two copies of cyclic group of order two and two copies of cyclic group of order four.

From Proposition 1, since for Q_2, $m = 2$ and $k = 2$, then

$$J(G) = \langle (x \otimes x)(x \otimes y)^2 (y \otimes y) \rangle \times \langle y \otimes y \rangle \times \langle (x \otimes y)(y \otimes x) \rangle \cong C_4 \times C_{2+2} \times C_2 \cong C_4^2 \times C_2.$$

Thus, (10) holds.

Conclusion

In this paper, the nonabelian tensor squares and homological functors have been computed for all 2-Engel groups of order eight.

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