SOLVING MIXED BOUNDARY VALUE PROBLEM VIA AN INTEGRAL EQUATION WITH THE GENERALIZED NEUMANN KERNEL IN BOUNDED DOUBLY CONNECTED REGION

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To my beloved parents, siblings

and all of my friends for your support, care and guidance in my life.

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ABSTRACT

This dissertation determines solution of a certain class of a mixed boundary value problem in bounded doubly connected region by using the method of boundary integral equations. The method depends on reformulating the boundary value problem with mixed Dirichlet - Neumann condition to the Riemann - Hilbert problem. Our approach in this dissertation is to work out in detail the reformulation of the mixed boundary value problem into the Riemann - Hilbert problem and study the efficiency of the proposed numerical scheme on challenging geometries, in particular when the boundaries are closed to each other. As an examination of the proposed method, some numerical examples for some different test regions are presented. These examples include comparison between the numerical result and the exact solutions. Numerical examples reveal that the proposed method offers an effective solution technique for the mixed boundary value problem when the boundaries are close to each other.

ABSTRAK

Disertasi ini menentukan penyelesaian tehadap suatu kelas masalah nilai sempadan yang bercampur-campur di dalam kawasan berkait dua kali ganda dengan menggunakan kaedah persamaan kamiran sempadan. Kaedah tersebut bergantung kepada perwakiln semula masalah nilai sempadan dengan syarat Dirichlet-Neumann bercampur kepada masalah Riemann-Hilbert. Pendekatan kami dalam kajian ini adalah untuk mengkaji keberkesanan skim berangka yang dicadangkan pada geometri yang lebih mencabar, khususnya apabila sempadan adalah berdekatan dengan satu sama lain. Sebagai suatu ujian kepada kaedah yang dicadangkan, beberapa contoh berangka untuk beberapa kawasan ujian yang berbeza dibentangkan. Contoh-contoh perbandingan antara keputusan berangka dan penyelesaian yang tepat dilampirkan. Contoh-contoh berangka menunjukkan bahawa kaedah yang telah dicadangkan memberi satu teknik penyelesaian yang berkesan untuk nilai nempadan bercampur apabila sempadan adalah berdekatan antara satu sama lain.

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CHAPTER 1

RESEARCH FRAMEWORK

1.1 Introduction

Partial differential equations play a significant role in science and technology. Many of the fundamental theories of physics and engineering are expressed by means of system of partial differential equations. Fluid mechanics is often formulated by the Euler equations of motion or the so-called Navier-Stokes equations. While, electricity and magnetism are formulated by Maxwells equations, and the general relativity by Einstein's field equations. It is therefore important to develop techniques that can be used to solve a wide variety of partial differential equations.

Integral equation method offer an attractive alternative to conventional finite difference, finite element and spectral methods for finding numerical solutions. They offer several notable advantages: complex boundaries are handled naturally, high- order accuracy is easier to attain, the correct boundary conditions at infinity are guaranteed, and ill-conditioning associated with directly discretizing the PDE is avoided. Nevertheless, since discretizations of integral equations result in dense linear systems, these methods have been less popular than others. However, with the advancement of fast algorithms, integral equation methods have become popular for solving large-scale problems.

There are many phenomena in these fields that can be described as boundary value problems. However, formulating and solving such problems are not easy when we talk about real modelling of those phenomena. Furthermore, it is also important to study existence and uniqueness of the solution of these problems. These issues were and still occupy the minds of mathematicians and engineers.

A boundary value problem is a problem involves finding the solution of a differential equation or system of differential equation which meets certain specified requirements or boundary conditions at the end points or along a boundary, usually connected with the physical condition for certain values of the independent variable.

The investigation of boundary value problems (denoted as BVPs for short) of ordinary differential equations is of great significance. On one hand, it makes a great impact on the studies of partial differential equations in Lian *et al.* (1996). On the other hand, BVPs of ordinary differential equations can be used to describe a large number of mechanical, physical, biological, and chemical phenomena (see Shi *et al.* (1997), Horgan *et al.* (2002), Amster and Rgers (2007), Amara (2009)). So far a lot of work has been carried out, including second-order, third-order, and higher-order BVPs with various boundary conditions.

One example of a boundary value problem is the Dirichlet problem. In potential theory it is defined as follows: Let Ω be a region in two-dimensional space, let Γ be its boundary and u be a function defined and continuous over $\Omega \cup \Gamma$. The Dirichlet problem is to find a solution of Laplace's equation $\bigtriangledown^2 u = 0$ that is harmonic in Ω, continuous in $\Omega \cup \Gamma$ and satisfies the equation $u = \phi$ at the boundary as shown in Figure 1.1 (Sneddon, 1966).

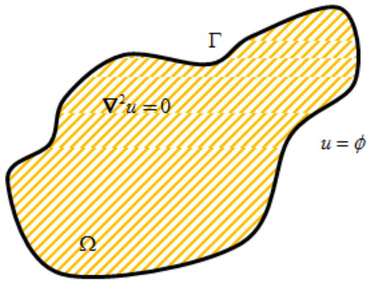

Figure 1.1: Dirichlet Problem

Another example of a boundary value problem is the Neumann problem, which is defined as follows: Let Ω be again a region in two - dimensional space and Γ be its boundary and u be a function defined and continuous over $\Omega \cup \Gamma$. The Neumann problem is to find solution of Laplace's equation $\bigtriangledown^2 u = 0$ which is harmonic in Ω, continuous in $\Omega \cup \Gamma$, and which satisfies a normal derivative equation $\frac{\partial u}{\partial n} = \phi$ on the boundary as illustrated in Figure 1.2 (Sneddon, 1966).

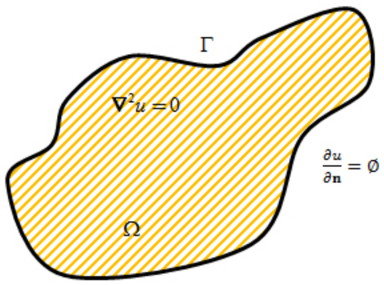

Figure 1.2: Neumann Problem

1.2 Background of the Study

A boundary value problem that has a mixture of Dirichlet and Neumann conditions is called a mixed boundary value problems. Three examples of such boundary value problems are shown in the Figures 1.3, 1.4 and 1.5.

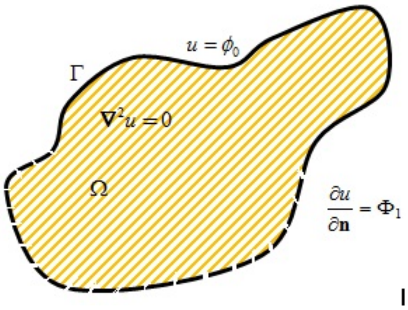

Figure 1.3: Mixed Dirichlet - Neumann Problem

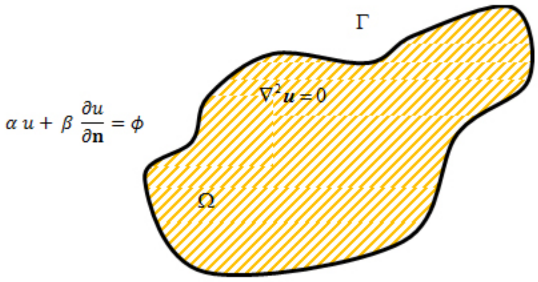

Figure 1.4: Mixed Dirichlet - Neumann Problem

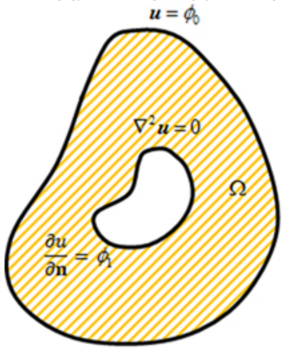

Figure 1.5: Mixed Dirichlet - Neumann Problem

A boundary condition that involves Neumann and Dirlichet conditions, is also called Robin condition. The potential theory is one of the areas that could represent this theory. It has various applications in electrostatics, heat transfer, linear elasticity and fluid flow (Sneddon, 1966).

Mixed boundary value problems occur, in a natural way, in varieties of branches of physics and engineering and several mathematical methods have been developed to solve this class of problems of applied mathematics.

Hence, the solution of mixed boundary value problem is highly needed. Since in general the solution of the problem cannot be obtained in closed form, we have to resort to numerical methods to approximate the solution considered. A number of numerical methods have been devised. The common ones, used by many researchers and studied extensively, are the finite difference methods, finite element methods and the boundary integral methods.

The solution of mixed boundary value problems requires considerable mathematical skill. Although the analytic solution begins using a conventional technique such as separation of variable or transform methods, the mixed boundary condition eventually leads to a system of equations, involving series or integrals, that must be solved. The solution of these equations often yields a Fredholm integral equation of the second kind. Because these integral equations usually have no closed form solution, numerical methods must be employed (Duffy, 2008).

It is hard to find integral equations for mixed problems that are second kind with operators that are compact on the entire boundary. Some attempts towards this direction are given by Greenbaum *et al.* (1993), Haas and Brauchli (1991), Liu (2005) and Helsing (2009).

Haas and Brauchli (1991) applied conformal mapping to transform the mixed problem onto the unit disk to form a corresponding Riemann-Hilbert problem in terms of Cauchy integrals. Greenbaum *et al.* (1993) considered Laplace's equation and Dirichlet-Neumann map in multiply connected regions. Liu (2005) determined Dirichlet-to-Neumann map for the Helmholtz equation with mixed boundary condition. Helsing (2009) has developed an integral equation with compact operators on almost entire boundary for solving mixed problems.

Recently, the interplay of Riemann-Hilbert problems (RH problems) and integral equation with the generalized Neumann kernel has been investigated in Wegmann *et al.* (2005) for simply connected regions with smooth and piecewise smooth boundaries and in Wegmann and Nasser (2008) for bounded and unbounded multiply connected regions. It has been shown that the problem of conformal mapping , Dirichlet problem , Neumann problem and mixed Dirichlet-Neumann problem can all be treated as RH problems as discussed in Nasser (2009b), Murid and Nasser (2009), Nasser *et al.* (2011) and Alhatemi *et al.* (2011).

Alhatemi *et al.* (2011) have solved the mixed boundary value problem on doubly connected region by using the method of boundary integral equation. Their approach was to reformulate the mixed boundary value problem into the form of Riemann - Hilbert problem. The Riemann - Hilbert problem is then solved using a uniquely solvable Fredholm integral equation on the boundary of the region. The efficiency of the proposed numerical scheme in Alhatemi *et al.* (2011) does not seem to be tested on really challenging geometries. What happens when the circles come even closer? Will the accuracy deteriorates? Are more discretization points needed to achieve a given accuracy? How does its computing cost and the achievable accuracy scale with much larger value of discretization points?

1.3 Statement of the Problem

This research will work out in detail the reformulation of the mixed boundary value problem into the Riemann - Hilbert problem given in Alhatemi *et al.* (2011) and study the efficiency of the proposed numerical scheme on challenging geometries, in particular when the boundaries are closed to each other.

1.4 Objectives of the Study

The objectives of this research are:

1. To work out in detail the reformulation of the boundary value problem with mixed Dirichlet-Neumann condition on bounded doubly connected region to corresponding Riemann-Hilbert problem.
2. To solve numerically the integral equation with the generalized Neumann kernel related to the Riemann-Hilbert problem using Nyström method with the trapezoidal rule.
3. To obtain numerical solutions of the mixed boundary value problem using Mathemayica and make some comparisons with Alhatemi *et al.* (2011).

1.5 Scope of the Study

There are several methods for solving mixed boundary value problems such as conformal mapping, integral equations, Green's function, separation of variables, transform methods, and much more.This research will consider solving mixed boundary value problems only on doubly connected regions with smooth boundary using integral equation with the generalized Neumann kernel.

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