THE n^{th} COMMUTATIVITY DEGREE OF NONABELIAN METABELIAN GROUPS OF ORDER AT MOST 24

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A dissertation submitted in partial fulfillment of the
requirements for the award of the degree of
Master of Science (Mathematics)

Faculty of Science
Universiti Teknologi Malaysia

JANUARY 2013

To my beloved mother, father and family

ACKNOWLEDGEMENTS

First of all, thanks to Allah, who has given me the strength to complete this dissertation. In preparing this thesis, I was in contact with many people. They have contributed towards my understanding and thoughts. In particular, I wish to express my sincere appreciation to my dissertation supervisor, Dr. Nor Muhainiah Mohd Ali for her encouragement, guidance and invaluable suggestions. She has helped me in many ways besides making sure that I had all necessary information as a reference to complete this dissertation.

Moreover, I would like to express my thanks to my parents, Abd Halim Othman and Zakiah Wan Abdullah for their long lasting love. Lastly, I would like to thank my siblings, my friends and Mohd Efiez Mustafa. They really supported me and played an important role in the completion of my thesis. I would like to thank them for their encouragement, love and emotional support.

ABSTRACT

A group G is metabelian if and only if there exists an abelian normal subgroup A such that the factor group, G/A is abelian. Meanwhile, for any group G, the commutativity degree of a group is the probability that two randomly selected elements of the group commute and denoted as $P(G)$. Furthermore, the n^{th} commutativity degree of a group G is defined as the probability that the n^{th} power of a random element commutes with another random element from the same group, $P_n(G)$. In this research, $P(G)$ and $P_n(G)$ for nonabelian metabelian groups of order up to 24 are computed and presented. The n^{th} commutativity degree of a group are found by using the formula of $P_n(G)$.

ABSTRAK

Suatu kumpulan G adalah metabelan jika dan hanya jika wujud satu subkumpulan normal yang abelian, A dengan syarat kumpulan faktornya, G/A adalah abelian. Sementara itu, bagi suatu kumpulan G, darjah kekalisan tukar tertib bagi kumpulan itu ialah kebarangkalian dua unsur dipilih secara rawak dari kumpulan itu adalah berkalis tukar tertib dan ditandakan sebagai $P(G)$. Tambahan lagi, darjah kekalisan tukar tertib ke-n ditakrifkan sebagai kebarangkalian kuasa ke-n bagi suatu unsur rawak berkalis tukar tertib dengan suatu unsur rawak yang lain dari kumpulan yang sama, $P_n(G)$. Dalam kajian ini, $P(G)$ dan $P_n(G)$ bagi kumpulan metabelan tak abelan dihitung dan diperkenalkan. Darjah kekalisan tukar tertib ke-n dicari dengan menggunakan formula $P_n(G)$.

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LIST OF SYMBOLS

$P_n(G)$	n^{th} commutativity degree
$P(G)$	Commutativity degree
GAP	Groups, Algorithms and Programming
G	A group G
$\|G\|, \|x\|$	Order of the group G, the order of the element x
$\mathbb{Z}$	Set of integers, the finite cyclic group
$\mathbb{Z}_n$	Cyclic group of order n
$H \leq G$	H is a subgroup of G
$G \cong H$	G is isomorphic to H
$G \times H$	Direct product of G and H
G'	The commutator subgroup of G
$\in$	Element of
$\times$	Direct product
$\rtimes$	Semidirect product
Q_n	Quarternion group of order $2n$
D_n	Dihedral group of order $2n$

CHAPTER 1

INTRODUCTION

1.1 Introduction

A group G is abelian if it satisfies the commutative law, namely $xy = yx$ holds for every $x, y \in G$. However, not all groups are abelian. A group that has some of the elements that do not commute is called a nonabelian group, also known as noncommutative group.

Let G be a group and suppose x and y are elements of G. By considering the total number of pair (x, y) for which x and y are commute (such that $xy = yx$) and divide it by the total number of pair (x, y) which is possible, the result will give the abelianness or commutativity degree of a group G.

Furthermore, by extending the notion of $xy = yx$ to $x^n y = yx^n$, the n^{th} commutativity degree is obtained. In details, the n^{th} commutativity degree is equal to the quotient of the total number of pair (x, y) for which x^n and y commute with the total number of pair (x, y) which is possible.

1.2 Research Background

A metabelian group is a group whose commutator subgroup is abelian. Equivalently, a group G is metabelian if and only if there exists an abelian normal subgroup A such that the quotient group G/A is abelian.

In 2010, Abdul Rahman [1] determined all metabelian groups of order at most 24. There are 59 groups of order less than 24 and 15 groups of order 24 including abelian and nonabelian groups. From the results obtained, all groups of order less than 24 are metabelian. Among these 59 metabelian groups only 25 of them are nonabelian. However, among 15 groups of order 24, only two groups that are not metabelian which are symmetric group of order 24, S_4 and special linear group of 2 by 2 matrices over field of three elements, $SL(2,3)$ and the rest 13 groups are all metabelian in which 10 of them are nonabelian and three groups are abelian.

The probability that two elements of the group G (chosen randomly with replacement) commute is also known as the commutativity degree of a group G and denoted as $P(G)$. This probability can be written as

$$P(G) = \frac{\text{Number of ordered pairs } (x,y) \in G \times G \ni xy = yx}{\text{Total number of ordered pairs } (x,y) \in G \times G}$$
$$= \frac{|\{(x,y) \in G \times G \mid xy = yx\}|}{|G|^2}.$$

The commutativity degree of nonabelian metabelian groups of order at most 24 has been determined by Che Mohd [2] in 2011. Her results are used throughout this research in order to find the n^{th} commutativity degree of a group, $P_n(G)$ where $P_n(G)$ is defined as,

$$P_n(G) = \frac{|\{(x, y) \in G \times G \mid x^n y = yx^n\}|}{|G|^2},$$

and G is a nonabelian metabelian group of order at most 24.

1.3 Problem Statement

What are the n^{th} commutativity degrees of all nonabelian metabelian groups of order at most 24?

1.4 Research Objectives

The main objectives of this research are:

1) to study and present the basic concepts of metabelian groups,
2) to present the notion of commutativity degree and the n^{th} commutativity degree of groups,
3) to determine the n^{th} commutativity degree of nonabelian metabelian groups of order at most 24.

1.5 Scope of Research

This research will focus only on nonabelian metabelian groups of order at most 24 and their n^{th} commutativity degree.

1.6 Significance of Study

The results obtained can be beneficial for computing the commutativity degree and the n^{th} commutativity degree for other groups. Besides, the results of the commutativity degree and the n^{th} commutativity degree can be transferred to non-commuting graph where this kind of graph can be used to characterize the group theory properties of a group.

1.7 Thesis Organization

This dissertation is organized into five chapters. The first chapter is the introduction chapter which contains the research background, problem statement, research objectives, scope of research, significance of study, thesis organization and conclusion of the chapter.

The second chapter includes some literature review about metabelian groups, commutativity degree and the n^{th} commutativity degree of a group. Some definitions and theorems that are used throughout the research are also listed.

In the following chapters, the result of n^{th} commutativity degree for nonabelian metabelian groups of order less than 24 and equal to 24 are presented. The results are splitted into two chapters, which are $P_n(G)$ for G is a metabelian group of order less than 24 (25 groups) and G is a metabelian group of order 24 (10 groups).

Finally, the summary and conclusion of this research are included in Chapter 6.

1.8 Conclusion

In this chapter, the introduction of this research are discussed followed by the background of the research. After that, the statement of the problem, objectives of research, scope of research and the significance of study are included. Finally, thesis organizations are stated. In the next chapter, literature reviews related to this research are discussed especially for metabelian groups, the commutativity degree and the n^{th} commutativity degree of the groups.

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