SOLUTION AND INTERPOLATION OF ONE-DIMENSIONAL HEAT EQUATION BY USING CRANK-NICOLSON, CUBIC SPLINE AND CUBIC B-SPLINE

WAN KHADIJAH BINTI WAN SULAIMAN

A thesis submitted in fulfillment of the
requirements for the award of the degree of
Masters of Science (Mathematics)

Faculty of Science
Universiti Teknologi Malaysia

JANUARY 2013

Especially for my beloved parents,

Wan Sulaiman Bin Wan Ismail and Wan Maimunah Binti Wan Ismail.

ACKNOWLEDGEMENTS

I wish to express my sincere appreciation to my main dissertation supervisor, Tuan Hamisan Rahmat, for a lot of encouragement, guidance, advises and motivation in order to complete my project. Tuan Haji Hamisan always managed to read and gives great care for the improvement of the presentation for this project.

I also would like to thank my beloved parents, siblings, nieces, nephews, and Mr. Muhammad Muinuddin Mat Jusoh for being great supporter in my life. I always love all of you, never 'decrease' always 'increase', up to infinity...

To my beloved and the best friend forever, Siti Nor Athirah Sharif and Siti Musliha Mat Rasid, never give up in your research. Hopefully, both of you will graduate on time. To Siti Fatehah Aduka, be the best teacher and I hope that you're always will be loved by your students!=) To Nurul Syaiima Othman, I hope your life is always full with happiness... =)

To course mates a.k.a best friends who always make my days happy and cheerful, especially Raja Nadiah Raja Mohd Nazir, Ikka Afiqah Amir, and Nurul Fariha Mokhter, never 'wipe out' our moments together. To Farhana Osman, thanks for your support and encouragement to finish my dissertation project and my study.

Last but not least, dear twins, [Nor Athirah Izzah + Nor Afifah Hanim]*Zulkifli, Nur Atiqah Dinon, Nor Aziran and all friends who always encourage me and provide me helpful suggestions to complete this project, thank you very much!
Be the best, among the best!
All the best in the future!

ABSTRACT

The purpose of this study is to apply the technique of Cubic Spline, Cubic B-Spline and Crank-Nicolson in one-dimensional heat equations with Dirichlet boundary conditions. Then, their accuracy of numerical methods are compared by computing their absolute error and relative error. Those results of the methods are calculated by using Matlab 2008 and Microsoft Visual Studio 2010 (C++). As the results, Crank-Nicolson is a good approximation solution since the result of relative error is quite close to the zero. Besides that, for interpolation method, cubic B-spline interpolation is found to give better results compare to the cubic spline interpolation since the relative error of cubic B-spline is better than cubic spline. Regarding to the findings, it can be seen clearly that the cubic spline, cubic B-spline and Crank-Nicolson are well approximated and give better results with smaller step size.

ABSTRAK

Kajian ini adalah bertujuan untuk menggunakan kaedah splin kubik, B-Splin kubik dan kaedah Crank-Nicolson dalam persamaan haba satu dimensi dengan syarat sempadan Dirichlet. Kemudian, ketepatan kaedah-kaedah berangka tersebut dibandingkan dengan mengira ralat mutlak dan ralat relatif masing-masing. Kaedah-kaedah berangka ini dikira dengan menggunakan Matlab 2008 dan Microsoft Visual Studio 2010 (C++). Hasilnya, Crank-Nicolson adalah penyelesaian penghampiran yang baik kerana ralat relatif yang diperolehi menghampiri sifar. Selain itu, bagi kaedah interpolasi, interpolasi B-splin kubik memberikan keputusan yang lebih baik berbanding dengan interpolasi splin kubik berdasarkan keputusan ralat relatif B-Splin kubik adalah lebih baik berbanding splin kubik. Berdasarkan penemuan kajian, ia boleh dilihat dengan jelas bahawa splin kubik, B-Splin kubik dan kaedah Crank-Nicolson memberikan nilai anggaran dan keputusan yang lebih baik dengan saiz langkah yang kecil.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

PDE	-	Partial Differential Equation
RMC	-	Research Management Centre
IVP	-	Initial Value Problem
BVP	-	Boundary Value Problem
OSC	-	Orthogonal Spline Collocation
RBF	-	Radial Basis Function
EMD	-	Empirical Mode Decomposition
GCBSF	-	General Cubic B-Spline Function
TEM	-	Transmission Electron Microscopy

CHAPTER 1

INTRODUCTION

1.1 Introduction

Partial differential equations (PDE) are very important in many branches of mathematics, science and engineering such as hydrodynamics, elasticity, quantum mechanic and electromagnetic theory. In mathematics, PDE is an equation that contains an unknown function of several variables, and one or more of its partial derivatives [1]. Besides that, according to J.D. Logan, PDE models are the evolution of a system in both time and space. The system can be observed both in time and in spatial region which may be one, two or three dimensional. PDE models may also be independent of time, but depend on several spatial variable [2]. The PDE can be solved either manually by hand or computer programming software such as Matlab, Mathematica, Maple and Microsoft Visual Studio(C++) Programming.

This chapter includes the introduction of the study which starts with the research background. Then, it will be followed by the statement of the problem which is simply introduced. Next, this study has few objectives that need to be achieved by follow the scope of the study. Then, the significance of the study also presented in this chapter.

1.2 Background of the Study

Heat equation is part of partial differential equations where it supposes to be linear or non-linear form. It is an important PDE which describe the distribution of heat or variation in temperature in a given region over time. The heat equation can be solved either by analytical approximation or numerical approximation methods. Analytically, heat equation can be solved by using Fourier series where the technique is proposed by Joseph Fourier in 1822. Finite difference method is one example of numerical approach that can solve heat equation. Analytical approximation methods often provide extremely useful information but tend to be more difficult to apply compare to the numerical methods [3]. Numerically, spline interpolation has been chosen to interpolate the heat problem.

In thermodynamics, heat is defined as the energy that across the boundary of a system when this energy transport occurs due to a difference temperature between the systems and its surrounding. The second law of thermodynamics states that heat always flow over the boundary of the system in the direction of falling temperature [4]. We are interested to study about heat equation where it will be solved numerically.

A spline is a mechanical device used by draftsmen to draw a smooth curve consisting of a strip or rod of some flexible material to which weights are attached, so that it can be constrained to pass through or near certain plotted points on a graph. The term of spline function is intended to suggest that the graph of such a function is similar to a curve drawn by a mechanical spline [5]. Spline is useful because it is piecewise polynomial function that satisfies certain continuity requirements for both the derivatives and the function that furnish the graphs of polynomials and polygonal paths [6].

Besides that, B-splines or bell-shaped splines is also a piecewise polynomial function but differ from spline interpolation which is first suggested by Schoenberg in 1946 [7]. These are basis function of cubic spline which allows the degree of the resulting curve to be changed without any changed in the data. The B-spline can be of any degree but, in computer graphics and other applications, B-splines of degree two or three are generally found to be sufficient [8].

Crank-Nicolson is a finite difference method for numerical solving heat equation and similar partial differential equation. It is an implicit method which was developed by John Crank and Phyllis Nicolson in 1947 [7]. Implicit method differs from explicit method which for implicit method, the values to be computed are not just a function of values at the previous time step, but also involve the values at the same time step which are not readily available. Then for explicit method, all starting values are directly available from initial and boundary conditions and each new value is obtained from the values that already known [9]. For this study, we only considered implicit method namely Crank-Nicolson method.

This study focused on cubic spline interpolation, cubic B-spline interpolation and Crank-Nicolson method where the methods is applied in heat problem. The accuracy of cubic spline interpolation, cubic B-spline interpolation and Crank-Nicolson are presented.

1.3 Statement of the Problem

In this study, cubic spline and cubic B-spline are approached in order to overcome the weakness of polynomials which are having strongly oscillating properties and consist of a high number of arithmetical operations involved in the calculations of the polynomials. Crank-Nicolson method also considered in the study in order to determine the accuracy of the method. Meanwhile, the accuracy of cubic spline and cubic B-spline interpolation are determined to compare with each other.

1.4 Objectives of the Study

The objectives of this project are:

(i) To study in general cubic spline interpolation, cubic B-spline interpolation and Crank-Nicholson method.

(ii) To derive the algorithm of cubic spline, cubic B-spline interpolation and Crank-Nicolson.

(iii) To apply the technique of Crank-Nicolson in one-dimensional heat equations and calculate the interpolation by using cubic spline and cubic B-spline.

(iv) To determine the accuracy of cubic spline interpolation, cubic B-spline interpolation and Crank-Nicolson method by computing their absolute error and relative error.

1.5 Scope of the Study

In this study, we are applying the technique of Crank-Nicolson in one-dimensional heat equations and calculate the interpolation by using cubic spline and cubic B-spline. Specifically, for finite difference method, the Crank-Nicolson method is considered in this study since it is better than explicit method. Then, the methods of cubic spline and cubic B-spline interpolation based on collocation approach are being used in order to interpolate the one-dimensional heat equation since they are piecewise polynomial which produced smooth curve. Then, the accuracy of each method will be determined and compared.

1.6 Significance of the Study

The accuracy of the Crank-Nicolson methods, Cubic spline and cubic B-spline interpolation will be discovered by conducting this study. The cubic spline and cubic B-spline are compared each other in order to determine which method give well interpolation approximation solution. The methods are very useful since the obtained results give well approximate solution. The methods also can be applied in other field such as physics and engineering field.

1.7 Thesis Organization

This dissertation has been organized into five chapters. The first chapter serves as an introduction to the whole project. This chapter introduces the heat equation and spline interpolation which is the main focused on the study. Chapter 1 also includes all the important points such as statement of problem, objectives of the study, scope of study and significance of the study.

Chapter 2 presents the literature review of this research. Various works by different researchers regarding heat equation, the spline interpolation including the cubic spline interpolation and cubic B-spline interpolation and Crank-Nicolson method. It is important to know how the heat equation, cubic spline, cubic B-spline interpolation and Crank-Nicolson are developed. This chapter also describes the applications and some contributions given by the researchers towards the development of the methods that are conducted in this study.

Chapter 3 discusses details about the derivation of cubic spline interpolation, cubic B-spline interpolation and Crank-Nicolson method including the heat equation. Besides that, in this chapter also includes the algorithms of the methods that have been applied in this study.

Chapter 4 performs an example of one-dimensional heat problem where it is solved by analytical method and numerical method. Besides that, this chapter also presents about the results which has been determined by the Crank-Nicolson method, cubic spline interpolation and cubic B-spline interpolation that are applied in the one dimensional heat problem. Furthermore, the graph of comparison between the numerical solution that have been applied in this study also showed. The absolute error and relative error for those methods are tabulated in tables and the accuracy of the methods are shown. Finally, the summary of the results obtained are included.

Finally, Chapter 5 summarizes the obtaining results. Then, suggestions and recommendations for the future research in this study are also included.

REFERENCES

1. Constanda, C. *Solution Techniques for Elementary Partial Differential Equations*. United States of America: Chapman and Hall/CRC. 2002.

2. Logan, J. D. *Applied Partial Differential Equations*. New York: Springer-Verlag. 2004.

3. Smith, G. D. *Numerical Solution of Partial Differential Equations*. United States: Oxford University Press 1985.

4. Baehr, H. D. and Stephen, K. *Heat and Mass Transfer*. Germany: Springer. 2006.

5. Greville, T. N. E. *Theory and Applications of Spline Functions*. New York: Academic Press. 1975.

6. Holland, A. S. B. and Sahney, B.N. *The General Problem of Approximation and Spline Functions*. New York: Robert E. Krieger Publishing Company Co., Inc. 1979.

7. Burden, R. L. and Faires, J. D. *Numerical Analysis: Eight Edition*. United States of America: Bob Pirtle. 2005.

8. Sastry, S.S. *Introductory Methods of Numerical Analysis*. New Delhi: Eastern Economy Edition. 2005.

9. Balagurusamy, E. *Numerical Methods*. India: Tata McGrawHill Publishing Company Limited. 1999.

10. Kadalbajoo, M. K., Tripathi, L. P. and Kumar, A. A Cubic B-Spline Collocation Method for a Numerical Solution of the Generalized Black-Scholes Equation. *Mathematical and Computer Modelling*. 2012. 55: 1483–1505.

11. Ekolin, G. Finite Difference Methods for a Nonlocal Boundary Value Problem for Heat Equation. *BIT*. 1991. 31: 245–261.

12. Dehghan, M. A Finite Difference Method for a Non-local Boundary Value Problem for Two-Dimensional Heat Equation. *Applied Mathematics and Computation*. 2000. 112: 133–142.

13. Jumarhon, B. A Numerical Method for the Heat Equation with Non-Smooth Corner Conditions. *Communications in Numerical Methods in Engineering*. 2001. 17: 727–736.

14. F. Ternat, O. Orellana and P. Daripa. Two Stable Methods with Numerical Experiments for Solving the Backward Heat Equation. *Biomedical Engineering Online*. 2011. 61: 266–284.

15. Hairer, M. Singular Perturbation to Semilinear Stochastic Heat Equations. *Probability Theory Related Field*. 2012. 152: 265–297.

16. Bourchtein, A. and Bourchtein, L. On Iterated Crank-Nicolson Methods for Hyperbolic and Parabolic Equations. *Computer Physics Communications*. 2010. 181: 1242–1250.

17. Jeong, D. and Kim, J. A Crank-Nicolson scheme for the Landau-Lifshitz Equation without Damping. *Journal of Computational and Applied Mathematics*. 2010. 234: 613–623.

18. Tavakoli, F., Mitra, S. K. and Olfert, J. S. Aerosol Penetration in Microchannels. *Journal of Aerosol Science*. 2011. 42: 321–328.

19. Sastry, S.S. *Introductory Methods of Numerical Analysis*. India: Prentice Hall. 2003.

20. Bickley, W. G. Piecewise Cubic Interpolation and Two-Point Boundary Value Problems. *Comp. J*. 1968. 11: 206–208.

21. Jain, P. C. and Lohar, B. L. Cubic Spline Technique for Coupled Non-Linear Parabolic Equations. *Computer and Mathematics with Applications*. 1979. 5: 179–185.

22. Al-Said, E. A. The Use of Cubic Spline in the Numerical Solution of a System of Second-Order Boundary Value Problems. *Computers and Mathematics with Applications*. 2001. 42: 861–869.

23. Al-Said, E. A. and Noor, M. A. Cubic Spline Method for a System of Third-Order Boundary Value Problems. *Applied Mathematics and Computation*. 2003. 142: 195–204.

24. Al-Said,E. A., Noor, M. A. and Rassias, T. M. Cubic Spline Method for Solving Fourth-Order Obstacle Problems. *Applied Mathematics and Computation*. 2006. 174: 180–187.

25. Bialecki, B. and Fairweather, G. Orthogonal Spline Collocation Methods for Partial Differential Equations. *Journal of Computational and Applied Mathematics*. 2001. 128: 55–82.

26. Li, B., Fairweather, G. and Bialecki, B. Discrete-Time Orthogonal Spline Collocation Methods for Vibration Problems. *SIAM Journal Numerical Analysis*. 2002. 39(6): 2045–2065.

27. Bialecki, B. A Fast Solver for the Orthogonal Spline Collocation Solution of the Biharmonic Dirichlet Problems on Rectangles. *Journal of Computational Physics*. 2003. 191: 601–621.

28. Danumjaya, P. and Pani, A. K. Orthogonal Spline Collocation Methods for the Extended Fisher-Kolmogorov Equation. *Journal of Computational and Applied Mathematics*. 2005. 174: 101–117.

29. Bialecki, B. and Fernandes, R. I. An Alternating-Direction Impicit Orthogonal Spline Collocation Methods for Non-Linear Parabolic Problems on Rectangular Polygons. *SIAM Journal Science Computer*. 2006. 28(3): 1054–1077.

30. Zhu, C. G. and Kang, W. S. Applying Cubic B-Spline Quasi-Interpolation to Solve Hyperbolic Conservation Laws. *U.P.B, Sci. Bull, Series D*. 2010. 72(4): 49–58.

31. Jiang, Z. and Wang, R. An Improved Numerical Solution of Burger's Equation by Cubic B-Spline Quasi Interpolation. *Journal of Information and Computational Science 7*. 2010. 5: 1013–1021.

32. Goh, J., Majid, A. A. and Ismail, A. I. M. Numerical Method Using Cubic B-Spline for the Heat and Wave Equation. *Computers and Mathematics with Applications*. 2011. 62: 4492–4498.

33. Liu, X., Huang, H. and Xu, W. Approximate B-Spline Surface Based on RBF Neural Networks. *Springer-Verlag*. 2005. LNCS 3514: 995–1002.

34. Chen, Q. *et al*. A B-Spline Approach for Empirical Mode Decompositions. *Advanced in Computational Mathematics*. 2006. 24: 171–195.

35. Minh, T. N., Shiono, K. and Masumoto, S. Application of General Cubic B-Spline Function (GCBSF) for Geological Surface Simulation. *International Symposium on Geoinformatics for Spatial Infrastucture Development in Earth and Allid Sciences*. 2006.

36. Dauget, J. *et al*. Alignment of Large Image Series using Cubic B-Spline Tessellation: Application to Transmission Electron Microscopy Data *Springer-Verlag*. 2007. LNCS 4792: 710–717.

37. Prenter, P. M. *Splines and Variational Methods*. Canada: John Wiley and Sons Inc. 1975.

38. Mitchell, A. R. *Computational in Partial Differential Equations*. England: John Wiley and Sons Ltd. 1977.