NYSTRÖM METHOD FOR SOLVING NON-UNIQUELY SOLVABLE INTERIOR RIEMANN-HILBERT PROBLEM ON REGION WITH CORNERS VIA INTEGRAL EQUATION

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To Prophet Muhammad (pbuh),

A person whom I love more than myself,

And my loving mother Hj Habsa Mustafa and two children

(Shko & Shyar) and my Wife and in memory my late father

(Hassan Jaff)

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ABSTRACT

This work involve a boundary integral equation method to find the non-uniquely solvable numerical solution of the Interior Riemann-Hilbert problem on a region with corners. The integral equation was derived based on the Fredholm integral equation of the second kind with continuous kernel and the solvability of the integral equation and its equivalence to the problem is reviewed the derived integral equation in this research for the non-uniquely solvable interior Riemann-Hilbert problem on a region with corners will be computed in achieving this aim, this study developed two numerical formulas where the Nystrom method with the Gaussian quadrature rule are implemented. So that, the singularities are eliminated during numerical integration. Numerical examples on four test regions with 0ff-corners are presented to demonstrate the effectiveness of this formulation.

ABSTRAK

Kerja ini membangunkan satu kaedah sempadan persamaan kamiran untuk penyelesaian berangka masalah Interior Riemann-Hilbert di rantau dengan off-sudut berasal dalam disertasi ini berdasarkan persamaan Fredholm penting jenis kedua dengan kernel berterusan. Kelarutan persamaan kamiran dikaji dan bukti kesetaraan kepada masalah ini disediakan. Persamaan kamiran yang diperolehi kemudiannya digunakan untuk menyelesaikan berangka bukan unik larut dalam Riemann-Hilbert masalah. Dalam mencapai matlamat ini, tepat skim berangka maju di mana kaedah Nystrom dengan peraturan kuadratur Gaussian dilaksanakan. Oleh itu, singularities dihapuskan semasa kamiran berangka. Di samping itu, satu formula digunakan untuk penyelesaian menyisipkan pada setiap titik sudut luar yang menggunakan nilai-nilai yang diperolehi di luar penjuru mata, iaitu formula interpolasi Nystrom. Contoh berangka kepada empat kawasan ujian dengan 0ff-sudut dibentangkan untuk menunjukkan keberkesanan penggubalan ini.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURS

LIST OF APPENDICES

CHAPTER 1

INTRODUCTION

1.1 Introduction

The theory of boundary value problems for analytic functions is one of the major important branches of complex analysis. It has wide application because a lot of practical problems in physics, mechanics, and engineering can be modeled as such, problems or integral equations which are closely linked to the boundary value problems for analytic functions.

Generally, a boundary value problem of applied mathematics is a problem of finding a function satisfying certain equations in a specified region and satisfying more than a few prescribed conditions on the boundary of the region, and the well-known classical boundary value problem is Dirichlet problems and Neumann problems. Boundary value problems of this type are of great practical importance and they are called boundary value problems of the first kind, and of the second kind, respectively (Henrici, 1977; Jeffre, 2006).

A class of fundamental boundary value problems for analytic functions is Riemann problem, later known as the Riemann–Hilbert problem, briefly RH problem. The first formulation of the RH problem appeared in Bernhard Riemann's Doctor of

Philosophy (*Ph.D.) Thesis in 1851* (*Wegert, 1992*) and it turns out to be one of the basic subjects in the theory of analytic functions. It is a living subject with a fascinating history and interesting applications. It has a rich theory, interrelation to functional analysis, quite a lot of fields of complex analysis of one and quite a lot of variables, Solutions were derived using complex functions for particular problems such as potential, plane electrostatic, Platter problem. Other applications of the RH problem are in fluid dynamics and gas dynamics (Nasser, 2007; Munakhov1986). An important example where the need for RH problem arises in a mixed boundary value problem. For example plane potential problems consisting of Dirichlet and Neumann conditions given on the boundary can be reduced to a RH problem (Hass, 1991).

The RH problem on a region with corners i.e., An arbitrary simply connected region bounded by a curve having a continuously turning tangent except possibly at a finite number of corners whose interior angle are well defined, where there may occur a jump discontinuously of the first derivative in the complex plane. Can be solved by the integral equation method, and the integral equation that has been derived are classified as a linear Fredholm integral equation of the second kind with continuous kernels on region with corners is the union of a finite number of non-intersecting smooth arcs with contiguous ends forming corner points. The boundary value problem in a closed region is solved through the boundary integral equation. The unknown function $\mu(t)$ occurs both inside and outside of the integral and is classed as a Fredholm integral equation of the second kind, which has the form

$$\mu(x) - \lambda \int_a^b K(x,t)\mu(t)dt = f(x) \tag{1.1}$$

Where a and b are constants, λ is a constant which, when the value is known is sometime engrossed into $K(x, t)$, a function of two variables called the Kernel or nucleus of the integral equation, while $f(x)$ is known as the forcing function or the driving term of the equation.

1.2 Background of the problem

The earlier methods for solving the RH problem are mostly limited to the RH problem on circular regions. And that the RH problem is invariant under the conformal transplantation. The RH problems on simply connected regions other than the circular regions were solved by reducing it, by means of conformal mappings, to a RH problem in circular regions.

The two most frequently encountered methods for RH problem are the (Gakhov, 1966) and (Muskhelishvili, 1977) methods which are not based on the integral equation. Muskhelishvili's method provided a direct reduction of the RH problem to the Hilbert problem and the solution of the Hilbert problem in the circular regions are known; hence the solution of the RH problem can then be constructed from the solution of the Hilbert problem, Gakhov's method gives the solution of the RH problem in closed form in terms of the real regularizing factor and the Schwarz operator and a practical determination of regularizing factors seemed to be desired (Gakhov, 1966).

The first two methods based on integral equations for the RH problem are Hilbert and Sherman methods, Sherman's method is related to the RH problem on simply connected region, while Hilbert's method is limited to the circular region, and it has serious difficulties in terms of its solvability, these difficulties and limitations make them, not the preferred methods for solving the RH problem. The first full method based on boundary integral equation, avoiding these difficulties in the Hilbert and Sherman methods for solving RH problem on a smooth arbitrary simply connected region for general indices was developed by Murid *et al.* (2003),(Murid, 2006) and (Nasser, 2006), and they implemented the Nyström method for the numerical solution of the problem.

Recently, a method for solving the interior and the exterior RH on region with corners briefly, RHC; were developed by (Ismail, 2007) and (Zamzamir, 2011), respectively derived integral equation for both unique and non-uniquely solvable

interior and exterior RHC and their solvability and equivalence to was established theoretically, Hence the solution of the integral equation is the solution of the RHC.

They constructed numerical formula by Picard iteration method and Nyström method for these problems.The exciting singularities of their integral equation must be eliminated during numerical implementation and this successfully done for both unique and non unique interior and exterior RHC by using the Picard Iteration in Ismail (Ismail, 2007) and Zamzamiar (Zamzamiar, 2011) respectively .Nyström method was also implemented by Zamzamiar's Doctor of Philosophy (*Ph.D*)(Zamzamiar, 2011) for both uniquely and non-uniquely solvable RHC.

While Ismail's Doctor of Philosophy (*Ph.D*)(Ismail, 2007) employed the Nyström method only for the uniquely solvable integral equation of the interior RHC. Hence, in this research we will implement the Nyström Method for the non-uniquely solvable integral equation for the interior RHC.

1.2   Statement of the Problem

This research will construct two numerical formulas by Nyström method for the non-uniquely interior RHC. The first formulas will be constructed following the non-uniquely solvable integral equation for the exterior RHC. (Zamzamiar, 2011) and the second will be constructed the same was as for the uniquely solvable integral equation for the interior RHC. (Ismail, 2007). Some numerical experiment will be performed to investigate the accuracy of the result.

1.3   Objectives of the study

The objectives of this research are as follows:

- To construct examples of interior non-uniquely solvable RHC.
- To construct two numerical formulas based on Nyström method for the non-unique   interior RHC.
- To perform numerical experiment for the solution of non-uniquely solvable RHC.

1.5   Scope of the Research

The recent available method for solving RHC are Nyström method   and numerical formulas by Picard Iteration Method has been done in uniquely and non-uniquely solvable integral equation for both interior and exterior RHC by (Ismail, 2007) and (Zamzamiar, 2011). While the formula for non-uniquely solvable integral equation for interior RHC has not been done. Hence, this research aims to construct this formula based Nyström method as well as perform some numerical example,   and will not consider the Picard iteration method.

1.6   Significance of the Research

Zamzamiar completed the study for the uniquely and nonuniquely solvable integral equation for exterior RHC wheis both Picard itration method and Nyström method, were employed (Zamzamiar, 2011). Previously Ismail (Ismail,2007) employed the Picard iteration method for both uniquely and non-uniquely solvable interior RHC and only employed the Nyström Method for the uniquely solvable interior RHC (Ismail, 2007).

Hence, based on (Ismail, 2007)and (Zamzamiar,2011) The significance of this research is to employed Nyström method for non-uniquely solvable interior RHC and some numerical formulas.   So this work will be given a complete study for solving exterior and interior RHC via integral equations by   the Nyström method.

REFERENCES

Ahlfors, L. V. (1979). *Complex Analysis.International Student Edition*. Singapore: McGraw-Hill.

Asmar, N. H. (2005). *Partial Differential Equations with Fourier Series and Boundary Value Problems*. Upper Saddle River, NJ: Prentice Hall.

Begehr, H. G. W. (1994). *Complex Analytic Methods for Partial Differential Equations*. World Scientific.

Gakhov, F. D. (1966). *Boundary Value Problems. English Translation of Russian Edition 1963*. Oxford: Pergamon Press.

Graham, I. G. and Chandler, G. A. (1988).*High-Order Methods for Linear Functionals of Solutions of Second Kind Integral Equations*. *SIAM J. Numer Anal.*25: 1118–1173.

Henrici, P. (1977) *Applied and Computational Complex Analysis*, Vol. 3. New York: John Wiley.

Haas, R. and Brauchli, H. (1991). *Fast Solver for Plane Potential Problems with Mixed Boundary Conditions.Comput.Methods Appl. Mech. and Engrg.*89: 543–556.Hille, E. (1959). *Analytic Function Theory.Vol. 1*.Blaisdell, New York.

Ismail, M. (2007).*Numerical Solution of the Interior Riemann-Hilbert Problem on Region with Corners via Boundary Integral Equation*.Doctor of Philosophy Thesis.UniversitiTeknologi Malaysia, Skudai. Ismail.

Ismail, M., Sanugi, B. and Murid, A. H. M. (2006b).*Numerical Solution of Riemann Problem via Boundary Integral Equation with Corners.International Journal of Pure and Applied Mathematics*. 31(3): 379–400.

Ismail, M., Murid, A. H. M., &Sanugi, B. B. (2006a).An integral equation approach for the numerical solution of the Riemann problem on a simply connected region with corners. *International Journal of Simulation and Process Modelling*, *2*(1), 25-32.

Jeffrey, A. (2006). *Complex Analysis and Applications, 2nd Ed.* Chapman and Hall/CRC: Boca Raton.

Korach, L. D. (1984). *Boundary-Value Problems*.Addison-Wesley.

Kress, R. (1989). *Linear Integral Equations*. New York Berlin Heidelberg: Springer.

Kythe, P. K. and Puri, P. (2002).*Computational Method for Linear Integral Equations*. Birkhauser Boston.

Monakhov, V. N. (1986). *Boundary Value Problem with Free Boundaries for Elliptic Systems of Equations.English Translation of Russian Edition 1977*. American Mathematical Society.

Murid, A. H. M. and Razali, M. R. M. (1995).*On Boundary Integral Equations of the Second Kind for Symmetric Regions*. Technical Report, Dept.of Mathematics,UTM. 30.

Murid, A. H. M. and Nasser, M. M. S. (2003).*Eigenproblem of the Generalized Neumann Kernel.Bull. Malaysian Math. Sc. Soc.* 26(2): 13–33. Murid.

Muskhelishvili, N. I. (1977). *Singular Integral Equations. English Translation of Russian Edition 1953*. Leyden: Noordhoff.

Nasser, M. M. S. and Murid, A. H. M. (2006).*Numerical Solution of the Riemann-Hilbert Problems Using Boundary Integral EquationMethod*.UniversitiTeknologi Malaysia, Skudai. *Unpublished*.

Nasser, M. M. S. (2007). *Boundary Integral Equations with the Generalized Neumann Kernel for the Neumann Problem.MATEMATIKA*.23, 83-98.

Sherman, D. I. (1945). *On Some Problems of the Theory of Potential* (In Russian). *Prikl.Matem.I Mekh.*9: 479–488.

Swarztrauber, P. N. (1972). *On the Numerical Solution of the Dirichlet Problem for a Region of General Shape.SIAM Journal of Numerical Analysis.*9(2):300–306.

Wegert, E. (1992). *Nonlinear Boundary Value Problems for Holomorphic Functions and Singular Integral Equations*. Berlin: AkademieVerlag.

Wegmann, R., Murid, A. H. M. and Nasser, M. M. S. (2005).*The Riemann-Hilbert Problem and the Generalized Neumann Kernel.Journal of Computational and Applied Mathematics*. 182, 388-415.

Zamzamiar.Z . (2011). *Numerical Solution of the Exterior Riemann-Hilbert Problem on Region with Corners via Boundary Integral Equation*. Doctor of Philosophy Thesis. Universiti Teknologi Malaysia, Skudai. Zamzamiar.