NUMERICAL MODELLING OF FULLY QUANTUM, SPONTANEOUS PARAMETRIC DOWN CONVERSION PROCESS AND COINCIDENCE DETECTION

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To my mother, father
and my beloved wife

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ABSTRAK

Fokus kajian adalah untuk memodelkan proses penukaran penurunan berparameter secara spontan (SPDC) di dalam kristal tak linear yang menghasilkan foton-foton yang saling berkait sifatnya antara satu sama lain mengunakan kaedah berangka dengan pendekatan fizik kuantum sepenuhnya. Berdasarkan kajian sebelum ini, model yang dihasilkan mengfokuskan nombor foton pam laser sehingga n = 4 sahaja manakala dalam projek ini, model yang dihasilkan adalah sehingga nombor pam laser n = 8. Kesemua persamaan pembeza bagi amplitud keadaan hasilan, perubahan keadaan hasilan selepas pembelah alur cahaya, pengutub cahaya dan pengesanan foton diselesaikan secara simbolik dan berangka menggunakan pakej sistem algebra berkomputer sumber terbuka yang dikenali sebagai Maxima. Pakej ini juga digunakan untuk mengira nilai kebarangkalian yang diperolehi daripada analisis selain dapat menghasilkan hasil analisis dalam bentuk grafik. Keputusan menunjukkan bahawa bagi kes parameter perolehan ϑ dan juga amplitud pam α_3 yang kecil, keberangkalian untuk berlakunya proses ini adalah didominasi oleh keadaan hasilan yang tidak berlaku sebarang penurunan berbanding dengan keadaan yang berlaku proses penurunan sepenuhnya. Walau bagaimanapun, bila ϑ ditingkatkan, proses SPDC pada amplitud pam α_3 yang rendah, keseluruhan proses dikuasai oleh penghasilan yang mengandungi foton isyarat dan pemelahu. Keputusan ini selari dengan apa yang ditemui oleh kajian sebelum ini. Taburan kebarangkalian paling maksimum bagi tanpa proses penurunan, sebahagian proses penurunan dan proses di mana berlakunya penurunan sepenuhnya sentiasa bertepatan dengan nilai α_3 . Kebarangkalian pengesanan foton secara bersama antara foton isyarat dan pemelahu dapat dihasilkan dengan menyalurkan foton berbilang hasil daripada proses SPDC ke interferometer. Ianya menunjukkan bahawa corak dan bentuk taburan kebarangkalian bagi pengesanan dua-foton dan empat-foton adalah sama sebagaimana yang diterbitkan oleh kajian sebelum ini.

ABSTRACT

This work focuses on numerical modeling of the spontaneous parametric down conversion (SPDC) process in nonlinear crystal which produces correlated photons using fully quantum approach. From previous work, the modelling only focus the pump photon number until n = 4 while in this project the modelling were extended to pump photon numbers n = 8. All the coupled linear differential equations for the product state amplitudes as well as the transformations of the product states through the beam splitter, polarizers and photon detectors were solved numerically and symbolically using an open-source Computer Algebraic System (CAS) software called Maxima. In addition the package was also used to calculate various probabilities generated during the analyses and to produce graphical outputs of the results. The results show that for small gain parameter ϑ and pump amplitude α_3, the probability of generating multi-particle product states with zero down conversion consistently dominate the SPDC process compared to that of the product states with full down conversion in pump photons. When the gain parameter ϑ is increased however, the SPDC process at low pump amplitude α_3 begins to favour the product states containing certain number of signal and idler photons which is in agreement with the findings from the previous works. Furthermore the peak of probability distributions for all the zero, partial and fully down conversion product states always coincide with the mean photon number of the pump photons. The applications of the multi-particle states to a standard interferometer lead to the expressions for the probability of coincidence detection. Again the results show that both the trends and shapes of the two-photon and four-photon coincidence probability distributions are also in agreement with the published by previous research.

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LIST OF SYMBOLS

$\boldsymbol{E}$	-	Electric field
$\boldsymbol{B}$	-	Magnetic field
$\boldsymbol{J}$	-	Current densities
σ	-	Charge densities
ϵ_0	-	Electric permittivity of free space($8.854\text{x}10^{-12}\,\text{Fm}^{-1}$)
ϵ_r	-	Relative permittivity of the medium
$\boldsymbol{D}$	-	Electric displacement
$\boldsymbol{P}$	-	Electric polarization
χ	-	Electric susceptibility
χ_M	-	Magnetic susceptibility
μ_0	-	Magnetic permeability of the vacuum $(4\pi\, x 10^{-7}\, Hm^{-1})$
μ_r	-	Relative magnetic permeability of a medium
$\boldsymbol{M}$	-	Magnetization of a medium
$\boldsymbol{r}$	-	Relative position
t	-	Time
c	-	Speed of light
v	-	Speed of electromagnetic waves
ω	-	Angular frequency
ϕ	-	Optical phase
k	-	Wave vector
Z	-	Wave impedance
$\boldsymbol{I}$	-	Poynting vector
f	-	Frequency
λ	-	Wavelength
A_c	-	Coherence area
n_o	-	Ordinary refractive index
n_e	-	Extraordinary refractive index
m	-	Mass
$\boldsymbol{p}_x$	-	Linear momentum

ψ	-	Wave functions
$\mathbf{E}$	-	Energy
$\mathbf{H}$	-	Hilbert space
$\hbar$	-	Planck constant
H	-	Hamiltonian
$\hat{D}(\alpha)$	-	Unitary displacement operator
$\hat{a}$	-	Annihilation operator
$\hat{a}^{\dagger}$	-	Creator operator

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CHAPTER 1

INTRODUCTION

This chapter focuses on the ideas of single photon, single-photon sources, entangled photons and common nonlinear optical phenomena which include the spontaneous parametric down-conversion (SPDC). In addition, this chapter also covers the background of the problem, the statements of the problem, the objectives, the hypothesis, the significance, and the scope of this research.

1.1 Introduction

The idea of photon was first introduced by Planck (1901) when he worked on black-body radiation experiment and suggested that the energy in electromagnetic waves could only be released energy in a packet. Later Einstein (1965) suggested that the electromagnetic waves could only exist in discrete wave-packet which he called quanta. Lewis (1926), a physical chemist, published a speculative theory that photons were “uncreateable and indestructible”. Although his theory was contradicted by many experiments, the name photon was adopted and being used by many physicists to explain the discrete energy of light. When the first idea of photon arise, the generation of single photons was not being considered.

Many experimental physicists in early stage of understanding photon used attenuated laser beam to ensure that the probability of having more than one photon became negligible. Although this method is acceptable for some experiments, it is still questionable because the attenuated laser beam is not a true single-photon source. When using attenuated laser beam, the vacuum probability is much higher that the probability to detect a photon, so the detection of no photon regime is always higher than the single photon itself. The probability to detect two photons also is never zero. So the attenuated laser beam cannot be assumed as a single photon source.

Nowadays, the advance in quantum information science has increased the demand for the optical sources which produce ultra bright single photons. In particular secured quantum cryptography and linear optical quantum computing depend on the availability of such single-photon sources. The combination of strict requirements for single photons plus new technologies are driving an exciting research effort into single-photon generation.

Quantum dots in pillar micro cavities, falling neutral atoms and trapped ions in cavities, defects in diamond nanocrystals, single molecule in a solid and parametric down conversion are among the methods used to produce single photons and photon pairs (Grangier *et al.*, 2004). The most commonly used method is called the parametric down conversion process which was first introduced by Klyshko (1988). This method was first called photon fluorescence and it produced photon pairs.

The photons in the photon pairs are said to be entangled or correlated to each other and they carried nonlocal information which is beyond the domain of classical physics and always considered as a paradox in physics like what Einstein *et al.* (1935) claimed in their paper. But then, this paradox has now become clear and acceptable after Bell (1964) discovered the nonlocal properties and introduced the Bell's inequalitiy, which can be verified by experimental work. Recently there are growing interests on the applications of entangled or correlated photons, for example, the usage of the correlated photons in quantum cryptography (Ekert, 1991), quantum teleportation (Bennett *et al.*, 1993), and quantum computation (Ekert and Jozsa, 1995).

The photon-pair generation is a second order nonlinear process in which a pump photon disappears leading to the creation of two photons with lower energy. This generation is driven by an optical pump field oscillating at the frequency ω_p and occurs spontaneously to produce twin photons namely the signal and idler photons with frequencies ω_s and ω_i respectively which are lower than ω_p. This process can be conceptually illustrated by the diagram in Figure 1.1.

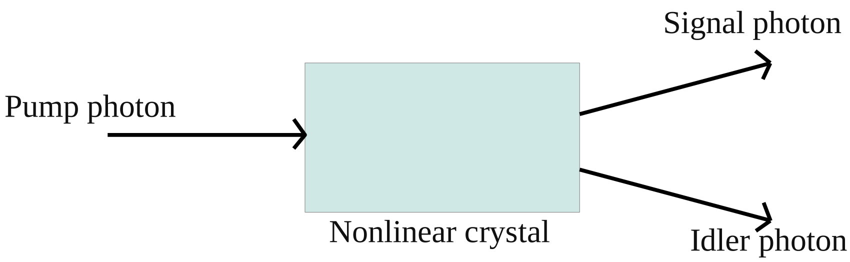

Figure 1.1 : Twin photon generation process

The signal and idler photons are said to be entangled to each others in frequency domain (De Martini and Sciarrino, 2005). The non-classical correlation between the intensities of the generated two- photon states has been observed for the first time by Burnham and Weinberg (1970), and has become widely used in the experiments of quantum optics (QO). Thanks to the seminal work of Leonard Mandel and his collaborators (Hong and Mandel, 1986, Hong et al., 1987, Ghosh and Mandel, 1987 and Mandel, 1999), optical parametric oscillators (OPO) based on processes of down-conversion in a cavity have proved to be efficient sources of frequency tunable light with a range of unique properties.

Figure 1.2 shows the other three types of non-linear interactions that being described in recent paper (De Martini and Sciarrino, 2005). They are the quantum-injected optical parametric amplification, the stimulated emission by a biphoton state and the frequency up-conversion.

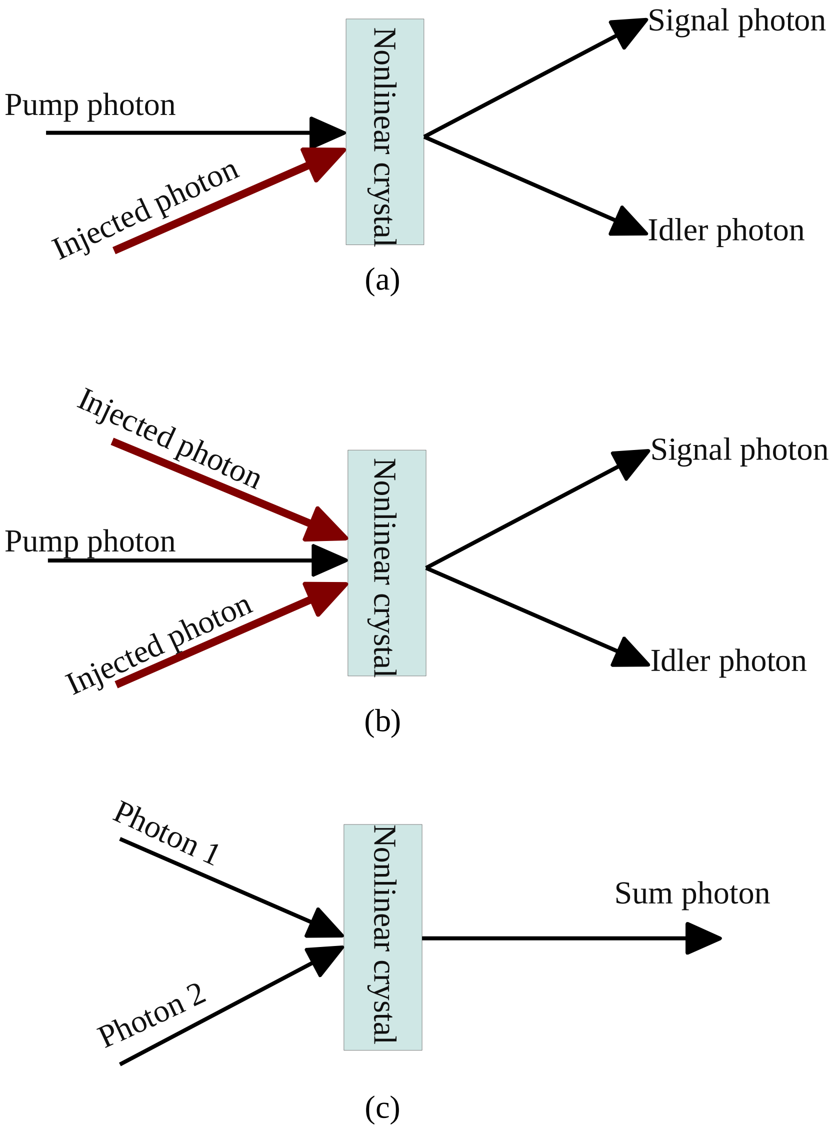

Figure 1.2 : Schematic Diagram of Nonlinear Optical Processes;
(a) quantum-injected optical parametric amplification,
(b) stimulated emission by a biphoton state,
(c) frequency up-conversion

1.2 Background of the Problem

Early SPDC theory was proposed by D. N. Klyshko in 1966 and the research on his theory still exists with new, unexplored possibilities of using SPDC for the discovery of new phenomena at that time (Mandel and Wolf, 1995). Most research works were experimental and several others were done by numerical methods.

Fully quantum theoretical treatment of SPDC process is based mainly on the work of Podoshvedov *et al.* (2005). They derived the expressions to calculate the probability distributions of photons from SPDC process. However, in their work, the expressions for the photon correlation probabilities were derived only for cases with maximum of four photons in the pump.

For SPDC experimentalists who conduct experiments on parametric down conversion in small scale, there will be a need for a pump photon with a low photon number to be down converted into two or more entangled or correlated photons. Thus a SPDC theory fully based on quantum theory that works with low photon number is needed even though many SPDC experiments have been conducted by using laser beam which normally is treated as classical optics rather than quantum optics.

In this project, the expressions for the probability distributions of correlated photons in full quantum SPDC process for cases with more than four photons are derived. The problems are also solved numerically by using open source software, Maxima, a software freely available for others to use. It is hoped that the numerical studies on SPDC process will become much easier and reliable to conduct in the future by incorporating the quantum properties of light before as well as after propagating through the nonlinear crystal.

1.3 Statement of Problem

Even though most SPDC experimental works have used attenuated laser beams as the photon sources, more precise works are known to require single-photon sources. The classical treatment of the SPDC theory is commonly considered to be the approximation for cases when pump photon number involved is very large. The SPDC process in nonlinear crystal therefore should be analyzed fully using quantum theory.

1.4 Research Objectives

In this project there are two main objectives to be completed. The first objective is to carry out numerical modeling of the SPDC process in fully quantum way for cases where the pump field contains more than four photons. The second objective is to derive, plot and analyze the multi-particle coincidence probability distributions based on the correlated photons generated from the SPDC process.

1.5 Statement of Hypothesis

It is hypothesized that, fully quantum theoretical treatment on SPDC process can be simulated using Maxima and the results for cases with up to four photons in the pump field should be consistent with those published by Podoshvedov *et al.* (2005) and Maxima can further simulate the fully quantum SPDC process for cases where pump field contains higher photon number.

1.6 Significance of Research

This work will give us an insight about the theoretical model of the quantum field interaction of light in nonlinear optical crystal. In addition it can be an alternative means of investigating the SPDC process. It also helps us to advance our knowledge in quantum optics as well as in the computational method used to study the SPDC process.

1.7 Scope of Study

The scope of the study is to investigate the fully quantum theory of the SPDC process in nonlinear crystal and then to calculate the probability distributions of the correlated states in the process and the detection probabilities in coincidence detection using open source computer algebraic system (CAS) software, Maxima. The results will be compared with to those published by Podoshvedov *et al.* (2005).

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