NONLINEAR EVOLUTION EQUATIONS IN HIROTA'S AND SATO'S THEORIES VIA YOUNG AND MAYA DIAGRAMS

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Dedicated to my lovely husband, my beloved parents, sisters and little brothers.

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ABSTRACT

This work relates Hirota direct method to Sato theory. The bilinear direct method was introduced by Hirota to obtain exact solutions for nonlinear evolution equations. This method is applied to the Kadomtsev-Petviashvili (KP), Korteweg-deVries (KdV), Sawada-Kotera (S-K) and sine-Gordon (s-G) equations and solved to generate multi-soliton solutions. The Hirota's scheme is shown to link to the Sato theory and later produced the Sato equation. It is also shown that the τ-function, which underlies the form of the soliton solutions, acts as the key function to express the solutions of the Sato equation. By using the results of group representation theory, particularly via Young and Maya diagrams, it is shown that the τ-function is naturally being governed by the class of physically significant nonlinear partial differential equations in the bilinear forms of Hirota scheme and are closely related to the Plucker relations. This framework is shown for Kadomtsev-Petviashvili (KP), Korteweg-deVries (KdV), Sawada-Kotera (S-K) and sine-Gordon (s-G) equations.

ABSTRAK

Kerja ini mengaitkan kaedah langsung Hirota dengan teori Sato. Kaedah langsung bilinear diperkenalkan oleh Hirota untuk memperoleh penyelesaian tepat bagi persamaan evolusi tidak linear. Kaedah ini diaplikasi kepada persamaan Kadomtsev-Petviashvili (KP), Korteweg-deVries (KdV), Sawada-Kotera (S-K) dan sinus-Gordon (s-G) dan diselesaikan untuk menghasilkan pernyelesaian multi-soliton. Kaedah Hirota dibuktikan mempunyai perkaitan dengan teori Sato dan kemudiannya menghasilkan persamaan Sato. Kaedah Hirota ini juga menunjukkan bahawa fungsi τ, yang mendasari penyelesaian soliton, berfungsi sebagai fungsi utama untuk mengungkapkan penyelesaian persamaan Sato. Dengan menggunakan keputusan teori perwakilan kumpulan, teruatamanya melalui rajah Young dan Maya, dibuktikan bahawa fungsi τ adalah dengan bersahaja diterajui oleh persamaan separa gelombang tidak linear berkepentingan fizikal dalam bentuk bilinear kaedah Hirota dan mempunyai kaitan rapat dengan hubungan Plucker. Kerangka ini dipamerkan melalui persamaan Kadomtsev-Petviashvili (KP), Korteweg-deVries (KdV), Sawada-Kotera (S-K) dan sinus-Gordon (s-G).

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CHAPTER 1

INTRODUCTION

In this chapter, we begin to explain the soliton theory (since we use soliton theory) and its historical background. It is then followed by the background of the problem, statement of problem, the objectives, the scope and significance of the study.

1.1 Soliton Theory and its Historical Background

In 1834, John Scott Russell determined the most efficient design for canal boats. While conducting experiments, he discovered a phenomenon that he described as the wave of translation (*see* Figure 1.1). In fluid dynamics the wave is now called a Scott Russell solitary wave or well known as soliton. Russell reported his observations to the British Association in his 1844 'Report on Waves' in the following words:

I believe I shall best introduce the phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, the suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or *diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel* (Drazin and Johnson, 1989).

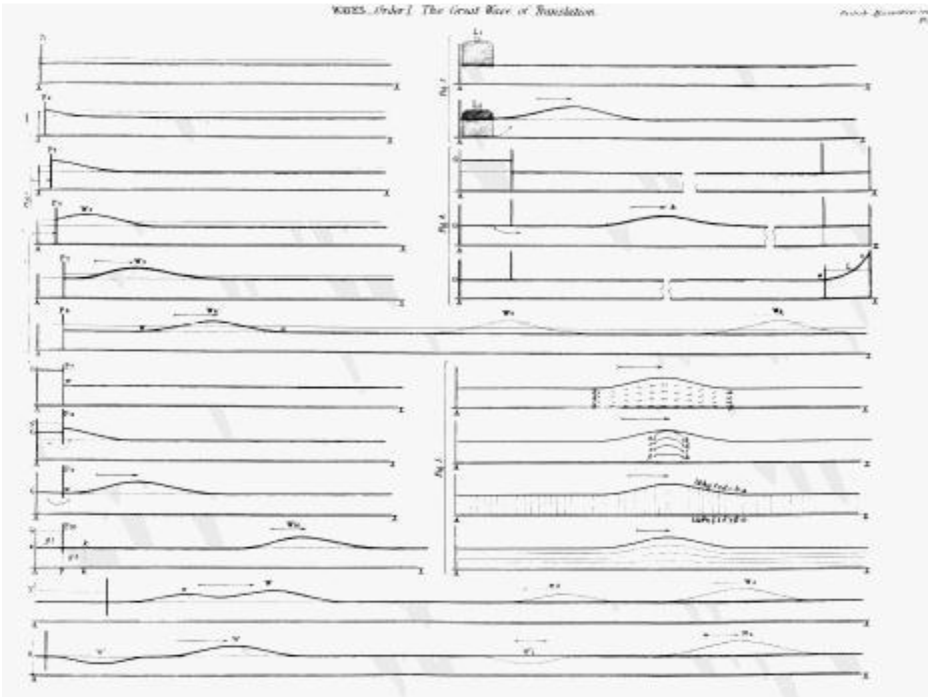

Figure 1.1 Russell's Waves of Translation.

He was convinced that the solitary wave was of fundamental importance, but prominent nineteenth and early twentieth century scientists thought otherwise (Druitt, 2005). Their contemporaries spent some time attempting to extend the theory but it would take until 1895, when Diederik Korteweg and Hendrik de Vries published a theory of shallow water waves that reduced Scott Russell's problem to its essential features. One of their results was the nonlinear partial differential equation which would play a key role in soliton theory (Korteweg and de Vries, 1895).

In 1965, Norman Zabusky of Bell Labs and Martin Kruskal of Princeton University first established soliton behavior in media subjected to the Korteweg-de Vries equation (KdV) in a computational investigation using a finite difference approach. They also demonstrated how this behavior explained the puzzling earlier work of Fermi, Ulam and Pasta in 1955 (Fermi-Pasta-Ulam nonlinear lattice oscillations). While in 1967, Gardner, Greene, Kruskal and Miura discovered an inverse scattering transform enabling analytical solution of the KdV equation (Gardner *et al.,* 1967).

The work of Peter Lax on Lax pairs and the Lax equation has since been developed to the solution of many related soliton generating systems where in 1968, Lax simplified his idea in his paper of integrals of nonlinear equations of evolution and solitary waves (Lax, 1968). While in 1972, Zakharov and Shabat illustrated that the inverse scattering technique could also be generalised for other soliton equations, such as KdV equation and nonlinear Schrodinger equation (Zakharov and Shabat, 1972). A direct method was first proposed by Hirota in his paper in 1971 which is known as 'Hirota's direct method' (Hirota, 1971).

There are multiple applications of soliton theory in physics. Numerous interesting nonlinear wave phenomena can occur and produce surface waves include the pattern-forming standing waves called Faraday waves (*see* Figure 1.2) (Krasnopolskaia *et al*., 2009-2010). Due to their invariance properties, solitons are of great potential use in light wave communication technology. Other applications of solitons include internal gravity waves in a stratified fluid and also natural creations in the atmosphere that are known as Rossby waves (*see* Figure 1.3) (Drazin and Johnson, 1989), (Druitt, 2005).

Figure 1.2 Stable pattern of Faraday waves on the water surface.

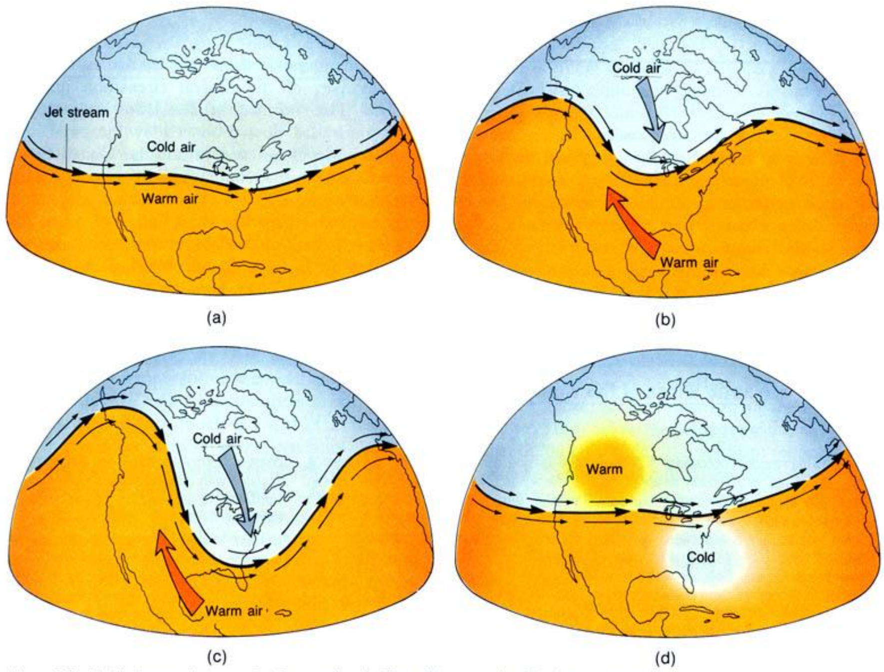

Figure 8•9 Cyclic changes that occur in the upper-level airflow of the westerlies. The flow, which has the jet stream as its axis, starts out nearly straight and then develops meanders that are eventually cut off. *(After J. Namias, NOAA)*

Figure 1.3 Rossby waves.

1.2 Problem Background

In 1981, Sato (1981) had unveiled the structures of Grassmann manifold in soliton equations by means of the method of algebraic analysis. He noticed that the τ-function of the Kadomtsev-Petviashvili (KP) equation is closely related to the Plucker coordinates appearing in the theory of Grassmann manifold and then discovered that the solutions of the KP equation as well as of its generalization constitutes an infinite Grassmann manifold. This project would like to show that these novel ideas can be expressed in terms of the Hirota *D*-operators, and thus be linked with Hirota's method. Sato's works were continued by Ohta *et al.* (1988) who proved that the generalized Lax equation, the Zakharov-Shabat equation and the IST scheme can be generated from the Sato equation that eventually yield the τ-function. Later, this τ-function is shown to be governed by the partial differential equations in bilinear forms which have the strong connection to the Plucker relations.

We note that Hirota's method makes an efficient tool for solving problems in mathematics of nonlinear evolution equations. Hirota's method is a powerful tool that can be employed together with a deep knowledge of the mathematics that lies beneath, namely Sato's theory. This is not to say that what lies beneath is intricately complicated, in fact, we will see that Sato's theory affords a deeper and beautiful understanding of soliton theory from a unified viewpoint.

1.3 Problem Statements

The project is to study in details the Hirota direct method and the link with the underlying concepts within Sato equation. This is then used rigorously in solving well-known nonlinear partial differential equation such as Kadomtsev-Petviashvili (KP) hierarchy which includes Kadomtsev-Petviashvili (KP) equation, Korteweg de-Vries (KdV) equation, Sawada-Kotera (S-K) equation and in a different setting, sine-Gordon (s-G) equation. We also investigate in details on how Grassmann manifold plays the role in obtaining Plucker relations and connect it with the Kadomtsev-Petviashvili (KP) equation, Korteweg de-Vries (KdV) equation, Sawada-Kotera (S-K) equation and sine-Gordon (s-G) equation.

1.4 Objectives of Study

i. To demonstrate that Hirota's scheme can be generated from Sato theory and produced the Sato equation in terms of τ-function.

ii. To show that the τ-function is naturally being governed by certain nonlinear partial differential equations in the bilinear form of Hirota scheme, which can be represented in the form of Plucker relations.

iii. To show that Sato equation can also be generated from Grassmann manifold, which is the basic underpinning in producing the Plucker relations, and thus linking to the Hirota's method.

iv. To obtain the bilinear form of certain well known nonlinear partial differential equations from the Plucker relations.

1.5 Scope of the Study

The research concentrates on showing the close relations between Hirota direct method, Sato Formalism and Grassmann manifold. These closed relations are shown to apply on some nonlinear evolution equations via Young and Maya diagrams.

1.6 Significance of Study

It is shown that the Hirota scheme and Sato theory are able to generate certain physically significant nonlinear partial differential equations. It is also shown that an infinite number of nonlinear evolution equations (KP hierarchy), share the multi-soliton solutions. The τ-function is the key function to express these solutions. By employing the results of the representation theory of groups, we show that the physically significant nonlinear partial differential equations governing the τ-function are closely related to the Plucker relations and can be written in bilinear form of Hirota scheme. The solutions of the Sato equation, and consequently those of the KP hierarchy in the Hirota scheme, can be explicitly expressed by the τ-function.

1.7 Thesis Outline

The thesis consists of six chapters including the conclusion. In Chapter 1, we start with a brief introduction of soliton theory and its historical background, followed by the problem background, problem statements, the objectives of study, scope of the study and last but not least the significance of study.

In Chapter 2, we present the literature review relating to the development of the direct method that was introduced by Ryogo Hirota. In the second section, it concerns with the progress of Sato formalism and finally focuses on the relation between the Hirota's method and Sato theory.

In Chapter 3, the Hirota direct method is introduced and together with the basic idea of Hirota's method. The essentials of the direct method are given, the Hirota's *D*-operator form is introduced and the bilinearizations of the nonlinear differential equations are obtained. This chapter also reviews on the Sato formalism where the basics of Sato theory are given in order to obtain the Sato equation. Hence, the τ-function is found to act as the key function to express the solutions of the Plucker relations. Later, the Grassmann manifold is shown to be the basic underpinning of Plucker relations.

Chapters 4 and 5 illustrate the relationship between the Hirota direct method and the Plucker relations. In Chapter 4, the Young diagram acts as the method that solves the KP, KdV, S-K and s-G equations whiles in Chapter 5, the method of Maya diagram is used to solve the KP, KdV, S-K and s-G equations. The complete proof will be demonstrated in detail.

Finally, the conclusions of this study are given in Chapter 6. It also encloses some ideas for future research.

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