SINGULAR VALUE DECOMPOSITION AND STRUCTURED TOTAL LEAST NORM FOR APPROXIMATE GREATEST COMMON DIVISOR OF UNIVARIATE POLYNOMIALS

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To mak, ayah

abang, akak, adik, kak t

hamzah, irfan

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ABSTRACT

This study presents experimental works on approximate GCD of univariate polynomials. The computation of approximate GCD is required in the case of imperfectly known or inexact data which emerges from physical measurements or previous computation error. SVD of Sylvester matrix is used to determine the degree of the approximate GCD. Then, STLN method is employed to generate an algorithm for solving the minimization problem that is to find the minimum perturbation such that the perturbed polynomials have a nonconstant GCD. The formulation of the STLN algorithm lead to the formation of LSE problem and this is solved using QR factorization. Every computation is done using MATLAB. Once the minimum perturbation is found, a MATLAB toolbox called Apalab is used to determine the coefficients of the approximate GCD. Results from experimental work performed reveal that the STLN algorithm presented is as efficient as the existing minimization algorithms.

ABSTRAK

Kajian ini membentangkan kerja – kerja analisa bagi pembahagi sepunya terbesar anggaran melibatkan polinomial pemboleh ubah tunggal. Pengiraan pembahagi sepunya terbesar adalah diperlukan dalam situasi melibatkan data yang tidak diketahui secara sempurna hasil dari sukatan fizikal atau berlakunya ralat dalam pembundaran data. Penguraian nilai tunggal bagi matriks Sylvester digunakan untuk menentukan darjah pembahagi sepunya terbesar anggaran. Seterusnya, kaedah jumlah norma terkecil berstruktur digunakan untuk menjana algoritma bagi menyelesaikan masalah peminimuman iaitu untuk mencari pengusikan minimum supaya polinomial terusik yang dihasilkan mempunyai pembahagi sepunya terbesar bukan malar. Formulasi algoritma jumlah norma terkecil berstruktur menghasilkan pembentukan masalah kuasa dua terkecil dengan pembatas kesamaan dan ini diselesaikan dengan menggunakan kaedah pemfaktoran QR. Setiap pengiraan dilakukan dengan menggunakan perisian MATLAB. Apabila pengusikan minimum telah didapati, sebuah modul perisian MATLAB yang digelar Apalab digunakan untuk menentukan pekali bagi pembahagi sepunya terbesar anggaran tersebut. Hasil dari kerja – kerja pengiraan yang dilakukan menunjukkan algoritma jumlah norma terkecil berstruktur yang dibentangkan adalah secekap dengan algoritma yang sedia ada.

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LIST OF ABBREVIATIONS

BFGS	Broyden – Fletcher – Goldfard – Shanno
GCD	Greatest Common Divisor
GKO	Gohberg – Kailath – Olshevsky
LS	Least Squares
LSE	Least Squares with Equality Constraint
PRS	Polynomial Remainder Sequence
STLN	Structured Total Least Norm
STLS	Structured Total Least Squares
SVD	Singular Value Decomposition
TLS	Total Least Square

CHAPTER 1

INTRODUCTION

1.1 Motivation

The evolution of knowledge in the field of science and technology lead to the advancement in the invention of mathematical tools and software. The tedious and troublesome works that were once occupied in the field of mathematics can be easily solved with the existence of numerous mathematical tools and software. This advancement is providing assistance to human being especially in the area of numerical analysis and optimization.

Numerical analysis or particularly numerical linear algebra and optimization frequently relates to one another. For instance, the LS problem in which it requires the residual or error to be minimized. Moreover, when a function minimizer needs to be determined, it requires a system of linear equation to be solved, in which it is a matter of linear algebra. Recent development disclosed the problem of GCD requires knowledge in both field, that is, linear algebra and optimization.

The problem of finding the GCD of univariate polynomials arises in most of engineering fields and also in some of mathematics branches and computer science (Zeng, 2004). Some of these are control linear system, network theory, computer – aided design and image processing (Pan, 2001; Lecumberri *et al*., 2009). In most of these applications, input data are derived from the results of physical experiments or previous computations. Moreover, most of the input data are represented as floating point numbers which are affected by errors.

Euclid's algorithm yields the correct GCD of a set of polynomials if they are known exactly and symbolic computation is used (Lecumberri *et al*., 2009). Nonetheless, as explained previously, most of the input data emerges from physical measurement and therefore may be inexact or imprecise. And when it comes to the GCD computation, this results to an ill – posed problems. For instance, let $h(x)$ be a nonconstant divisor of a polynomial $f(x)$. Apparently, $\gcd\left(f(x), h(x)\right) = h(x)$. But, if it happened to be a small perturbation in $f(x)$, which is represented by $\delta > 0$, then, $\gcd\left(f(x) + \delta, h(x)\right) = 1$. It can be seen that an arbitrary small perturbations reduced a nontrivial GCD to a constant. These cause to the rise of the problem of approximate GCD where the input polynomials have to be manipulated so that they have a nontrivial GCD.

Several techniques to solve the matter of approximate GCD have been developed in the past recent years. Schönhage (1985) and Noda and Sasaki (1991) are some of the authors who modified the conventional Euclid's algorithm in application to the inexact case. On the other hand, some other authors such as Corless *et al*. (1995), Rupprecht (1999) and Bini and Boito (2007) produced algorithms for approximate GCD in employment of resultant matrix. Particularly, Corless *et al*. (1995) produced an SVD based algorithm for approximate GCD. Moreover, Pan (2001) used Padé approximation to estimate the degree of approximate GCD.

Optimization approach is also being considered in the matter of approximate GCD. Corless *et al*. (1995) has further suggested that an optimization technique can be used to compute the approximate GCD. Then, Karmarkar and Lakshman (1998), Chin *et al*. (1998), Kaltofen *et al*. (2006), Winkler and Allan (2008a) and Terui (2009) are some of the authors who further worked on this approach. They produced algorithms with the main objective of finding the minimum perturbation such that the perturbed polynomials have a nonconstant GCD.

With this as a motivation, this study is done with the purpose to reproduce the existing algorithm of approximate GCD of univariate polynomial, particularly in the context of optimization. SVD, Sylvester matrix and STLN method to solve the minimization problem will be used in this study. Then, a MATLAB toolbox that is used to determine the approximate GCD will be introduced.

1.2 Problem Statement

The notion of approximate GCD leads to the rise of the problem of determining the highest degree of approximate divisor and finding minimum perturbation so that the perturbed polynomials have a nontrivial GCD. Therefore, in this study, an SVD based algorithm is employed to determine the degree of the approximate GCD. Furthermore, an algorithm based on the method of STLN is used to compute the minimum perturbation.

1.3 Objective of the Study

The objectives of this study are :

i. To employ the SVD of Sylvester matrix to the computation of degree of approximate GCD for several test problems.

ii. To apply the method of STLN to the computation of minimum perturbation.

iii. To reproduce and modify the existing algorithm for computing minimum perturbation by using QR factorization rather than the method of weight to solve the LSE problem.

iv. To compute the approximate GCD using a MATLAB toolbox once the minimum perturbation is known.

v. To investigate the existence of the family of approximate GCD and its corresponding minimum perturbation.

1.4 Scope of the Study

This study particularly focuses on the SVD of the Sylvester matrix and the method of STLN for solving the minimization problem of approximate GCD. Only univariate polynomial pair is considered in this study. The SVD of the Sylvester matrix is used to determine the highest possible degree of the approximate divisor. Then, the method of STLN is used to compute the minimum perturbation such that the input polynomial pair has an approximate GCD. Each computation is performed using MATLAB. Subsequently, a MATLAB toolbox called Apalab is used to determine the coefficients of the approximate GCD. Also, the notion of the family of approximate GCD is clarified and several results are exhibited.

1.5 Significance of the Study

The problem of finding a minimum perturbation such that the perturbed polynomials have GCD leads the direction of this study. It is fully hope from this empirical work, an algorithm to determine the minimum perturbation can be reproduced. This can then be employed in the practical application such as in engineering fields and other mathematics substances such as in the matter of simplifying rational expressions.

Furthermore, the study on approximate GCD of polynomials can initiate further investigation on the subject matter.

1.6 Research Methodology

1.6.1 Research Design and Procedure

This study begins by learning the concept of matrix decomposition of SVD and polynomial GCD. Related literature on SVD and polynomial GCD is being reviewed and related theorems and definitions are being identified. Then, the notion of approximate polynomial GCD is being studied. Existing works done on this matter are being identified and discussed. These include the application of resultant matrix, particularly, the application Sylvester matrix in the matter of approximate GCD. Moreover, the application of optimization approaches in the matter of approximate GCD is being reviewed. Recent development on the methods which efficiently solve the matter of approximate GCD, i.e. to find the minimum perturbation is being identified. This leads to the study on the method of STLN.

Then, the process of study continues by reproducing algorithms in relation to the matter of approximate GCD. Two algorithms are being reproduced. First is the algorithm to determine the highest possible degree of approximate divisor in application of SVD of Sylvester matrix of the corresponding input polynomials. The second algorithm concerned on the computation of the minimum perturbation for the input polynomials so that the perturbed polynomials have a nontrivial GCD. Modification is made to the existing algorithm where QR factorization is used to solve the LSE problem which is a part of the algorithm.

These algorithms are performed using MATLAB and its efficiency is observed by employing the algorithms to several test problems. Finally, from this experimental works done, analysis of the results is done and conclusion is made.

1.6.2 Operational Framework

Start

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Understand the concept of SVD and polynomial GCD.

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Literature review on SVD, polynomial GCD, Euclid's algorithm, approximate GCD, resultant matrix, optimization techniques and STLN.

↓

Formulation of problem : Identifying problem, objectives, scopes, topic and methods employed.

↓

SVD of Sylvester matrix and its application to determine the degree of approximate GCD. Related theorems, definitions are introduced. Algorithm to determine the degree of approximate GCD is being reproduced.

↓

STLN and its application to the matter of approximate GCD i.e. to determine the minimum perturbation for the input polynomials so that the perturbed polynomials have a nontrivial GCD. Algorithm is being reproduced and some modification on the algorithm is done.

↓

Performing experimental works by employing both algorithms that are being reproduced. For a given univariate polynomials, the degree of the approximate divisor is computed using the first algorithm. Then, minimum perturbation is computed using second perturbation. Several cases and situation are considered.

↓

Summarizing the study done, conclusion on the results obtained and suggestions for future works.

↓

Finish

1.7 Dissertation Outline

This study presents method of solving the problem of approximate GCD. Of particular interest are the applications of SVD and STLN to this problem. Six chapters are presented in this dissertation.

Chapter 2 focuses on the related literatures of GCD. A brief description on the history and several applications of SVD are made. Furthermore, a brief development on the Euclid's algorithm is clarified. Then, the term of approximate GCD is defined and several methods to solve the problem are described.

Chapter 3 provides a detailed study of SVD and relevant matter which relates to the problem of approximate GCD. Then, the algorithm to determine the degree of approximate GCD is clarified.

Chapter 4 describes the application of STLN to solve the problem of approximate GCD. Furthermore, the algorithm to find the minimum perturbation for the input polynomials is clarified. Then, a MATLAB toolbox that is used for finding the coefficients of approximate GCD is introduced.

Chapter 5 presents several experimental works on approximate GCD. These include polynomial roots computation and observation on the effect of error tolerance and data perturbation. Then, a notion of family of approximate GCD is introduced and some of them are exhibited.

Finally, Chapter 6 concludes the dissertation with a summarization on the work done, conclusion and possible directions for future work.

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