Gas-Kinetic BGK Scheme for Supersonic Channel Flow Simulation

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Abstract: In this paper, the gas-kinetic BGK (Bhatnagar-Gross-Krook) scheme is developed for simulating compressible inviscid supersonic channel flow. BGK scheme is an approximate Riemann solver that uses the collisional Boltzmann equation as the governing equation for flow evolutions. For efficient computations, particle distribution functions in the general solution of the BGK model are simplified and used for the flow simulations. Second-order accuracy is achieved via the reconstruction of flow variables using the MUSCL (Monotone Upstream-Centered Schemes for Conservation Laws) interpolation technique together with a multistage Runge-Kutta method. The number of grid points used for the domain is 241x131 which is generated using a structured algebraic grid generation method. The computational values compared very well to those of the analytical solution.
Keywords: finite difference method, BGK scheme, high-order accuracy, compressible inviscid flow.

1. Introduction

Tremendous efforts have been devoted and great progress has been achieved in the field of computational fluid dynamics for compressible inviscid flows in the past decade. The key goal is to design a numerical scheme that fulfills all the ideal computational requirements, which are high degree of accuracy, robustness, and efficiency. Although great effort and advances have been achieved toward this goal, however, none of them seem to be perfect enough to pass all the above said conditions. Among those notable and successful are the Godunov-type schemes and flux vector splitting schemes. In the family of Godunov-type schemes, Roe's FDS (Flux Difference Splitting) scheme [1] is the most popular owing to its accuracy for compressible inviscid and viscous flow simulations. However, the occurrence of transverse shock instability and negative internal energy of Godunov-type schemes hinder their usage in the computation of high speed flows with strong shock waves and expansion fans [2,3,4]. Owing to this reason, contemporary intention for the development of numerical schemes is focused on combining the accuracy of Godunov-type schemes and the robustness of flux vector splitting schemes.

In conventional numerical schemes, the Euler equations are discretized for the solutions of the compressible inviscid flows. Besides these schemes, several gas-kinetic schemes have been developed based on the Boltzmann equation [5]. A particular strength of kinetic schemes lies precisely where Godunov-type FDS schemes often fail, such as carbuncle phenomena, entropy condition, and positivity [6,7,8]. There are mainly two kinds of gas-kinetic schemes, and the differences lie within the governing equations used in the gas evolution stage. One of the well-known kinetic schemes is called the KFVS (Kinetic Flux Vector Splitting) which is based on the collisionless Boltzmann equation and the other is based on the collisional BGK model [9,10,11]. Like any other FVS method, the KFVS scheme is very diffusive and less accurate in comparison with the Roe-type Riemann solver. The diffusivity of the FVS schemes is mainly due to the particle or wave-free transport mechanism, which sets the CFL (Courant-Friedrichs and Lewy) time step equal to particle collision time [12]. In order to reduce diffusivity, particle collisions have to be modeled and implemented into the gas evolution stage. One of the distinct approaches is to take particle collision into consideration in the gas evolution stage [5,13,14,15]. In this method, the collision effect is considered by the BGK model as an approximation of the collision integral in the Boltzmann equation. It is found that this gas-kinetic BGK scheme possesses accuracy that is superior to the flux vector splitting schemes and avoids the anomalies of Godunov-type schemes [2,3,5,6,10,13,16]. Recent advancements in the development of the gas-kinetic BGK scheme can be found in the work of Xu, Chae, Ruan, Kunik and Abdussalam [2,3,4,5,12,17,18,19,20,21].

This paper is organized as follows. The governing equations for the two-dimensional compressible inviscid flows are presented. The descriptions of the gas-kinetic BGK scheme are also addressed. Then, a proposed method of reconstructing the initial solutions has been described. An explicit multistage Runge-Kutta time integration method is adopted for solving the gas-kinetic BGK scheme. Hence, the scheme developed in the previous parts is applied to the supersonic channel flow problem to investigate its essential features. The computational results are compared with the analytical solutions. Lastly, concluding remarks are made.

2. Numerical Method

The two-dimensional Euler equations in Cartesian coordinates are written as

$$\frac{\partial W}{\partial t} = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}. \tag{1}$$

Where,

$$W = \begin{bmatrix} \rho \\ \rho U \\ \rho V \\ \rho \varepsilon \end{bmatrix}, \quad F = \begin{bmatrix} \rho U \\ \rho U^2 + p \\ \rho UV \\ (\rho\varepsilon + p)U \end{bmatrix}, \quad G = \begin{bmatrix} \rho V \\ \rho UV \\ \rho V^2 + p \\ (\rho\varepsilon + p)V \end{bmatrix}. \tag{2}$$

In Eq. (1), ρ, ρU, ρV and $\rho\varepsilon$ are the macroscopic mass, x-momentum, y-momentum and total energy density respectively, where P is the pressure. Subsequently, Eq. (1) is transformed into curvilinear coordinates (ξ, η). These are detailed in Hoffmann and Chiang [22].

A standard gas-kinetic BGK scheme is based on the collisional Boltzmann equation [2,3,5,13] and it is written in two dimensions as

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = \frac{(g - f)}{\tau}. \tag{3}$$

Where f is a real particle distribution function, u and v are the particle velocities, and the right hand side stands for the relaxation model suggested to approximate the collision term. In which, g is an equilibrium particle distribution function which f approaches through particle collisions within a collision time scale τ. A general solution f of Eq. (3) at the cell interface $(x_{i+1/2}, y_j) = (0, 0)$ in two-dimensions is obtained as

$$f(0,0,t,u,v,\varsigma) = \left(1 - e^{-t/\tau}\right)g_o + e^{-t/\tau} f_o(-ut, -vt). \tag{4}$$

With the definition of $e^{-t/\tau} = \varphi$, the above distribution function f is written as

$$f(0,0,t,u,v,\varsigma) = (1 - \varphi)g_o + \varphi f_o(-ut, -vt). \tag{5}$$

Finally, the gas-kinetic BGK numerical flux across the cell interface in the x-direction can be computed as

$$\begin{aligned} F_x &= \int u\, f(0,0,t,u,v,\varsigma)\Psi\, d\Xi \\ F_x &= (1-\varphi)F_x^e + \varphi F_x^f \end{aligned}. \tag{6}$$

where F_x^e is the equilibrium flux function and F_x^f is the non-equilibrium or free stream flux function. Thus, in order to derive the numerical flux for the BGK scheme at the cell interface in the x-direction following the steps as shown in Eq. (6), yield the following relation:

$$F_{i+1/2,j} = (1-\varphi)F_{i+1/2,j}^e + \varphi F_{i+1/2,j}^f. \tag{7}$$

While the numerical flux at the cell interface in the y-direction is obtained in a similar manner and the resulting relation is presented as

$$G_{i,j+1/2} = (1-\varphi)G_{i,j+1/2}^e + \varphi G_{i,j+1/2}^f. \tag{8}$$

For the high-order spatial accuracy, a method known as the MUSCL approach [23] is adopted along with van Leer's limiter. Hence, the left and right states of the primitive variables ρ, U, V, p at a cell interface could be obtained through the non-linear reconstruction of the respective variables and are given as

$$\begin{aligned} Q_l &= Q_{i,j} + \frac{1}{2}\phi\left(\frac{\Delta Q_{i+1/2,j}}{\Delta Q_{i-1/2,j}}\right)\Delta Q_{i-1/2,j} \\ Q_r &= Q_{i+1,j} - \frac{1}{2}\phi\left(\frac{\Delta Q_{i+3/2,j}}{\Delta Q_{i+1/2,j}}\right)\Delta Q_{i+1/2,j} \end{aligned}. \tag{9}$$

Where Q is a primitive variable and the subscript l, and r correspond to the left and right side of a considered cell interface. In addition, $\Delta Q_{i+1/2,j} = Q_{i+1,j} - Q_{i,j}$. The van Leer's limiter used in the reconstruction of flow variables in Eq. (9) is given as

$$\phi(\Omega) = \frac{(\Omega + | \ \Omega \ |)}{(1+\Omega)} \tag{10}$$

For the time integration, an explicit method known as the multistage Runge-Kutta method is employed for the flow simulation and the details of this method can be obtained easily from various literatures [23,24].

3. Results and Discussions

1) Supersonic Channel Flow Problem

This particular flow problem is taken from Hoffmann and Chiang [24] where a channel which includes both compression and expansion corners is used to illustrate the formation of oblique shock and expansion waves and their reflection and interaction. The rationalization behind this selection is simply because analytical results can be calculated and used to validate the numerical results computed from the BGK scheme.

The physical dimensions of the domain is given by the grid and shown in Fig. 1. The compression/expansion angle for the channel is 10^o. The number of grid points for the domain is 241 x 131. The following free stream conditions are used to initialize the flow:

$$p = 100\,\text{kPa} \quad T = 300\text{K} \quad M = 2.0\,.$$

Where, *p*, *T* and *M* are pressure, temperature and Mach number, respectively. The boundary conditions used for this problem consist of a supersonic inlet and outlet at the left and right of the physical domain. While an inviscid wall condition is applied at the lower and upper wall.

The pressure and density contours showing the formation of the oblique shock, expansion wave and their reflection and interaction are shown in Figs. 2 and 3, respectively. In addition, the computed pressure and density distributions along the lower surface are compared to the analytical solutions calculated from Ref. [24]. The former is shown in Fig. 4 while the latter is shown in Fig. 5. The computational values shown on both of the plots compare very well to those of the analytical solutions. Also, it can be observed from these results that the BGK scheme is able to capture and resolve the shock accurately without producing any pre- or post-shock oscillation. Besides, the accuracy of the BGK scheme is clearly demonstrated by its ability to describe the entire flow effectively.

4. Conclusion

In this paper, a numerical method based on the two-dimensional gas-kinetic BGK scheme was developed and subsequently applied to the simulation of the supersonic channel flow problem to investigate its computational accuracy. The outcomes of the supersonic channel flow problem substantiated the computational characteristics of the BGK scheme to be accurate and having high shock resolution capabilities.

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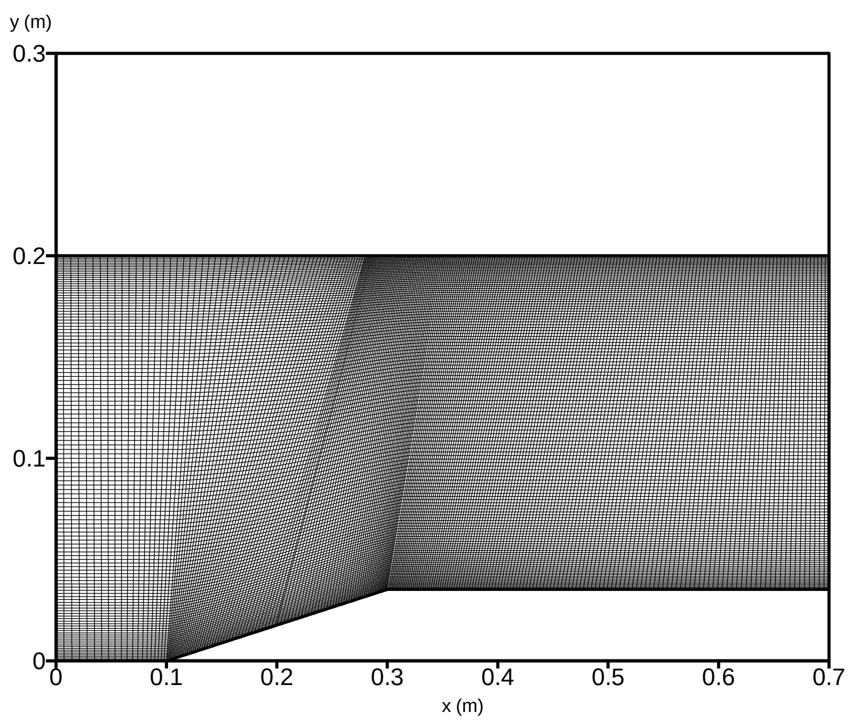

Figure 1: Physical mesh for the supersonic channel.

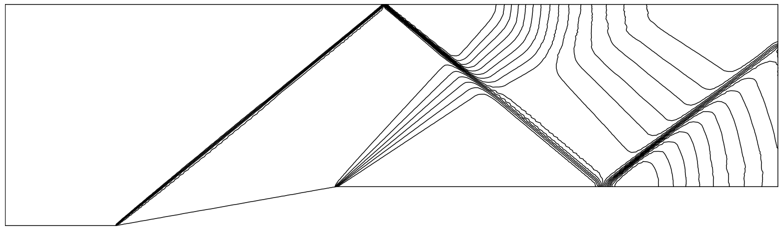

Figure 2: Pressure contours for M = 2.0.

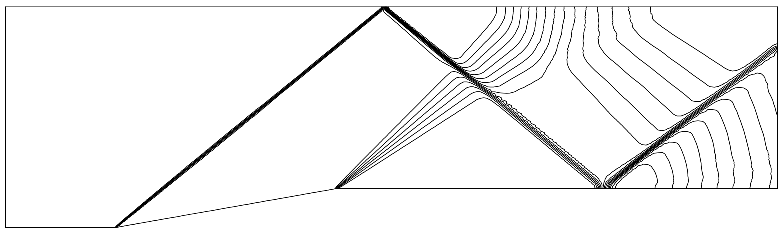

Figure 3: Density contours for M = 2.0.

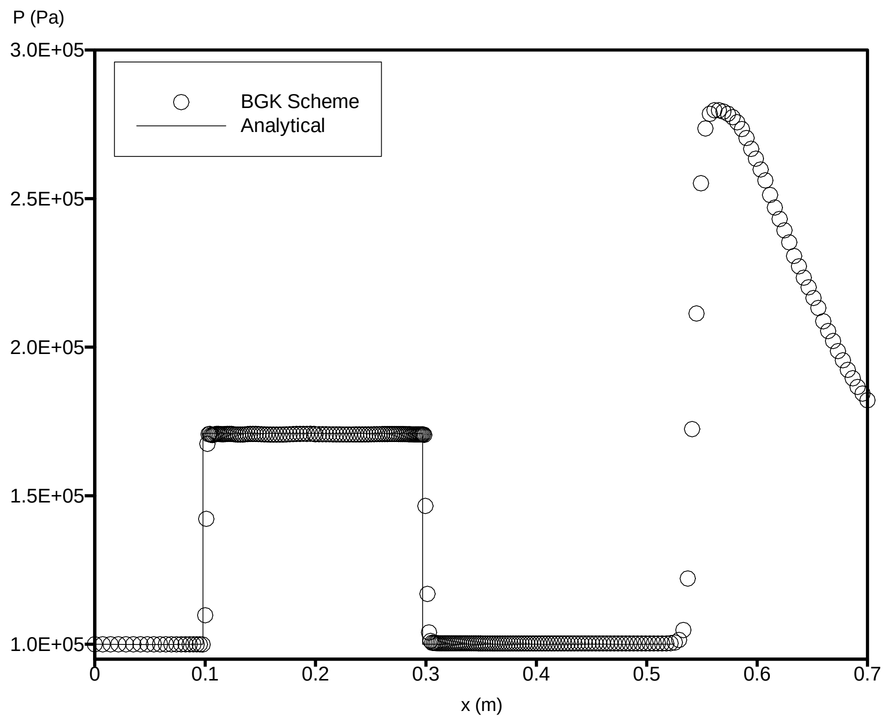

Figure 4: Comparison of the pressure distributions on the lower surface at M = 2.0.

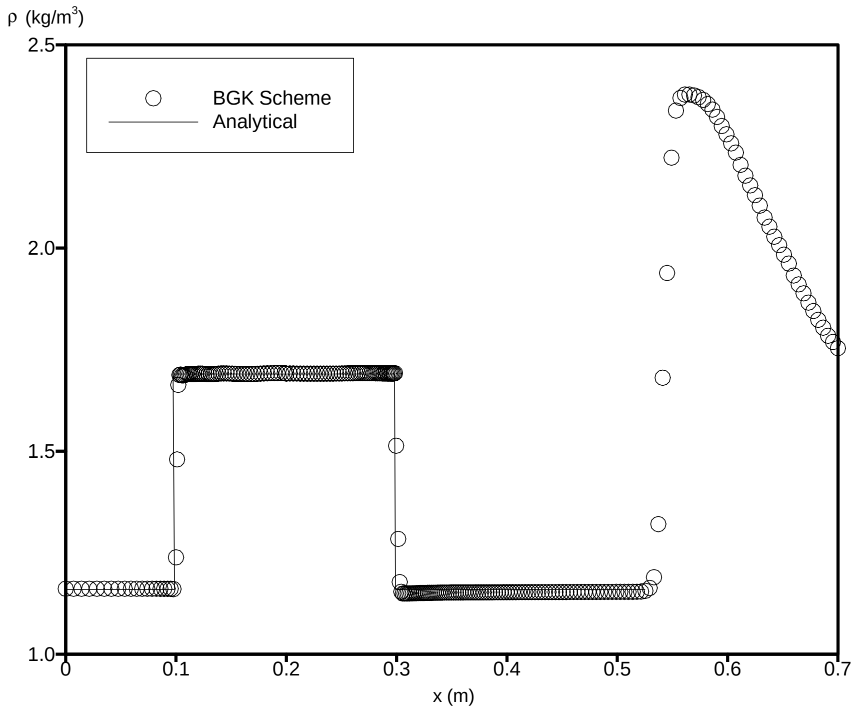

Figure 5: Comparison of density distributions on the lower surface at M = 2.0.