HIROTA-SATO FORMALISM VIA MAYA DIAGRAMS ON KP, KdV AND S-K EQUATIONS

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Abstract.

This article illustrates Hirota-Sato formalism by establishing that Hirota's direct method is derivable from Sato theory. This formalism is considered via Maya diagrams and used to describe the Kadomtsev-Petviashvili (KP), Korteweg-de Vries (KdV) and Sawada-Kotera (S-K) equations. This is done by expressing the Hirota bilinear forms of KP, KdV and S-K equations in terms of Maya diagrams. These results are then shown to be closely linked to the Plucker relations in Sato theory. Thus Hirota-Sato formalism via this conceptual framework provides a deeper understanding of soliton theory from a unified viewpoint.

Keywords Hirota bilinear form, Maya diagrams, Plucker relations, Kadomtsev-Petviashvili (KP) equation, Korteweg-de Vries (KdV) equation, Sawada-Kotera equation

1. Introduction

There are various methods of examining the integrable nonlinear waves equations that have soliton solutions, where each method has its own suppositions and areas of usage. For example, the inverse scattering transform (IST) can be used to solve initial value problems, but it uses powerful analytical methods and quantum scattering theory (e.g. [1]), and therefore makes strong assumptions about the nonlinear waves equations. On the lesser extreme, one can find a travelling wave solution to almost all equations by a straightforward substitution which reduces the equation to an ordinary differential equation (e.g. [2]). Between these two extremes lies Hirota's direct/bilinear method. Although the transformation was intrinsically inspired by IST, Hirota's method does not need the same mathematical assumption and, as a consequence, the method is applicable to a wider class of equations than IST (e.g. [3]). At the same time, because it does not use such sophisticated techniques, it usually produces a smaller class of solutions, the multi-soliton solutions. It is particularly efficient for constructing multisoliton solutions to integrable nonlinear waves equations. The advantage of it over others is that it is algebraic rather than analytic. In many problems the key to further developments is a detailed understanding of soliton scattering, and in such cases Hirota's method is the optimal tool.

Hirota's method is an effective tool as it can be employed without a deep knowledge of the mathematics that lies beneath, namely Sato's theory [4-5]. This is not to say that what lies beneath is valueless, in fact in this article, we will establish that Sato's theory allows Hirota's method to have a deeper and beautiful understanding of soliton theory from a unified viewpoint. Here we note that Hirota's method makes an efficient tool for an applied mathematicians' toolbox, and furthermore, we may credit it (along with the pioneering IST) for inspiring a departure into soliton theory.

Hirota direct method was first introduced by Hirota in his well-known 1971 paper [6]. The first step of this method is to transform the nonlinear partial differential equation into a quadratic form in dependent variables. The new form of the equation is called 'bilinear form'. In the second step, we write the bilinear form of the equation as a polynomial of a special differential operator known as Hirota D-operator. This polynomial of D-operator is called 'Hirota bilinear form'. In this article we shall only focus in solving on some physically significant nonlinear wave equations which includes KP [7], KdV [8] and Sawada-Kotera equations [9]. The KdV and KP equations have been used extensively as a model for one- and two-dimensional shallow water waves of long wavelength with weakly non-linear restoring forces (e.g. [10]) and ion-acoustic waves in plasmas (e.g. [11]), and Sawada-Kotera in some mathematical approaches to tsunami (e.g. [12])

Grassmannian manifolds are known as the basics of Sato's theory where the τ-function was obtained from the derivation of the Sato's equation (e.g. [13-14]). The manipulations of Schur functions and Maya diagrams later produce the Plucker relations. In this article we show how the Plucker relations in Sato's theory can be transformed into the Hirota bilinear form of KP, KdV [15], and Sawada-Kotera equations [16], [17] for their respective τ-functions via the Maya diagrams, and thus verify this conceptual framework.

2. Hirota bilinear forms of KP, KdV and S-K equations

Consider the KP equation

$$(4u_t - 12uu_x - u_{xxx})_x - 3u_{yy} = 0. \tag{1}$$

By applying the dependent variable transformation (or logarithmic transformation, refer to [2-3]), the Hirota bilinear form of KP equation is

$$\begin{aligned}&\left(4D_xD_t - D_x^4 - 3D_y^2\right)\tau\,.\,\tau\\ &= \left(\tau_{xxxx} - 4\tau_{xx_3} + 3\tau_{x_2x_2}\right)\tau - 4\left(\tau_{xxx} - \tau_{x_3}\right)\tau_x + 3(\tau_{xx} - \tau_{x_2})(\tau_{xx} + \tau_{x_2})\\ &= 0\,,\end{aligned} \tag{2}$$

where the notation $t_n \equiv x_n$, $n \in \mathbb{Z}^+$ and $x = x, y = x_2, t = x_3$. This result is shown to link with the Plucker relations in [22].

Next, we let the KdV equation as

$$u_t + 6u\, u_x + u_{xxx} = 0. \qquad (3)$$

We then re-write the independent variables using the notation as

$$t_n \equiv x_n \;\; n \in \mathbb{Z}^+ \; and \; x = x, t = x_3.$$

By applying the logarithmic transformation on the KdV equation, then we have

$$2(\tau\, \tau_{xx_3} - \tau_x\, \tau_{x_3}) + 6\tau_{xx}^2 + 2\tau\, \tau_{xxxx} - 8\tau_x\, \tau_{xx} = 0.$$

Then after making use of the D-operator properties, we obtain

$$(D_x D_t + D_x^4)\, \tau \cdot \tau = 0. \qquad (4)$$

This result for the KdV equation is shown to link with the Plucker relations in [22].
The operator on the left hand side of (4) may be factorized. It may then be written as

$$D_x(D_t + D_x^3)\, \tau \cdot \tau = 0, \qquad (5)$$

where equation (5) is the bilinear form of KdV equation.

Next, we let the Sawada-Kotera equation as

$$u_t + 15uu_{x\,x\,x} + 15u_x u_{x\,x} + 45u^2 u_x + u_{x\,x\,x\,x\,x} = 0. \qquad (6)$$

For the Sawada-Kotera equation, we then re-write the independent variables using the notation as

$$t_n \equiv x_n \;\; n \in \mathbb{Z}^+ \; and \; x = x, t = x_5.$$

By applying the logarithmic transformation on the Sawada-Kotera equation, then we have

$$\left(18\tau_{xx_5} + 2\tau_{xxxxxx}\right)\tau - \left(18\tau_{x_5} + 12\tau_{xxxxx}\right)\tau_x + 30\tau_{xx}\tau_{xxxx} - 20\tau_{xxx}\tau_{xxx} = 0 \qquad (7a)$$

By using the properties of Hirota's D-operator, the Sawada-Kotera equation now becomes

$$D_x D_t\;\; \tau \cdot \tau = 2\tau\tau_{xx_5} - 2\tau_x\tau_{x_5}, \qquad (7b)$$

$$D_x^6\, \tau \cdot \tau = 2\tau\tau_{xxxxxx} - 12\tau_x\tau_{xxxxx} + 30\tau_{xx}\tau_{xxxx} - 20\tau_{xxx}\tau_{xxx} = 0, \qquad (7c)$$

Substituting (7b), (7c) into (7a), the bilinear form of Sawada-Kotera equation becomes

$$(9D_x D_t + D_x^6)\, \tau \cdot \tau = 0 \qquad . \qquad (7d)$$

This result for the S-K equation is shown to link with the Plucker relations in [22].

We are able to express the N-soliton solution as the $N \times N$ Wronskian (refer [3], [4] and [20]),

$$\tau_N = \begin{pmatrix} h_1^{(0)} & h_1^{(1)} & \dots & h_1^{(N-1)} \\ h_2^{(0)} & h_2^{(2)} & \dots & h_2^{(N-1)} \\ \vdots & \vdots & \dots & \vdots \\ h_N^{(0)} & h_N^{(2)} & \dots & h_N^{(N-1)} \end{pmatrix} \tag{8}$$

where $h_i^{(n)}$ is defined by

$$h_i^{(n)} = \frac{\partial^n h_i}{\partial x^n} \tag{9}$$

and each h_i $(i = 1, 2, \dots)$ satisfies the differential equations

$$\frac{\partial h_i}{\partial x_n} = \frac{\partial^n h_i}{\partial x^n}. \tag{10}$$

It is confirmed that τ_N satisfies the KP, KdV and Sawada-Kotera equations as were shown in (2), (5) and (7) respectively, and were shown to link with Plucker relations in [22].

3. KP, KdV and S-K equations in terms of Maya diagrams and Plucker relations in Sato theory.

We now derive the derivative of τ_N with respect to x with the help of Maya diagrams.
The derivative with respect to x of the matrix τ_N is equal to the sum of determinants for $i = 1, 2, 3, \dots, N$ in which the ith column of τ_N is replaced by its derivative. But the remaining derivative column which is the first column is equal to the second column; the derivative of the second column is equal to the third column and so on. As a result, only the determinant with the last column differentiated remains.

By taking the derivative of the last column of τ_N , we are able to obtain the derivative of $\tau_{N,x}$ such as

$$\tau_{N,x} = \begin{pmatrix} h_1^{(0)} & h_1^{(1)} & \dots & h_1^{(N-2)} & h_1^{(N)} \\ h_2^{(0)} & h_2^{(2)} & \dots & h_2^{(N-2)} & h_2^{(N)} \\ \vdots & \dots & \dots & \dots & \vdots \\ h_N^{(0)} & h_N^{(2)} & \dots & h_N^{(N-2)} & h_N^{(N)} \end{pmatrix} \tag{11}$$

where we have added one to the superscript $(N - 1)$ in τ_N. Only the number of derivatives in each column of τ_N is taking part in this differentiation whiles the numbers of rows are unaffected and remains the same.

Maya diagram can be defined as a placement of a bead (bead-filled or without) at each position in $\mathbb{Z} + {}^{1}/_{2}$, $\mathbb{Z}$ is an integer, subject to the condition that at all but definitely many positions $m < 0, m = 0, 1, \dots, N-4, N-3, \dots$, are filled with bead-filled positions and all but definitely many positions $m > 0$ are filled with empty positions (refer to [18-19]). For instance, Fig.1 shows a vacuum state corresponding to a system of fermionic particles with pure energy states, and is given by the following Maya diagram.

0	1	...	N-4	N-3	N-2	N-1	N	N+1	N+2	N+3	N+4	N+5	N+6
O	O	...	O	O	O	O							

Figure 1: Maya diagram of a vacuum state

We will apply the Maya diagrams to adopt the simple notation

$$\tau_N = [0, 1, \dots, N-1] = \tau, \tag{12}$$

$$\tau_{N,x} = [0, 1, \dots, N-2, N] = \tau_x. \tag{13}$$

These can be written in the form of the following Maya diagrams;

		0	1	...	N-4	N-3	N-2	N-1	N	N+1	N+2	N+3	N+4
τ	=	O	O	...	O	O	O	O					
τ_x	=	O	O	...	O	O	O		O				

Figure 2: Maya diagrams for notations (12) and (13)

The diagram for τ above shows that the fermion only occupy on the left hand sides of the diagrams while the right hand sides of the cell is empty, which we may considered this situation to be the vacuum state. In the same way, the diagram for τ_x represents the state where the fermion occupying the $(N-1)$th cells or the last column is shifting one place to the right.

In obtaining the higher order derivatives of τ_N such as τ_{xx}, it involved the shifting derivative of the second last and last columns. Fermion occupying the $(N-1)$th cell is excited into the $(N+1)$th cell. While fermion occupying the $(N-2)$th cell is excited into the Nth cell. Continuing on this pattern for higher derivatives, it is obvious that the derivatives only correspond to a shift in certain fermions in the Maya diagrams.

		0	1	...	N-4	N-3	N-2	N-1	N	N+1	N+2	N+3	N+4
τ_{xx}	=	O	O	...	O	O	O			O			
	+	O	O	...	O	O		O	O				

		0	1	...	N-4	N-3	N-2	N-1	N	N+1	N+2	N+3	N+4
τ_{xxx}	=	O	O	...	O	O	O				O		
	+2	O	O	...	O	O		O		O			

		0	1	...	N-4	N-3	N-2	N-1	N	N+1	N+2	N+3	N+4
	+	O	O	...	O		O	O	O				

		0	1	...	N-4	N-3	N-2	N-1	N	N+1	N+2	N+3	N+4
τ_{xxxx}	=	O	O	...	O	O	O					O	
	+3	O	O	...	O	O		O			O		
	+2	O	O	...	O	O			O	O			
	+3	O	O	...	O		O	O		O			
	+	O	O	...		O	O	O	O				

		0	1	...	N-4	N-3	N-2	N-1	N	N+1	N+2	N+3	N+4
τ_{xxxxx}	=	O	O	...	O	O	O						O
	+4	O	O	...	O	O		O				O	
	+5	O	O	...	O	O			O		O		
	+6	O	O	...	O		O	O			O		
	+5	O	O	...	O		O		O	O			
	+4	O	O	...		O	O	O		O			

Figure 3: Maya diagrams for the derivatives

Consider the derivative τ_{x_2}. By referring to (10), we can see that each column is differentiated twice as below

$$\frac{\partial h_i}{\partial x_2} = \frac{\partial^2 h_i}{\partial x^2}$$

Therefore, the derivative of τ_N with respect to x_2 is

$$\tau_{x_2} = [0, 1, \dots, N-3, N-2, N+1] + [0, 1, \dots, N-3, N, N-1]$$

$$= [0, 1, \dots, N-3, N-2, N+1] - [0, 1, \dots, N-3, N-1, N]. \qquad (14)$$

The minus sign comes from the swapping of any pair of columns, and which means that the re-ordering of the columns of the determinant will result in the increase of the number of the superscript.

Expressing in terms of the Maya diagrams for τ_{x_2}, we have

		0	1	...	N-4	N-3	N-2	N-1	N	N+1	N+2	N+3	N+4
τ_{x_2}	=	O	O	...	O	O	O			O			
	−	O	O	...	O	O		O	O				

		0	1	...	N-4	N-3	N-2	N-1	N	N+1	N+2	N+3	N+4
$\tau_{x_2x_2}$	=	O	O	...	O	O	O					O	
	−	O	O	...	O	O		O			O		
	+2	O	O	...	O	O			O	O			
	−	O	O	...	O		O	O		O			
	+	O	O	...		O	O	O	O				

		0	1	...	N-4	N-3	N-2	N-1	N	N+1	N+2	N+3	N+4
τ_{x_3}	=	O	O	...	O	O	O				O		
	−	O	O	...	O	O		O		O			
	+	O	O	...	O		O	O	O				

		0	1	...	N-4	N-3	N-2	N-1	N	N+1	N+2	N+3	N+4
τ_{x_3x}	=	O	O	...	O	O	O					O	
	−	O	O	...	O	O			O	O			
	+	O	O	...		O	O	O	O				

Figure 4: Maya diagrams for the derivatives of τ_{x_2} and τ_{x_3}.

In the following, we show how the Plucker relations in Sato's theory and the Hirota bilinear forms of KP, KdV and S-K equations are linked by their respective τ-functions via the Maya diagrams, and thus verify this conceptual framework.

3.1 KP equation

Therefore, in solving the KP equation, we have

		N-2	N-1	N	N+1
τ	=	O	O		
τ_x	=	O		O	
$\tau_{xx} - \tau_{x_2}$	= 2		O	O	

$$\tau_{xx} + \tau_{x_2} = 2 \begin{array}{|c|c|c|c|}\hline \mathrm{O} & & & \mathrm{O} \\ \hline\end{array}$$

$$\tau_{xxx} - \tau_{x_3} = 3 \begin{array}{|c|c|c|c|}\hline & \mathrm{O} & & \mathrm{O} \\ \hline\end{array}$$

$$\tau_{xxxx} - 4\tau_{xx_3} + 3\tau_{x_2x_2} = 12 \begin{array}{|c|c|c|c|}\hline & & \mathrm{O} & \mathrm{O} \\ \hline\end{array}$$

where we have neglected cells common to all the Maya diagrams. Substituting these results into the KP equation, we obtain another expression

$$\begin{array}{ccccccccccc} & \text{N-2} & \text{N-1} & \text{N} & \text{N+1} & & \text{N-2} & \text{N-1} & \text{N} & \text{N+1} \\ & \begin{array}{|c|c|c|c|}\hline \mathrm{O} & \mathrm{O} & & \\ \hline\end{array} & & & & \times & \begin{array}{|c|c|c|c|}\hline & & \mathrm{O} & \mathrm{O} \\ \hline\end{array} \\ - & \begin{array}{|c|c|c|c|}\hline \mathrm{O} & & \mathrm{O} & \\ \hline\end{array} & & & & \times & \begin{array}{|c|c|c|c|}\hline & \mathrm{O} & & \mathrm{O} \\ \hline\end{array} \\ + & \begin{array}{|c|c|c|c|}\hline \mathrm{O} & & & \mathrm{O} \\ \hline\end{array} & & & & \times & \begin{array}{|c|c|c|c|}\hline & \mathrm{O} & \mathrm{O} & \\ \hline\end{array} \end{array}$$

Figure 5: Maya diagrams and Plucker relations for KP equation

These are actually the Plucker relations that were obtained from the bilinear form of KP equation (refer [22]).

3.2 KdV Equation

Summarizing the above results in solving the KdV equation, we have

$$\tau = \begin{array}{|c|c|c|c|}\hline \mathrm{O} & \mathrm{O} & & \\ \hline\end{array}$$

$$\tau_x = \begin{array}{|c|c|c|c|}\hline \mathrm{O} & & \mathrm{O} & \\ \hline\end{array}$$

$$\tau_{xx} = \begin{array}{|c|c|c|c|}\hline \mathrm{O} & & & \mathrm{O} \\ \hline\end{array} + \begin{array}{|c|c|c|c|}\hline & \mathrm{O} & \mathrm{O} & \\ \hline\end{array}$$

$$\tau_{x_2} = \begin{array}{|c|c|c|c|}\hline \mathrm{O} & & & \mathrm{O} \\ \hline\end{array} - \begin{array}{|c|c|c|c|}\hline & \mathrm{O} & \mathrm{O} & \\ \hline\end{array}$$

$$\tau_{xxx} - \tau_{x_3} = 3 \begin{array}{|c|c|c|c|}\hline & \mathrm{O} & & \mathrm{O} \\ \hline\end{array}$$

$$\tau_{xxxx} - 4\tau_{xx_3} = -3 \begin{array}{|c|c|c|c|}\hline \mathrm{O} & & & \\ \hline\end{array} + 3 \begin{array}{|c|c|c|c|}\hline & \mathrm{O} & & \\ \hline\end{array} + 6 \begin{array}{|c|c|c|c|}\hline & & \mathrm{O} & \mathrm{O} \\ \hline\end{array}$$

$+3$	O	O		O
-3	O	O	O	

Substituting these expressions into the bilinear form of KdV equation, we obtain

-3	O				$\times$	O	O		
$+3$		O			$\times$	O	O		
$+6$			O	O	$\times$	O	O		
$+3$	O	O		O	$\times$	O	O		
-3	O	O	O		$\times$	O	O		
-12		O		O	$\times$	O		O	
$+$	O			O	$\times$	O			O
$+2$	O			O	$\times$		O	O	
$+$		O	O		$\times$		O	O	

Figure 6: Maya diagrams and Plucker relations for KdV equation

It is clearly shown that the above result is the basis [21] of the Plucker relations where we have obtained before [18-19].Therefore; it is proven that KdV equation is connected to the Plucker relations.

3.3 Sawada-Kotera equation

By making use of the Maya diagrams in solving the S-K equation, we have

τ	$=$	O	O		
τ_x	$=$	O		O	
τ_{xx}	$=$	O			O
	$+$		O	O	
τ_{xxx}	$=$	O			

$$+2 \quad \begin{array}{|c|c|c|c|}\hline & O & & O \\ \hline\end{array}$$

$$+ \quad \begin{array}{|c|c|c|c|}\hline O & O & O & \\ \hline\end{array}$$

$$\tau_{xxxx} \qquad = \quad \begin{array}{|c|c|c|c|}\hline O & & & \\ \hline\end{array}$$

$$+3 \quad \begin{array}{|c|c|c|c|}\hline & O & & \\ \hline\end{array}$$

$$+2 \quad \begin{array}{|c|c|c|c|}\hline & & O & O \\ \hline\end{array}$$

$$+3 \quad \begin{array}{|c|c|c|c|}\hline O & O & & O \\ \hline\end{array}$$

$$+ \quad \begin{array}{|c|c|c|c|}\hline O & O & O & \\ \hline\end{array}$$

$$18\tau_{xx_5} + 2\tau_{xxxxxx} \qquad = 20 \quad \begin{array}{|c|c|c|c|}\hline O & & & \\ \hline\end{array}$$

$$+50 \quad \begin{array}{|c|c|c|c|}\hline O & & O & \\ \hline\end{array}$$

$$+10 \quad \begin{array}{|c|c|c|c|}\hline & O & & \\ \hline\end{array}$$

$$+20 \quad \begin{array}{|c|c|c|c|}\hline O & O & & \\ \hline\end{array}$$

$$+10 \quad \begin{array}{|c|c|c|c|}\hline & & & O \\ \hline\end{array}$$

$$+10 \quad \begin{array}{|c|c|c|c|}\hline & O & O & O \\ \hline\end{array}$$

$$+20 \quad \begin{array}{|c|c|c|c|}\hline O & O & & \\ \hline\end{array}$$

$$18\tau_{x_5} + 12\tau_{xxxxx} \qquad = 30 \quad \begin{array}{|c|c|c|c|}\hline O & & & \\ \hline\end{array}$$

$$+30 \quad \begin{array}{|c|c|c|c|}\hline & O & & \\ \hline\end{array}$$

$$+90 \quad \begin{array}{|c|c|c|c|}\hline O & O & & \\ \hline\end{array}$$

$$+30 \quad \begin{array}{|c|c|c|c|}\hline O & O & & O \\ \hline\end{array}$$

$$+60 \quad \begin{array}{|c|c|c|c|}\hline & & O & \\ \hline\end{array}$$

$$+60 \quad \begin{array}{|c|c|c|c|}\hline O & & O & O \\ \hline\end{array}$$

Substituting these solutions into the Sawada-Kotera equation, we acquire another expression i.e.

$$20 \quad \begin{array}{|c|c|c|c|}\hline O & & & \\ \hline\end{array} \times \begin{array}{|c|c|c|c|}\hline O & O & & \\ \hline\end{array}$$

$$+50 \quad \begin{array}{|c|c|c|c|}\hline O & & O & \\ \hline\end{array} \times \begin{array}{|c|c|c|c|}\hline O & O & & \\ \hline\end{array}$$

$$+10 \quad \begin{array}{|c|c|c|c|}\hline & O & & \\ \hline\end{array} \times \begin{array}{|c|c|c|c|}\hline O & O & & \\ \hline\end{array}$$

+40	O	O			×	O	O		
+10				O	×	O	O		
+10		O	O	O	×	O	O		
−30	O				×	O		O	
−30		O			×	O		O	
−90	O	O			×	O		O	
−30	O	O		O	×	O		O	
−60			O		×	O		O	
−60	O		O	O	×	O		O	
+30	O				×	O			O
+90		O			×	O			O
+60			O	O	×	O			O
+90	O	O		O	×	O			O
+30	O	O	O		×	O			O
+30	O				×		O	O	
+90		O			×		O	O	
+60			O	O	×		O	O	
+90	O	O		O	×		O	O	
+30	O	O	O		×		O	O	
−20	O				×	O			
−40		O		O	×	O			
−20	O	O	O		×	O			
−20	O				×		O		O

-40		O		O	$\times$		O		O
-20	O	O	O		$\times$		O		O
-20	O				$\times$	O	O	O	
-40		O		O	$\times$	O	O	O	
-20	O	O	O		$\times$	O	O	O	

Figure 7: Maya diagrams and Plucker relations for S-K equation

According to [18], the fermionic Fock space F is an infinite dimensional vector space. This has a standard basis which can be indexed by a variety of objects [18-19]. Therefore, we can conclude that the bilinear form of S-K equation is also related to the Plucker relations.

4. Conclusion

From the nonlinear waves equation being considered, i.e. the KP, KdV and S-K equations, it was deduced that the τ-function in the bilinear forms of Hirota scheme for the respective nonlinear equations, via the Maya diagrams are closely associated to the Plucker relations in Sato theory. Therefore in terms of Maya diagrams, we may conclude that the τ-function is essential in indicating this relation between Hirota's direct method and Sato's theory. The above deliberations showed that Hirota's method is linked to Sato theory, and the Hirota-Sato formalism brings to light that the τ-function, which underlies the analytic form of soliton solutions of the related nonlinear waves equations. The Kadomtsev-Petviashvili (KP), Korteweg-de Vries (KdV) and Sawada-Kotera equations have been used to corroborate this theoretical framework.

Acknowledgements

This research is partially funded by MOHE FRGS Vote No. 78675. Aslinda is thankful to Universiti Teknologi Malaysia for the Zamalah/scholarship.

References

[1] Ablowitz, M., and Clarkson, P. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge: Cambridge University Press, 1991.

[2] Drazin, P.G., and Johnson, R.S. *Solitons: An Introduction*, Cambridge: Cambridge University Press, 1996.

[3] Hirota, R. *The Direct Method in Soliton Theory*. New York: Cambridge University Press, 2004.

[4] Ohta, Y., Satsuma, J., Takashi, D., and Tokihiro, T. An Elementary Introduction to Sato Theory, *Progress of Theoretical Physics Supplement*, 94: 210-241, 1988.

[5] Miwa, T., Date, E., and Jimbo, M. *Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras*, Cambridge: Cambridge University Press, 2000.

[6] Hirota, R. Exact Solution of the Korteweg-de Vries equation for Multiple Collisions of Solitons. *Phys. Rev.Lett*, 27:1192-1194, 1971.

[7] Kadomtsev, B. B., and Petviashvili, V. I. On the stability of solitary waves in weakly dispersive media. *Sov. Phys. Dokl.* 15: 539-541, 1970.

[8] Korteweg, D. J., and de Vries, G. On the Change of Form of Long Waves Advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves. *Philosophical Magazine* 39: 422-443, 1895.
[9] Sawada, K., and Kotera, T. A method for finding N-solitons of the KdV equation and KdV like equation. *Prog Theor Phys*. 51:1355-1367, 1974
[10] Segur, H., and Finkel, A. An analytical model of periodic waves in shallow water. *Stud. Appl. Math*. 73: 183-220, 1985.
[11] Infeld, E., and Rowlands, G. *Nonlinear waves, solitons and chaos*, Cambridge: Cambridge University Press, 2001
[12] Helal, M.A., and Mehanna, M.S. Tsunamis from nature to physics. *Chaos, Solitons and Fractals*. 36:787-796, 2008.
[13] Kolhatkar, R. *Grassmann Varieties*. McGill University: Master Thesis, 2004.
[14] Mikhailov, A.V., *Integrability*. Lecture Notes in Physics 767, Berlin: Springer-Verlag, 2009.
[15] Roelofs, G. H. M., and Martini R. Prolongation structure of the KdV equation in the bilinear form of Hirota. *Jour. Phys. A: Math. Gen*, 23:1877-1884, 1990.
[16] Krishnan, E. V. On Sawada-Kotera Equations. *IL Nuovo Cimento*, 92: 23-26, 1986.
[17] Changfu, L., and Zhengde, D. Exact Soliton Solutions for the fifth-order Sawada-Kotera equation. *Applied Mathematics and Computation*, 206: 272-275, 2008.
[18] Tingley, P., Notes on Fock Space. Preprint 16th November, 2011.
[19] Ariki, S., Representations of quantum algebras and combinatorics of Young tableaux, volume 26 of University Lecture Series. American Mathematical Society, Providence, RI, 2002
[20] Hirota, R., Soliton Equations as Dynamical System on a Infinite Dimensional Grassmann Manifolds. *RIMS Kokyuroku*, 439: 30-46, 1981.
[21] Anton, H., *Elementary Linear Algebra (Applications Version)* (9th ed.), London: Wiley International, 2005.
[22] Ali, N., and Aziz, Z.A., Hirota-Sato Formalism on Some Nonlinear Waves Equations. Preprint 25th February, 2012.