MATEMATIKA, 2009, Volume 25, Number 2, 167–176

A New Non-linear Multistep Method Based on Centroidal Mean in Solving Initial Value Problems

[1]**Nazeeruddin Yaacob & Teh Yuan Ying**
Department of Mathematics, Faculty of Science, Universiti Teknologi Malaysia
81310 UTM Skudai, Johor, Malaysia
e-mail: [1]ny@mel.fs.utm.my

Abstract A new 2-step fourth order implicit non-linear multistep method based on centroidal mean is considered in this paper. The new method is tested on some test problems; and numerical results show that the new method is able to produce acceptable numerical solutions for these test problems. Comparisons in terms of numerical accuracy between the new method and the classical 2-step Adams-Moulton method are carried out as well. Numerical experiments show that our new method performs better than the classical 2-step Adams-Moulton method in solving these test problems.

Keywords Initial value problems, non-linear multistep method; centroidal mean; Adams-Moulton method; 3-stage fourth order Lobatto IIIC method.

1 Introduction

Numerical methods from the class of linear multistep methods and the class of Runge-Kutta methods are defined by [1]

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{1}$$

and

$$\begin{aligned} y_{n+1} &= y_n + h \sum_{i=1}^{s} b_i k_i \\ k_i &= f\left(t_n + c_i h, y_n + h \sum_{j=1}^{s} a_{ij} k_j \right), \ i = 1, 2, \ldots, s. \end{aligned} \tag{2}$$

respectively. These methods are among the most common used numerical methods for the first order initial value problem of the form

$$y'(t) = f(t, y(t)), \quad y(a) = y_0, \quad t \in [a, b]. \tag{3}$$

A new research trend had emerged around the 1990's where researchers start to incorporate mean expressions into linear Runge-Kutta method in (2) to form a new kind of Runge-Kutta method that based on different kinds of means. This special type of Runge-Kutta method is considered as non-linear method due to the non-linear structures that arise from the implementation of various mean expressions. Articles which have discussed this type of method are such as [3] through [28].

In this article, we shall explore the possibility of deriving multistep method based on mean expressions for the numerical solution of (1). Our study is motivated by the success of [14] and [22] in deriving several non-linear multistep methods based on different mean expressions for the numerical solution of second order initial value problem. This article is organized as follows: In section 2, we present the procedure for obtaining the new 2-step implicit non-linear method based on centroidal mean. In Section 3, we present the local truncation error, consistency, zero-stability and convergence analysis of the new method. The stability polynomial and the regions of absolute stability for the new method are presented in Section 4. Section 5 shows the numerical implementations of the new implicit method to a variety of test problems and compares its performance with the classical 2-step implicit Adams-Moulton method in terms of numerical accuracy. Some remarks and conclusions will be given in Section 6.

2 Derivation of the 2-step Implicit Non-linear Method Based on Centroidal Mean

Firstly, we define the new 2-step implicit method as

$$\begin{aligned}\alpha_2 y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n = h \Bigg(& c_1 f_{n+2} + c_2 f_{n+1} + c_3 f_n + c_4 \tfrac{2\left(f_{n+2}^2 + f_{n+2} f_{n+1} + f_{n+1}^2\right)}{3(f_{n+2}+f_{n+1})} \\ & + c_5 \tfrac{2\left(f_{n+1}^2 + f_{n+1} f_n + f_n^2\right)}{3(f_{n+1}+f_n)} + c_4 \tfrac{2\left(f_{n+2}^2 + f_{n+2} f_n + f_n^2\right)}{3(f_{n+2}+f_n)} \Bigg)\end{aligned} \tag{4}$$

where $\alpha_0 = 0$, $\alpha_1 = -1$, $\alpha_2 = 1$ with c_1, c_2, c_3, c_4, c_5 and c_6 are constants that need to be determined. Note that $f_{n+2} + f_{n+1} \neq 0$, $f_{n+1} + f_n \neq 0$ and $f_{n+2} + f_n \neq 0$. On using Taylor series to expand both sides of equation (4) up to $O\left(h^4\right)$, and compare each coefficient, we obtain the following equations:

$$c_1 + c_2 + c_3 + c_4 + c_5 + c_6 = 1, \tag{5}$$

$$2c_1 + c_2 + \frac{3}{2}c_4 + \frac{1}{2}c_5 + c_6 = \frac{3}{2}, \tag{6}$$

$$2c_1 + \frac{1}{2}c_2 + \frac{5}{4}c_4 + \frac{1}{4}c_5 + c_6 = \frac{7}{6}, \tag{7}$$

$$\frac{1}{12}c_4 + \frac{1}{12}c_5 + \frac{1}{3}c_6 = 0, \tag{8}$$

$$-\frac{1}{8}c_4 - \frac{1}{24}c_5 - \frac{1}{3}c_6 = 0, \tag{9}$$

$$\frac{1}{4}c_4 + \frac{1}{12}c_5 + \frac{2}{3}c_6 = 0. \tag{10}$$

Using MATHEMATICA 5.0 in solving the system of equations given in (5) – (10), we obtain a set of solutions in terms of a free parameter c_6 shown as follows:

$$c_1 = \frac{5}{12} + \frac{c_6}{2}, \quad c_2 = \frac{2}{3} + 2c_6, \quad c_3 = -\frac{1}{12} + \frac{c_6}{2}, \quad c_4 = -2c_6, \quad c_5 = -2c_6.$$

On substituting these c_i, $i = 1, 2, \ldots, 6$ and $\alpha_0 = 0$, $\alpha_1 = -1$, $\alpha_2 = 1$ into equation (4), the resulting method is a 2-step implicit non-linear method based on centroidal mean:

$$\begin{aligned}y_{n+2} - y_{n+1} = h \Bigg(& \left(\tfrac{5}{12} + \tfrac{c_6}{2}\right) f_{n+2} + \left(\tfrac{2}{3} + 2c_6\right) f_{n+1} + \left(-\tfrac{1}{12} + \tfrac{c_6}{2}\right) f_n \\ & + (-2c_6)\, \tfrac{2\left(f_{n+2}^2 + f_{n+2} f_{n+1} + f_{n+1}^2\right)}{3(f_{n+2}+f_{n+1})} + (-2c_6)\, \tfrac{2\left(f_{n+1}^2 + f_{n+1} f_n + f_n^2\right)}{3(f_{n+1}+f_n)} \\ & + c_6 \tfrac{2\left(f_{n+2}^2 + f_{n+2} f_n + f_n^2\right)}{3(f_{n+2}+f_n)} \Bigg)\end{aligned} \tag{11}$$

Method (11) is named Non-linear Multistep method based on centroidal mean of two steps and fourth order or shortly NLMMCeM(2,4). The local truncation error in terms of c_6 for NLMMCeM(2,4) is given by

$$h^4\left(-\frac{1}{24}f_n'''\right)+h^5\left(\frac{c_6\left(f_n'\right)^4}{12\left(f_n\right)^3}-\frac{c_6\left(f_n'\right)^2f_n''}{12\left(f_n\right)^2}+\frac{c_6\left(f_n''\right)^2}{12f_n}-\frac{17}{360}f_n^{(4)}\right)+O\left(h^6\right). \tag{12}$$

Since there is a free parameter c_6, we choose this parameter so that the local truncation error shown in (12) is in $O\left(h^6\right)$. From (12), we force the first two terms to zero that is

$$h^4\left(-\frac{1}{24}f_n'''\right)+h^5\left(\frac{c_6\left(f_n'\right)^4}{12\left(f_n\right)^3}-\frac{c_6\left(f_n'\right)^2f_n''}{12\left(f_n\right)^2}+\frac{c_6\left(f_n''\right)^2}{12f_n}-\frac{17}{360}f_n^{(4)}\right)=0. \tag{13}$$

After some algebraic manipulations, c_6 is obtained as follows:

$$c_6=\frac{\left(f_n\right)^3\left(15f_n'''+17hf_n^{(4)}\right)}{30h\left(\left(f_n'\right)^4-f_n\left(f_n'\right)^2f_n''+\left(f_n\right)^2\left(f_n''\right)^2\right)} \tag{14}$$

where $\left(f_n'\right)^4-f_n\left(f_n'\right)^2f_n''+\left(f_n\right)^2\left(f_n''\right)^2\neq 0$. Note that c_6 is a constant since all functions $f_{n+j}^{(i)}$, $i=0,1,2,3,4,5$are evaluated at the point t_n. Therefore, the local truncation error for NLMMCeM(2,4) with c_6 in (14) is given by

$$\begin{aligned}&\mathrm{LTE}\,(\mathrm{Centroidal})\\&=\frac{h^5}{720}\left(-\frac{2\left(3f_n'\left(\left(f_n'\right)^2-f_nf_n''\right)^2+\left(f_n\right)^2\left(\left(f_n'\right)^2-2f_nf_n''\right)f_n'''\right)\left(15f_n'''+17hf_n^{(4)}\right)}{f_n\left(\left(f_n'\right)^4-f_n\left(f_n'\right)^2f_n''+\left(f_n\right)^2\left(f_n''\right)^2\right)}\right)\\&\quad-h^6\left(\frac{7}{240}f_n^{(5)}\right)\;+O\left(h^7\right).\end{aligned} \tag{15}$$

3 Consistency, Stability and Convergence Analysis for NLMMCeM(2,4)

We extend the theory of consistency, zero-stability and convergence for the linear multistep method to the new method NLMMCeM(2,4). As usual, the first characteristic polynomial, $\rho\left(\zeta\right)$ and the second characteristic polynomial of NLMMCeM(2,4), $\sigma\left(\zeta\right)$ can be obtained from the left-hand side and right-hand side of equation (11) respectively; with the substitution of $y_{n+j}=f_{n+j}=\zeta^j$ and $f_{n+j}^{(i)}=\zeta^j$ for $i=0,1,2,3,4,5$ and $j=0,1,2$. Therefore, we obtain

$$\rho\left(\zeta\right)=\zeta^2-\zeta \tag{16}$$

and

$$\begin{aligned}\sigma\left(\zeta\right)=&\left(\tfrac{5}{12}+\tfrac{c_6}{2}\right)\zeta^2+\left(\tfrac{2}{3}+2c_6\right)\zeta+\left(-\tfrac{1}{12}+\tfrac{c_6}{2}\right)+\left(-2c_6\right)\tfrac{2\left(\left(\zeta^2\right)^2+\zeta^2\times\zeta+\zeta^2\right)}{3\left(\zeta^2+\zeta\right)}\\&+\left(-2c_6\right)\tfrac{2\left(\zeta^2+\zeta\times1+1^2\right)}{3\left(\zeta+1\right)}+c_6\tfrac{2\left(\left(\zeta^2\right)^2+\zeta^2\times1+1^2\right)}{3\left(\zeta^2+1\right)}\end{aligned} \tag{17}$$

From the assumption $f_{n+j}^{(i)} = \zeta^j$ for $i = 0, 1, 2, 3, 4, 5$, we note that c_6 in (14) is evaluated at the point t_n and therefore we have $f_n^{(i)} = \zeta^0 = 1$ for $i = 0, 1, 2, 3, 4, 5$. On substituting $f_n^{(i)} = \zeta^0 = 1$ for $i = 0, 1, 2, 3, 4, 5$ into equation (14) yield

$$c_6 = \frac{\left(1^3\right)\left(15\left(1\right) + 17h\left(1\right)\right)}{30h\left(\left(1\right)^4 - 1 \times \left(1\right)^2 \times 1 + \left(1\right)^2\left(1\right)^2\right)} = \frac{15 + 17h}{30h}.$$

The first derivative of equation (16) is

$$\rho'\left(\zeta\right) = 2\zeta - 1. \tag{18}$$

On substituting $\zeta = 1$ into equations (16), (17) and (18) we obtain the following results:

$$\rho\left(1\right) = 0, \quad \sigma\left(1\right) = 1 \quad \text{and} \quad \rho'\left(1\right) = 1. \tag{19}$$

Since conditions in (19) hold for NLMMCeM(2,4), then we can say that it is consistent.

To determine the zero-stability of NLMMCeM(2,4), we must make sure that no root of $\rho\left(\zeta\right)$ has modulus greater than one, and every root with modulus one is simple. Therefore, from (16), the roots of

$$\zeta^2 - \zeta = 0$$

are $\zeta_1 = 1$ and $\zeta_2 = 0$. Consequently, we have $|\zeta_1| = 1$ and $|\zeta_2| = 0$ which are not greater than one and simple. In view of this, we can say that NLMMCeM(2,4) is zero-stable.

Finally, we can claim that NLMMCeM(2,4) is convergent because it is shown to be consistent and zero-stable.

4 Absolute Stability of NLMMCeM(2,4)

In order to carry out the stability analysis for NLMMCeM(2,4), we must obtain the stability polynomial and its corresponding regions of absolute stability. We can obtain the stability polynomial of NLMMCeM(2,4) by applying the Dahlquist's test equation $y' = \lambda y$ to equations (11) and (14) [2]. Note that λ is a complex constant with negative real part. On substituting (14) into (11) and then substituting $f_{n+2} = \lambda y_{n+2}$, $f_{n+1} = \lambda y_{n+1}$, $f_n = \lambda y_n$, $f_n' = \lambda^2 y_n$, $f_n'' = \lambda^3 y_n$, $f_n''' = \lambda^4 y_n$, $f_n^{(4)} = \lambda^5 y_n$, $y_{n+2} = \zeta^2$, $y_{n+1} = \zeta$ and $y_n = 1$ into (11), we obtain the following stability polynomial for NLMMCeM(2,4) as follows:

$$\left(195 - 58z\right)\zeta^4 - \left(240 + 188z\right)\zeta^3 + \left(270 + 42z\right)\zeta^2 - \left(240 + 188z\right)\zeta + \left(15 + 32z\right) = 0 \tag{20}$$

where $z = h\lambda$. Here, ζ can be interpreted as the characteristic roots of the difference equation (11). The condition for the stability is that the roots of (20) i.e. ζ are all of absolute value less than 1. By taking z as complex number i.e. $z = x + \mathrm{i}y$, we plot the region which satisfies the condition that all roots of (20) are of absolute value less than 1 in Figure 1. The shaded region in Figure 1 is the region which satisfies the condition that all roots of (20) are of absolute value less than 1. Consequently, the shaded region is the region of absolute stability of NLMMCeM(2,4).

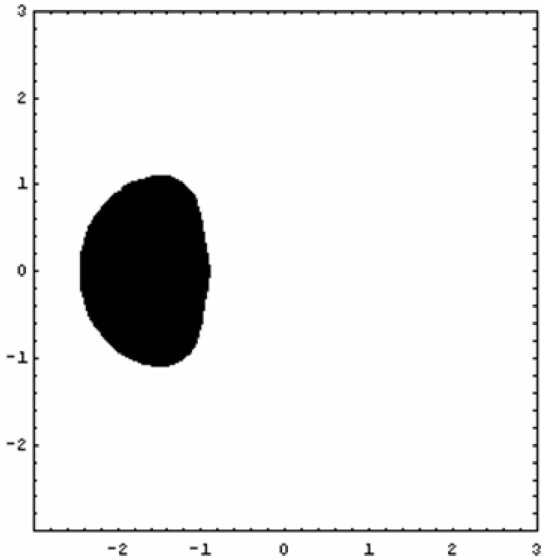

Figure 1: Stability Region of NLMMCeM(2,4)

5 Numerical Experiments and Comparisons

In this section, NLMMCeM(2,4) is used to solve some test problems in order to check its reliability and accuracy. We present i) the maximum absolute error over the integration interval given by

$$\max_{0 \leq n \leq N} \{|y(t_n) - y_n|\}$$

where N is the number of integration steps; and ii) the absolute error at the end-point of integration interval given by $|y(t_n) - y_N|$ for each test problem. Note that $y(t_n)$ represents the exact solution of a test problem at point t_n, while y_n is the approximations of the exact solution at point t_n of a test problem. The notation 1.26681(-5) indicates 1.26681×10^{-5}. Numerical results obtained using NLMMCeM(2,4) is compared with the numerical results obtained using the classical 2-step implicit Adams-Moulton method given by [29]

$$y_{n+2} - y_{n+1} = h\left(\frac{5}{12}f_{n+2} + \frac{2}{3}f_{n+1} - \frac{1}{12}f_n\right) \tag{21}$$

where the local truncation error of (21) is

$$\mathrm{LTE}\,(\mathrm{Adams} - \mathrm{Moulton}) = h^4\left(-\frac{1}{24}f_n'''\right) + h^5\left(-\frac{17}{360}f_n^{(4)}\right) + O\left(h^6\right). \tag{22}$$

The starting values y_1 for NLMMCeM(2,4) and 2-step Adams-Moulton are computed via the 3-stage fourth order Lobatto IIIC method shown in the following Butcher tableau [30]:

$$
\begin{array}{c|ccc}
0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\
\frac{1}{2} & \frac{1}{6} & \frac{5}{12} & -\frac{1}{12} \\
1 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\
\hline
 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6}
\end{array}
$$

Problem 1: [17]

$$y'(t) = y(t) - t^2 + 1, \quad y(0) = \frac{1}{2}, \quad t \in [0,1].$$

The exact solution for Problem 1 is given by $y(t) = (t+1)^2 - \frac{1}{2}e^t$.

Problem 2: [24]

$$y'(t) = y(t)\cos t, \quad y(0) = 1, \quad t \in [0,1].$$

The exact solution for Problem 2 is given by $y(t) = e^{\sin(t)}$.

Problem 3: [31]

$$\begin{aligned} y_1'(t) &= 0.2y_2(t), \quad y_1(0) = 1, \\ y_2'(t) &= -0.2y_1(t), \quad y_2(0) = 1. \end{aligned}$$

Problem 3 is solved numerically over the integration interval $t \in [0,1]$ and the exact solutions for Problem 3 are given by $y_1(t) = \cos 0.2t + \sin 0.2t$ and $y_2(t) = -\sin 0.2t + \cos 0.2t$.

Table 1 to Table 8 show that NLMMCeM(2,4) has no difficulty in solving all the test problems mentioned above; and it performs better than 2-step Adams-Moulton for different step length for Problem 1 and Problem 2. On the other hand, NLMMCeM(2,4) gives comparable accuracy to 2-step Adams-Moulton method in solving Problem 3 which is a system of first order differential equations.

Table 1: Maximum Absolute Error for Problem 1
With Respect to Step Length, h

h	Adams-Moulton	NLMMCeM(2,4)
1/16	1.26681(−05)	8.64014(−07)
1/32	1.65514(−06)	5.55726(−08)
1/64	2.11439(−07)	5.17533(−09)
1/128	2.67161(−08)	4.32471(−10)

Table 2: Error at the End-point for Problem 1
With Respect to Step Length, h

h	Adams-Moulton	NLMMCeM(2,4)
1/16	1.26681(−05)	8.64014(−07)
1/32	1.65514(−06)	5.55726(−08)
1/64	2.11439(−07)	5.17533(−09)
1/128	2.67161(−08)	4.32471(−10)

Table 3: Maximum Absolute Error for Problem 2
With Respect to Step Length, h

h	Adams-Moulton	NLMMCeM(2,4)
1/16	6.12206(-05)	1.36354(-05)
1/32	7.90321(-06)	9.23572(-07)
1/64	9.98723(-07)	9.06451(-08)
1/128	1.24430(-07)	9.36788(-09)

Table 4: Error at the End-point for Problem 2
With Respect to Step Length, h

h	Adams-Moulton	NLMMCeM(2,4)
1/16	6.12206(−05)	1.36354(−05)
1/32	7.90321(−06)	9.23572(−07)
1/64	9.98723(−07)	9.06451(−08)
1/128	1.24430(−07)	9.36788(−09)

Table 5: Maximum Absolute Error for Problem 3
With Respect to Step Length, $h(y_1(t))$

h	Adams-Moulton	NLMMCeM(2,4)
1/16	4.63483(−06)	4.63637(−06)
1/32	1.15738(−06)	1.15747(−06)
1/64	2.89350(−07)	2.89355(−07)
1/128	7.23379(−08)	7.23382(−08)

Table 6: Error at the End-point for Problem 3
With Respect to Step Length, $h(y_1(t))$

h	Adams-Moulton	NLMMCeM(2,4)
1/16	4.52400(−06)	4.59524(−06)
1/32	1.13190(−06)	1.14109(−06)
1/64	2.83275(−07)	2.8444(−07)
1/128	7.08570(−08)	7.10037(−08)

Table 7: Maximum Absolute Error for Problem 3
With Respect to Step Length, $h(y_2(t))$

h	Adams-Moulton	NLMMCeM(2,4)
1/16	9.37627(−07)	9.08091(−07)
1/32	2.32079(−07)	2.28239(−07)
1/64	5.77552(−08)	5.72659(−08)
1/128	1.44052(−08)	1.43435(−08)

Table 8: Error at the End-point for Problem 3
With Respect to Step Length, $h(y_2(t))$

h	Adams-Moulton	NLMMCeM(2,4)
1/16	9.37627(−07)	9.08091(−07)
1/32	2.32079(−07)	2.28239(−07)
1/64	5.77552(−08)	5.72659(−08)
1/128	1.44052(−08)	1.43435(−08)

6 Conclusions

We have presented a new 2-step fourth order non-linear multistep method based on centroidal mean (NLMMCeM(2,4)), that is suitable to solve first order initial value problems. Classical 2-step Adams-Moulton method is a third order method, but NLMMCeM(2,4) can achieved fourth order of accuracy by choosing the appropriate parameter c_6. This new method is shown to be consistent, zero-stable and convergent. Numerical results presented in Section 5 also suggest that NLMMCeM(2,4) is suitable to solve both single differential equation and systems of first order differential equations.

Acknowledgement

The authors would like to thank Universiti Teknologi Malaysia (UTM) for providing fund under the grant vot 78337 to conduct this research.

References

[1] J. D. Lambert, *Numerical Methods for Ordinary Differential Systems*, John Wiley & Sons, Chichester, 1991.

[2] J.D. Lambert, *Computational Methods in Ordinary Differential Equations*, John Wiley & Sons, London, 1973.

[3] D.J. Evans & B. B. Sanugi, *A comparison of numerical ODE solvers based on arithmetic and geometric means*, Int. J. Comput. Math. 23(1987), 37–62.

[4] D. J. Evans & B. B. Sanugi, *A nonlinear Runge-Kutta formula for initial value problems*, SIGNUN Newsletter 22 (3)(1987), 27 – 30.

[5] B.B. Sanugi, *A class of nonlinear methods based on Euler's formulae for IVP and its extension to the development of a new Runge-Kutta formula*, International Conference on Numerical Mathematics, Singapore, 1988.

[6] B.B. Sanugi & D. J. Evans, *A new fourth order Runge-Kutta formula for* $y' = Ay$ *with stepsize control*, Comput. Math. Appl. 15 (23)(1988), 991–995.

[7] D. J. Evans & B. B. Sanugi, *The use of nonlinear trapezoidal formulae with extrapolation for accurate solutions of initial value ODEs*, Int. J. Comput. Math. 27(1989), 199–217.

[8] B.B. Sanugi & D. J. Evans, *New predictor-corrector trapezoidal formulae for solving initial value problems*, Int. J. Comput. Math. 33(1990), 77–94.

[9] B.B. Sanugi, *A strategy for step size control in the implementation of Runge-Kutta methods for solving linear autonomous system of ODEs*, Second International Conference on Industrial and Applied Mathematics, Washing D. C., USA, 1991.

[10] N. Yaacob & D. J. Evans, *The Parallel Solution of IVP by RK Heronian Mean Method with Extrapolation*, Computer Studies Report 904, June, 1994, Loughborough University of Technology.

[11] D.J. Evans & N. Yaacob, *A fourth order Runge-Kutta method based on the Heronian mean formula*, Int. J. Comput. Math. 58(1995), 103–115.

[12] N. Yaacob & D. J. Evans, *A New 3-stage Fourth Order RK Composite Method for Solving Initial Value Problems*, Computer Studies Report 994, September, 1995, Loughborough University of Technology.

[13] B.B. Sanugi & N. Yaacob, *A New Fifth Order Five-stage Runge-Kutta Method for Initial Value Type Problems in ODEs*, Laporan Teknik Matematik M/LT No. 021, Mei, 1995, Universiti Teknologi Malaysia.

[14] B.B. Sanugi & N. Yaacob, *A Nonlinear Multistep Method for Second Order Differential Equations with Periodic Solution*, Laporan Teknik Matematik M/LT No. 028, November, 1995, University Teknologi Malaysia.

[15] N. Yaacob & B. B. Sanugi, *Strategi Pengawalan Ralat Dengan Kaedah Runge-Kutta Berasaskan Min Heronian Bagi Sistem Tak Kaku Dalam Masalah Nilai Awal*, Laporan Teknik Matematik M/LT No. 020, Februari, 1995, Universiti Teknologi Malaysia.

[16] N. Yaacob & B. B. Sanugi, *Penganggaran dan Pengawalan Ralat Dalam Kaedah HaM-RK(4) Untuk Penyelesaian Masalah Nilai Awal*, Laporan Teknik Matematik M/LT No. 023, Julai, 1995, Universiti Teknologi Malaysia.

[17] N. Yaacob & B. B. Sanugi, *A New 3-stage Fourth Order, RK-NHM34 Method for Solving* $y' = f(x, y)$, $y(x_0) = y_0$. Laporan Teknik Matematik M/LT No. 029, November, 1995, Universiti Teknologi Malaysia.

[18] N. Yaccob & B. B. Sanugi, *A New Fifth-Order Five-Stage Explicit Ham-RK5(5) Method for Solving Initial Value Problems in ODE's*, Laporan Teknik Matematik M/LT No. 034, Disember, 1995, Universiti Teknologi Malaysia.

[19] N. Yaacob, *New nonlinear Runge-Kutta methods for solving initial value problems*, Ph.D. thesis, Universiti Teknologi Malaysia, 1996.

[20] N. Yaacob & D. J. Evans, *A new fifth order explicit Runge-Kutta method with four stages for solving initial value problems in ODEs*, Int. J. Comput. Math. 65 (1997), 141–147.

[21] N. Yaacob & B. B. Sanugi, *A New Fourth-Order Embedded Method Based on the Harmonic Mean*, Matematika, 14(1998), 1–6.

[22] N. Yaacob & P. Chang, *New Nonlinear Multistep Metod Based On Contraharmonic Mean For Problems With Periodic Solutions*, Laporan Teknik Matematik LT/M Bil. 4/2001, Ogos, 2001, Universiti Teknologi Malaysia.

[23] D. J. Evans & A. R. Yaakub, *A fifth order Runge-Kutta RK(5,5) method with error control*, Int. J. Comput. Math. 79(11) (2002), 1179–1185.

[24] K. Murugesan, D. P. Dhayabaran, E. C. H. Amirtharaj & D. J. Evans, *A fourth order embedded Runge-Kutta RKACeM(4,4) method based on arithmetic and centroidal means with error control*, Int. J. Comput. Math., 79 (2(2002), 247–269.

[25] A. R. Yaakub & D. J. Evans, *New L-stable modified trapezoidal methods for the initial value problems*, Int. J. Comput. Math. 80(1) (2003), 95–104.

[26] N. Bi̧ldi̧k & M. Î̇nç, *On the numerical solution of initial value problems for nonlinear trapezoidal formulas with different types*, Int. J. Comput. Math., 80(1)(2003), 1175–1188.

[27] R. Ponalagusamy & S. Senthilkumar, *A Comparison of RK-Fourth Orders of Variety of Means and Embedded Means on Multilayer Raster CNN Simulation*, Journal of Theoretical and Applied Information Technology, 3(4) (2007), 7–14.

[28] S. Senthilkumar, *RK Starters for Multistep Methods on Hole-Filler CNN Simulation*, Journal of Theoretical and Applied Information Technology, 4(2)(2008), 170–177.

[29] J.D. Faires & R. Burden, *NumericalMethods*, 3rd ed., Thomson Learning, California, 2003.

[30] J.C. Butcher, *Numerical Methods for Ordinary Differential Equations*, John Wiley & Sons, West Sussex, 2003.

[31] N.D. Buono & C. Mastroserio, *Explicit methods based on a class of four stage fourth order Runge-Kutta methods for preserving quadratic laws*, J. Comput. Appl. Math., 140(2002), 231–243.