A STUDY OF THE BOUSSINESQ EQUATION AS A WAVE PROPAGATION IN SHALLOW WATER

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ABSTRACT

Shallow water waves have been expressed as a couple of equations by Whitham (1967). Many researchers have continued studies in this field by deriving the so-called Boussinesq equations. A straight forward derivation from the Whithams' shallow water equations shall immediately produce a coupled form of Boussinesq equation. In this paper we use the Lagrange coordinates in order to derive the single Boussinesq equation to represent wave motion on the surface of shallow water. We shall also discuss its solution by using the Hirota bilinear method and the KP hierarchy of equations.

ABSTRAK

Gelombang air cetek telah diungkapkan sebagai pasangan persamaan oleh Witham(1967). Ramai penyelidik telah meneruskan kajian di dalam bidang ini dengan menerbitkan persamaan yang dikenali sebagai persamaan Boussinesq. Penerbitan secara langsung daripada persamaan air cetek Witham akan terus menghasilkan Persamaan Boussinesq dalam bentuk berpasangan. Dalam kertas kerja ini, kita menggunakan koordinat Lagrange untuk menerbitkan persamaan Boussinesq tunggal yang mewakili pergerakan gelombang pada permukaan air cetek. Kita juga akan membincangkan penyelesaiannya dengan menggunakan kaedah bilinear Hirota dan hierarki KP.

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CHAPTER I

INTRODUCTION

1.1 Introduction

Standing on a beach and watching the waves roll in and break, we might guess that water is moving bodily towards the shore. But no water is pilling up on the beach. This phenomena has been the subject of study for many researchers since a few centries ago.

Study of wave propagation in fluids has become very important since a very long time ago. Witham(1974) has obtained a set of equations which describes a wave propagation in a shallow fluid. Since the model is a wave propagation in a shallow water, one would expect soliton solutions for such equations.

Many researchers have shown interest in this field. Among them are Groesen & Pudjaprasetya(1993), Hirota(1985), Nimmo & Freeman (1983), and others. Most of them dealt with coupled equations. In this project, we shall try to work with a single equation.

1.2 Objectives

1. To derive the Boussinesq equation from the shallow water wave equations.
2. To obtain the solution of the Boussinesq equation.

1.3 Methodology

1. Derivation of a new equation for waves propagating on surface of shallow water.
2. To use the obtained equation in order to describe the wave phenomena at the surface of shallow water.

1.4 Scope

The research will focus on the theoretical computation of the waves propagating on the surface of shallow fluids. The solution will be expressed in terms of some wronskian determinants. We shall also examine the properties of the solutions.

CHAPTER II

LITERATURE REVIEW

Most expositions of soliton theory outline the history of the Korteweg de Vries (KdV) equation, beginning with the physical observation of Scott Russell of a bow wave in a canal in 1834. The equations were first written down by Boussinesq in 1871 [Chen M. (2000)] and in 1895 by Korteweg and de Vries. In addition to describing water waves, the KdV equation also arises as a universal limit of lattice vibrations as the spacing goes to zero. The surprising numerical experiments of Fermi, Pasta and Ulam in 1955 on an anharmonic lattice and the ingenious explanation by Zabusky and Kruskal in 1965 in terms of solitons of the KdV equation were quickly followed by a ground-breaking paper of Gardner, Greene, Kruskal and Miura in 1967, which introduced the method of solving KdV using the inverse-scattering transform for the Hill's operator. This brings us into the modern era.

Several members of the Boussinesq System have been studied in the past, including the classical Boussinesq System. Peregrine (1972) consider the equations for water waves and the approximation behind then in wave on beaches and resulting sediment transport.

Then it followed by Schonbek (1981) , obtains the existence of solutions for the Boussinesq System of equations. In 1982, the exact solution of the classical Boussinesq equation was presented by Krishnan.

Manorajan, Mitchell and J. Morris (1984) expressed the numerical solution of the good Boussinesq equation by using Galerkin Methods. Then it followed quickly by Amick (1984) which comes out with the regularity and uniqueness of solutions to the Boussinesq system of equations. In 1999, an overview of the system was presented by

Bona, Chen and Saut, which discussed about Boussinesq equations for small-amplitude long wavelength water waves.

In 1998, Chen was presents an exact traveling-wave solution of Boussinesq systems. He obtained it is suffice to find a solution of an ordinary differential equation and the solution of the ordinary differential equation in a prescribed form can be found by solving a system of nonlinear algebraic equation.

Prabir Daripa (1999) obtained numerically and theoretically that the regularized equation admits non-local solitary wave solutions with oscillating tails at infinity by using dispersive regularization.

The existence of solitary wave solution with any phase speed $k>1$ was obtained by Min Chen (1999). He describe numerical method for searching multi-pulsed solutions and apply the method to the regularized Boussinesq system.

Yang Lei and Yang Kongqing (1999) obtain two kings of analytic singular solutions (finite-time and infinite-time singular solutions) of classical Boussinesq equation by using the improved homogenous balance (HB) method and the invariant-Backlund Transformation based on a special nonlinear transformation. An particular, a finite-time singular solution of the Boussinesq equation is obtained, which was produced from a non-singular physical field in the process of time evolution.

Boussinesq equation as a zero curvature representation of some third order linear differential equation and factorizing this linear differential equation was shown by M.A Jafarizadeh et. al (2001). They obtain the hierarchy of solution of the Boussinesq equation from the eigen spectrum of constant potentials.

S.R Pudjaprasetya et. al (2002) consider Boussinesq equation that describe wave elevation η and horizontal velocity u of the fluid particles at the surface. A comparison is made between the solitary waves from the two models (Boussinesq equation and KdV equation). The decoupled equation is used to describe the solitary wave splitting due to decreasing depth say from h_0 to h_1. By demanding the conserved quantities: mass and energy to be conserve during evolution, they were found the amplitude of two-soliton above h_1. This result agrees with result obtained using KdV model.

The derivation of four-parameter family of Boussinesq system from the two-dimension Euler equation for free surface flow was obtained by J.L Bone et. al (2002) and followed by Clyde M.D (2003) was presents the form of a "near-general" analytic solution for the Boussinesq equation. Obtained that only one characteristic function that satisfies the equation when analyticity is required. The solution developed is analytic in the classical complex variable sense (i.e is continous and single-valued in the region of interest) and is conservative in physics terminology.

Prabir Daripa et. al (2003) derived a class of model equation that describe the bi-directional propagation of small amplitude long waves on the surface of shallow water. The traveling solitary wave solutions are explicitly constructed for a class of lower order Boussinesq equation of higher-order, the appropriate equation to model solitary waves are derived under appropriate scaling in two specific cases : (i) $\beta \leq \left(\frac{1}{3}-\tau\right) \leq \frac{1}{3}$ and (ii) $\left(\frac{1}{3}-\tau\right) = O(\beta)$.

The derivation of the Boussinesq equation with the recursion form not only appearing in the main variables but in the coefficients was shown by C.H Kong and C.M Liu (2004). Parameters concerning the linear and nonlinear wave are also derived and by choosing a suitable water depth parameter, *m*, the optimal wave models are consequently determined. The model provides and easier and more flexible method to analyze the wave mechanics than previous studies based on the Pade approximation.

Baldwin et. al (2004) expressed the solutions of Boussinesq equation as polynomials of the hyperbolic tangent functions. They found that the tanh-method provides a straightforward algorithm to compute such particular solution for a large class of nonlinear PDE.

Tzirtzilakis et. al (2004) applied a combination of Fourier Spectral methods in space and finite difference in time to Boussinesq equation. They showed the interactions numerically and investigate their stability properties by varying the velocity parameter of the wave which appears in their analytical form.

CHAPTER III

MATHEMATICAL MODELLING

3.1 Introduction

In this paper, we shall use the coupled shallow water wave equations,

$$\eta_t + \{(1+\alpha\eta)\omega\}_x - \frac{\beta}{6}\omega_{xxx} + O(\alpha\beta, \beta^2 = 0), \tag{1}$$

$$\omega_t + \alpha\omega\omega_x + \eta_x - \frac{\beta}{2}\omega_{xxt} + O(\alpha\beta, \beta^2) = 0 \tag{2}$$

in order to derive the BE equation

$$u_{tt} - u_{xx} - 6(u^2)_{xx} - u_{xxxx} = 0. \tag{3}$$

We note that Eq.(1) and Eq.(2) were first derived by Witham (1974). The problem of finding a solution for Eq.(3) were initiated by Prabir Daripa (1999). He found a solitary wave solution,

$$u(x,t) = A\sec h^2\{A/6(x-ct)\}, \tag{4}$$

where A is the amplitude of the wave, and $c = \pm\sqrt{1 + 2A/3}$ is its speed. We shall also make use of the Hirota's bilinear method and the KP hierarchy of equations in order to find a relation between solution parameters p and q.

3.2 The Derivation

By using the chain rule, the x and t derivatives respectively are

$$\frac{\partial}{\partial x} = \frac{\partial X}{\partial x}\frac{\partial}{\partial X} + \frac{\partial T}{\partial x}\frac{\partial}{\partial T}$$

and

$$\frac{\partial}{\partial t} = \frac{\partial X}{\partial t}\frac{\partial}{\partial X} + \frac{\partial T}{\partial t}\frac{\partial}{\partial T}.$$

For our purpose in this paper, we use the transformation for U, X and T as below:

$$\left.\begin{aligned} U &= \eta - \alpha\eta^2 + O(\alpha^2), \\ X &= x + \alpha\int_{-\infty}^{X} \eta(x,t)dx \\ T &= t \end{aligned}\right\} \quad \text{and} \tag{5}$$

where X and T are in the form of Lagrangian coordinates.

Based on the chain rule, the derivatives for the last two equations of Eq.(5) are

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial x}\left[x + \alpha\int_{-\infty}^{X} \eta(x,t)dx\right]\frac{\partial}{\partial X} + \frac{\partial}{\partial x}(t)\frac{\partial}{\partial T} \\ &= (1+\alpha\eta)\frac{\partial}{\partial x}, \qquad \text{since}\ \frac{\partial T}{\partial x} = 0, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial}{\partial t}\left[x + \alpha\int_{-\infty}^{X} \eta(x,t)dx\right]\frac{\partial}{\partial X} + \frac{\partial}{\partial t}(t)\frac{\partial}{\partial T}. \\ &= \left[\alpha\int_{-\infty}^{X} \eta_t(x,t)dx\right]\frac{\partial}{\partial X} + (1)\frac{\partial}{\partial T}. \end{aligned}$$

In the following calculation, we shall only keep terms with $O(1), O(\alpha)$, and $O(\beta)$.

From Eq. (1),

$$\eta_t = -\{(1+\alpha\eta)\omega\}_x + \frac{\beta}{6}\omega_{xxx}$$
$$= -\omega_x + O(\alpha) + O(\beta)$$

Hence, we can write,

$$\frac{\partial}{\partial t} = \left(\alpha \int_{-\infty}^{X} -\omega_x dx\right)\frac{\partial}{\partial X} + \frac{\partial}{\partial T}$$
$$= -\alpha\omega\frac{\partial}{\partial X} + \frac{\partial}{\partial T}.$$

Now, the x and t derivatives are

and
$$\left.\begin{aligned} \frac{\partial}{\partial x} &= (1+\alpha\eta)\frac{\partial}{\partial X} \\ \frac{\partial}{\partial t} &= \frac{\partial}{\partial T} - \alpha\omega\frac{\partial}{\partial X} \end{aligned}\right\}. \tag{6}$$

Considering Eq,(1), the terms needed are

$$\eta_t = \frac{\partial\eta}{\partial t} = \frac{\partial\eta}{\partial T} - \alpha\omega\frac{\partial\eta}{\partial x} = \eta_T - \alpha\omega\eta_X,$$

$$[(1+\alpha\eta)\omega]_x = \frac{\partial}{\partial x}[(1+\alpha\eta)\omega] = (1+\alpha\eta)\frac{\partial}{\partial X}[(1+\alpha\eta)\omega] = (1+\alpha\eta)[(1+\alpha\eta)\omega]_X,$$

and

$$\omega_{xxx} = \frac{\partial^3\omega}{\partial x^3} = (1+\alpha\eta)\frac{\partial^3\omega}{\partial X^3} = (1+\alpha\eta)\omega_{XXX}.$$

Therefore, eq (1) now can be written as

$$(\eta_T - \alpha\omega\eta_X) + (1+\alpha\eta)\{(1+\alpha\eta)\omega\}_X - \frac{\beta}{6}(1+\alpha\eta)\omega_{XXX} = 0.$$

Note that

$$\{(1+\alpha\eta)\omega\}_X = (1+\alpha\eta)_{.X}\,\omega + (1+\alpha\eta)\omega_X$$
$$= \alpha\omega\eta_X + \omega_X + \alpha\eta\omega_X .$$

Then, Eq. (1) becomes

$$\eta_T - \alpha\omega\eta_X + (1+\alpha\eta)(\alpha\omega\eta_X + \omega_X + \alpha\eta\omega_X) - \frac{\beta}{6}(1+\alpha\eta)\omega_{XXX} = 0,$$

which then yields

$$\eta_T - \alpha\omega\eta_X + \alpha\omega\eta_X + \omega_X + \alpha\omega_X\eta + \alpha\omega_X\eta - \frac{\beta}{6}\omega_{XXX} + O(\alpha\beta, \alpha^2) = 0.$$

We can also write

$$\eta_T + \omega_X + 2\alpha\eta\omega_X - \frac{\beta}{6}\omega_{XXX} = 0.. \qquad (7)$$

Differentiating Eq.(7) with respect to T gives

$$\frac{\partial}{\partial T}\left(\eta_T + \omega_X + 2\alpha\eta\omega_X - \frac{\beta}{6}\omega_{XXX}\right) = \frac{\partial}{\partial T}(0),$$

which is

$$\eta_{TT} + \omega_{XT} + 2\alpha(\eta\omega_X)_T - \frac{\beta}{6}\omega_{XXXT} = 0. \qquad (8)$$

Now, consider Eq.(2). Listed below are the terms needed

$$\omega_t = \frac{\partial\omega}{\partial t} = \frac{\partial\omega}{\partial T} - \alpha\omega\frac{\partial\omega}{\partial X} = \omega_T - \alpha\omega\omega_X ,$$

$$\alpha\omega\omega_x = \alpha\omega\frac{\partial\omega}{\partial x} = \alpha\omega\left[(1+\alpha\eta)\frac{\partial\omega}{\partial X}\right] = \alpha\omega[(1+\alpha\eta)\omega_X] = \alpha\omega\omega_X + O(\alpha^2),$$

$$\eta_x = \frac{\partial\eta}{\partial x} = (1+\alpha\eta)\frac{\partial\eta}{\partial X} = (1+\alpha\eta)\eta_X ,$$

and

$$\frac{\beta}{2}\omega_{xxt}=\frac{\beta}{2}\frac{\partial}{\partial t}\left(\frac{\partial^2\omega}{\partial x^2}\right)=\frac{\beta}{2}\frac{\partial}{\partial t}\left[(1+\alpha\eta)\frac{\partial^2\omega}{\partial X^2}\right]$$

$$=\frac{\beta}{2}\left[\frac{\partial}{\partial T}(1+\alpha\eta)\omega_{XX}-\alpha\omega\frac{\partial}{\partial X}(1+\alpha\eta)\omega_{XX}\right]$$

$$=\frac{\beta}{2}\frac{\partial}{\partial T}\omega_{XX}+O(\alpha\beta,\alpha^2)$$

$$=\frac{\beta}{2}\omega_{XXT}.$$

Therefore Eq.(2) can be written as

$$\omega_T-\alpha\omega\omega_X+\alpha\omega\omega_X+(1+\alpha\eta)\eta_X-\frac{\beta}{2}\left(\frac{\partial}{\partial T}-\alpha\omega\frac{\partial}{\partial X}\right)[(1+\alpha\eta)\omega_{XX}]+O(\alpha\beta,\beta^2)=0$$

or

$$\omega_T+\eta_X+\alpha\eta\eta_X-\frac{\beta}{2}\omega_{XXT}+O(\alpha\beta,\beta^2)=0$$

which then reduces to

$$\omega_T+\eta_X+\frac{\alpha}{2}(\eta^2)_X-\frac{\beta}{2}\omega_{XXT}=0. \qquad (9)$$

The derivatives of ω are calculated from Eq. (7) and Eq.(9). From Eq.(7),

$$\omega_X=-\eta_T-2\alpha\eta\omega_X+\frac{\beta}{6}\omega_{XXX}$$

$$=-\eta_T-2\alpha\eta\left(-\eta_T-2\alpha\eta\omega_X+\frac{\beta}{6}\omega_{XXX}\right)+\frac{\beta}{6}\left(-\eta_T-2\alpha\eta\omega_X+\frac{\beta}{6}\omega_{XXX}\right)_{XX}$$

$$=-\eta_T-2\alpha\eta(-\eta_T)+\frac{\beta}{6}(-\eta_T)_{XX}+O(\alpha\beta,\alpha^2,\beta^2)$$

$$=-\eta_T+2\alpha\eta\eta_T-\frac{\beta}{6}\eta_{XXT}+O(\alpha\beta,\alpha^2,\beta^2)$$

While from Eq.(9), we have

$$\begin{aligned}\omega_T &= -\eta_X - \frac{\alpha}{2}\left(\eta^2\right)_X + \frac{\beta}{2}\omega_{XXT} \\ &= -\eta_X - \frac{\alpha}{2}\left(\eta^2\right)_X + \frac{\beta}{2}(\omega_T)_{XX} \\ &= -\eta_X - \frac{\alpha}{2}\left(\eta^2\right)_X + \frac{\beta}{2}\left(-\eta_X - \frac{\alpha}{2}\left(\eta^2\right)_X + \frac{\beta}{2}\omega_{XXT}\right)_{XX} \\ &= -\eta_X - \frac{\alpha}{2}\left(\eta^2\right)_X + \frac{\beta}{2}(-\eta_X)_{XX} + O(\alpha\beta, \alpha^2, \beta^2) \\ &= -\eta_X - \frac{\alpha}{2}\left(\eta^2\right)_X - \frac{\beta}{2}\eta_{XXX} + O(\alpha\beta, \alpha^2, \beta^2)\end{aligned}$$

Therefore,

$$\omega_{XT} = (\omega_T)_X = -\eta_{XX} - \frac{\alpha}{2}\left(\eta^2\right)_{XX} - \frac{\beta}{2}\eta_{XXXX}.$$

Also note that,

$$\omega_{XXXT} = -\eta_{XXXX} + O(\alpha, \beta).$$

Now, substituting all the required derivatives of ω into Eq.(8), we find

$$\eta_{TT} - \eta_{XX} - \frac{\alpha}{2}\left(\eta^2\right)_{XX} - \frac{\beta}{2}\eta_{XXXX} + 2\alpha[\eta(-\eta_T)]_T - \frac{\beta}{6}(-\eta_{XXXX}) + O\left(\alpha\beta, \beta^2\right) = 0.$$

or

$$\eta_{TT} - 2\alpha(\eta\eta_T)_T - \eta_{XX} - \frac{\alpha}{2}\left(\eta^2\right)_{XX} - \frac{\beta}{3}\eta_{XXXX} = 0. \tag{10}$$

Next, from Eq.(5)

$$\begin{aligned}\eta &= U + \alpha\eta^2 \\ &= U + \alpha(U + \alpha\eta^2)^2 \\ &= U + \alpha U^2 + O\alpha^2\end{aligned} \tag{11}$$

Also from Eq. (5),

$$U_{TT} = \eta_{TT} - \alpha\left(\eta^2\right)_{TT} + O\left(\alpha^2\right)$$

or

$$\eta_{TT} - \alpha\left(\eta^2\right)_{TT} = U_{TT} + O\left(\alpha^2\right). \tag{12a}$$

While from Eq.(11), we have

$$\eta_{XX} = U_{XX} + \alpha\left(U^2\right)_{XX} + O\left(\alpha^2\right). \tag{12b}$$

So

$$(\eta^2)_{XX} = \left(U^2\right)_{XX} + O(\alpha), \tag{12c}$$

and

$$\eta_{XXXX} = U_{XXXX} + O(\alpha). \tag{12d}$$

Using Eq.(12a), (12b), (12c), and (12d), Eq.(10) becomes

$$U_{TT} + \alpha\left(\eta^2\right)_{TT} - \alpha\left(\eta^2\right)_{TT} - \left[U_{XX} + \alpha\left(U^2\right)_{XX}\right] - \frac{\alpha}{2}\left(U^2\right)_{XX} - \frac{\beta}{3}U_{XXXX} = 0,$$

or

$$U_{TT} - U_{XX} - \alpha\left(U^2\right)_{XX} - \frac{\alpha}{2}\left(U^2\right)_{XX} - \frac{\beta}{3}U_{XXXX} = 0,$$

which can be written as

$$U_{TT} - U_{XX} - \frac{3\alpha}{2}\left(U^2\right)_{XX} - \frac{\beta}{3}U_{XXXX} = 0. \tag{13}$$

Eq.(13) is the required BE equation.

Now, let the linear transformation

$$U = au,\ X = bx \text{ and } T = ct.$$

This transformation implies $U_{TT} = \frac{a}{c^2}u_{tt}$, $U_{XX} = \frac{a}{b^2}u_{xx}, (U^2)_{XX} = \frac{a^2}{b^2}(u^2)_{xx}$ and $U_{XXXX} = \frac{a}{b^4}u_{xxxx}$.

Then, Eq.(13) can be shown to be

$$\frac{a}{c^2}u_{tt} - \frac{a}{b^2}u_{xx} - \frac{3\alpha}{2}\frac{a^2}{b^2}\left(u^2\right)_{xx} - \frac{\beta a}{3b^4}u_{xxxx} = 0.$$

If we divide the above equation with $\frac{a}{c^2}$, the above equation can be reduced to

$$u_{tt} - \frac{c^2}{b^2} u_{xx} - \frac{3\alpha}{2} \frac{ac^2}{b^2} \left(u^2\right)_{xx} - \frac{\beta c^2}{3b^4} u_{xxxx} = 0.$$

Compare with the BE equation, we choose a, b, c such that

$$\frac{c^2}{b^2} = 1, \qquad \frac{3}{2}\frac{ac^2}{b^2} = 6, \qquad \text{and} \qquad \frac{c^2}{3b^4} = 1.$$

Solving this equation will give,

$$\frac{3}{2}a = 6, \text{ or } a = 4,$$

and

$$\frac{1}{3b^2} = 1 \text{ or } b = \pm\sqrt{\frac{1}{3}}.$$

Hence $c = b = \pm\sqrt{\frac{1}{3}}$.

Futhermore, taking $\alpha = \beta = 1$, we eventually find the required BE equation

$$u_{tt} - u_{xx} - 6\left(u^2\right)_{xx} - u_{xxxx} = 0.$$

CHAPTER IV

SOLUTION

4.1 The Solution

In most of the literature, the form of BE equation considered is always in the form

$$u_{tt} - u_{xx} - 6(u^2)_{xx} - u_{xxxx} = 0. \tag{14}$$

Without loss of generality and for the purpose of our discussion, here we shall use a more convenient form of BE equation which is

$$u_{xxxx} + u_{xx} + 6(u^2)_{xx} + 3u_{tt} = 0. \tag{15}$$

We note that Eq.(14) is transferred into Eq.(15) by the simple transformation

$$x \rightarrow x \text{ and } t \rightarrow -\frac{i}{\sqrt{3}}t.$$

It is very well known that BE equation possess soliton solutions. A common practice in obtaining exact solutions for soliton equations is to assume

$$u = \frac{\partial^2}{\partial x^2}(\log F) \tag{16}$$

Applying Eq.(16) into Eq.(15), we can get the bilinear equation

$$\left(D_x^{\ 4} + D_x^{\ 2} + 3D_t^{\ 2}\right)F \cdot F = 0 \tag{17}$$

where

$$D_x^m D_t^n F \cdot F = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n F(x,t)F(x',t')\bigg|_{\substack{x'=x \\ t'=t}} .$$

In order to illustrate the use of the above definition of D operator, we shall now give some examples.

$$\begin{aligned} D_x^2 F \cdot F &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^2 F(x)F(x')\bigg|_{x'=x} \\ &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) F(x)F(x')\bigg|_{x'=x} \\ &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)\left(F_x(x)F(x') - F(x)F_{x'}(x')\right)\bigg|_{x'=x} \\ &= F_{xx}(x)F(x') - F_x(x)F_{x'}(x') - F_x(x)F_{x'}(x') + F(x)F_{x'x'}(x')\big|_{x'=x} \\ &= 2(F(x)F_{xx} - F_x(x)F_x(x)) \\ &= 2\left(FF_{xx} - F_x^2\right) \end{aligned}$$

Similarly we can always calculate

$$D_t^2 F \cdot F = 2\left(FF_{tt} - F_t^2\right)$$

and

$$D_x D_y F \cdot F = 2\left(FF_{xy} - F_x F_y\right).$$

The quickest way of solving Eq.(17) and hence the BE equation is to make use of the first equation in the KP hierarchy of equations [Jimbo & Miwa (1983)] which reads

$$\left(D_x^4 - 4D_x D_y + 3D_t^2\right) F \cdot F = 0 \tag{18}$$

And its single-soliton solution of the KP Eq.(18) is given by

$$F = Ae^{px+p^2t+p^3y} + Be^{qx+q^2t+q^3y} \tag{19}$$

for any non-zero arbitrary values of p and q.

The bilinear equation of BE Eq.(17) can be manipulated as follows:

$$\left(D_x^4 + D_x^2 + 3D_t^2\right)F \cdot F = \left(D_x^4 - 4D_xD_y + 3D_t^2\right)F \cdot F + \left(4D_xD_y + D_x^2\right)F \cdot F \quad (20)$$

Now, in order that Eq.(19) is to satisfy the BE Eq.(17), we would require

$$\left(4D_xD_y + D_x^2\right)F \cdot F = 0 \quad (21)$$

Eq.(21) is equivalent to

$$4\left(FF_{xy} - F_xF_y\right) + FF_{xx} - F_x^2 = 0 \quad (22)$$

Upon applying Eq.(19) into Eq.(22), we will find immediately that

$$4p^4 + 4q^4 + p^2 + q^2 = 4p^3q + 4q^3p + 2pq,$$

which can be arranged to give

$$\left(4p^4 + p^2 - pq - 4q^3p\right) + \left(4q^4 + q^2 - pq - 4p^3q\right) = 0.$$

Since p and q are arbitrary, they can be choosen such that

$$4p^4 + p^2 - pq - 4pq^3 = 0,$$

and

$$4q^4 + q^2 - pq - 4p^3q = 0.$$

The above choice will then yield

$$p + 4p^3 = q + 4q^3. \quad (23)$$

The relation of Eq.(23), though in a cubic form, will define the solution of the Boussinesq equation.

CHAPTER V

DISCUSSION AND CONCLUSION

5.1 Discussion

In this project, we have discussed the Boussinesq (BE) equation. In particular, we have successfully derived the BE equation

$$u_{tt} - u_{xx} - 6\left(u^2\right)_{xx} - u_{xxxx} = 0$$

starting with Withams' coupled shallow water wave equations

$$\eta_t + \{(1+\alpha\eta)\omega\}_x - \frac{\beta}{6}\omega_{xxx} + O\left(\alpha\beta, \beta^2 = 0\right),$$

$$\omega_t + \alpha\omega\omega_x + \eta_x - \frac{\beta}{2}\omega_{xxt} + O\left(\alpha\beta, \beta^2\right) = 0.$$

The derivation was done via the Lagrangian coordinates.

We have also produced the BE equation's solution. For that purpose, we used simple transformation and rewrite BE equation in the form of

$$u_{xxxx} + u_{xx} + 6\left(u^2\right)_{xx} + 3u_{tt} = 0.$$

The single BE equation was then written in the form comparable with the standard Jimbo and Miwa's hierarchy of equations. Using one of those equations, we were able to produce the so-called reduction formula

$$p + 4p^3 = q + 4q^3$$

for the BE equation. The relationship between p and q enabled us to write down the required N-soliton solution of the BE equation.

5.2 Conclusion

We have applied an analytical method to Withams' coupled equation to derive the single BE equation. The method consists of the Lagrange coordinates.

The solution of the derived BE equation is

$$u = \frac{\partial^2}{\partial x^2}(log\ F)$$

where

$$F = Ae^{px+p^2t+p^3y} + Be^{qx+q^2t+q^3y},$$

which is similar to the KP solution but different in the exponential conditions that is

$$p + 4p^3 = q + 4q^3.$$

5.2 Suggestion

In this project, we did not analyse the solution. That was due to our heavy work load during the duration of the project. We do hope to analyse our result in a future work. Besides, we plan to proceed this research to obtain the N-soliton solutions using the numerical methods instead of an analytical method.

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