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The Precise Value of Commutativity Degree in Some Finite Groups

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ABSTRACT

The commutativity degree of a finite group G, denoted by $P(G)$, is the probability that a selected chosen pair of elements of G commute. The object of this paper is to compute a precise value of commutativity degree of some finite metacyclic p-groups of class at least 3. In particular, we describe the commutativity degree of these groups in split and non-split case.

|Commutativity degree | Nilpotency class | Conjugacy class | Metacyclic group |

1. INTRODUCTION

The commutativity degree of a finite group G is the probability that a randomly selected pairs of elements of the group commute. The concept of commutativity degree or probability of commuting pairs of a group was established by Erdos and Turan [1], Gustafson [2].

Let G be a group of finite order n. The probability $P(G)$ that two elements selected at random from G are commutative is $\frac{|\Omega|}{n^2}$ where

$$\Omega = \{(a,b) \in G \times G | ab = ba\}.$$

In order to count the elements of Ω, we have for each $a \in G$ the number of elements of Ω of the form (a,b) is $|C_G(a)|$, where $C_G(a)$ is the centralizer of a in G. Hence we have $|\Omega| = \sum |C_G(a)|$, where the sum extends over all $a \in G$. We recall that if a and b are conjugate elements of G, then $C_G(a)$ and $C_G(b)$ are conjugate subgroups. Moreover, the number of elements in the conjugacy classes of a is $[G: C_G(a)]$. Hence, if $a_1, \dots, a_k$ are representatives of the conjugacy classes in G, then

$$|\Omega| = \sum_{i=1}^{k} [G: C_G(a_i)]\, |C_G(a_i)| = k \cdot n.$$

Thus $$P(G) = \frac{k(G)}{|G|} \quad (1)$$

where $k(G)$ is the number of conjugacy classes of G and $|G|$ is the order of G. This formula has been proved by Gustafson [2] and the method of the proof was used by Erdos and Turan [1]. Also, it has shown in [2] that if G is any non-abelian group, then the upper bound for $\frac{k(G)}{|G|}$ is $\frac{5}{8}$ thus $P(G) \le \frac{5}{8}$, which of course hold for all metacyclic p-groups .

Equation (1) shows that finding the commutativity degree of a group is equivalent to finding the number of conjugacy classes of the group.

There are several papers on the conjugacy classes and commutativity degree of finite p-groups see, for example [3,4,5,6,7]. Many authors achieved to significant results on the lower and upper bound for $k(G)$. For instance, Lopez in [5, Theorem 1] shows that if A is a maximal abelian subgroup of finite nilpotent group G and $|A| = p^\alpha$ then there is an integer $k \ge 0$ such that

$$k(G) = p^{2\alpha - m} + \frac{p^\beta (p+1)(p^{\alpha - m} - 1)}{p^{\alpha - m}} + \frac{k(p^2 - 1)(p - 1)}{p^{\alpha - m}},$$

where $|G| = p^m$ and $|Z(G)| = p^\beta$.

Indeed for $k \neq 0$, this formula also shows an upper bound for G and does not determine the exact number of $k(G)$. Also, several results have been verified about conjugacy classes of subgroups of metacyclic p-groups see [8,9,10]. For example, in [10, Theorem 1.3] it was shown that if G is any finite split metacyclic p-group for an odd prime p, that is, $G = H \ltimes K$ for subgroups H and K, and if $|H| = p^\alpha$ and $|K| = p^{\alpha+\beta}$, then there exist exactly

$$\frac{(\beta - \alpha + 1)(p^{\alpha+1} - 1)}{p - 1} + 4\sum_{i=0}^{\alpha-1} p^i (\alpha + i),$$

conjugacy classes of subgroups of G.

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Recently, in [11] a general formula for the exact number of conjugacy classes and commutativity degree of two generator p-groups has been computed. It has been shown that if G is a metacyclic p-group with the following presentations:

If G is nilpotent of class two, then

(1) $G \cong \langle a, b | a^{p^\alpha} = b^{p^\beta} = 1,\ [a, b] = a^{p^{\alpha-\gamma}} \rangle$,
where $\alpha, \beta, \gamma \in N, \alpha \geq 2\gamma, \beta \geq \gamma \geq 1$;

If p is odd and the class of G is greater than 2, then

(2) $G \cong \langle a, b | a^{p^\alpha} = b^{p^\beta} = 1,\ [b, a] = a^{p^{\alpha-\gamma}} \rangle$,
where $\alpha, \beta, \gamma \in N,\ \gamma - 1 < \alpha < 2\gamma,\ \beta \geq \gamma$;

(3) $G \cong \langle a, b | a^{p^\alpha} = 1,\ b^{p^\beta} = a^{p^{\alpha-\varepsilon}},\ [b, a] = a^{p^{\alpha-\gamma}} \rangle$,
where $\alpha, \beta, \gamma \in N,\ \gamma - 1 < \alpha < 2\gamma,\ \beta \geq \gamma,\ \alpha < \beta + \varepsilon$;

then, the commutativity degree of group G is equal to

$$p^\gamma + p^{-(\gamma+1)} - p^{-(2\gamma+1)}\ .$$

A group G is called metacyclic if it contains a normal cyclic subgroup N such that G/N is also cyclic. The metacyclic p-groups of class 2 have been classified in context with the classification of 2-generator p-groups of class 2 for $p > 2$ in [12] and $p = 2$ in [13]. Moreover, Beuerle [14] classified the non-abelian metacyclic p-groups of class at least 3 where p is any prime.

In this paper we focus on some metacyclic 2-groups of class at least 3. Our basic goal is to compute the exact value of commutativity degree of the generalized quaternion groups, dihedral groups, semi-dihedral groups and quasi dihedral groups.

The following presentation is the generalized quaternion group $Q_{2^{\alpha+1}}$.

Theorem 1. 1 [14] *Let G be a metacyclic 2- group . Then*
$$G \cong \langle a, b | a^{2^\alpha} = 1, b^2 = a^{2^{\alpha-1}}, [b, a] = a^{-2} \rangle,$$
where $\alpha \geq 3$.

Theorem 1.2 is a presentation of dihedral group $D_{2^{\alpha+1}}$.

Theorem 1.2 [14] If *G is a metacyclic group, then*
$$G \cong \langle a, b | a^{2^\alpha} = b^2 = 1, [b, a] = a^{-2} \rangle,$$
where $\alpha \geq 3$;

The following group is the semi-dihedral group $SD_{2^{\alpha+1}}$.

Theorem 1.3 [14] If *G is a metacyclic 2-group, then*
$$G \cong \langle a, b | a^{2^\alpha} = b^2 = 1, [b, a] = a^{2^{\alpha-1}-2} \rangle,$$
where $\alpha \geq 3$.

The group $QD_{2^{\alpha+1}}$, quasi-dihedral group has the following presentation.

Theorem 1.4 [14] If *G is a metacyclic group, then*
$$G \cong \langle a, b | a^{2^\alpha} = b^2 = 1, [b, a] = a^{2^{\alpha-1}} \rangle,$$
where $\alpha \geq 2$.

2. SOME BASIC RESULTS

In this section we state some results which are need to prove our theorems. First we should introduce some notations. We denote
$$G(p, \alpha, \beta, \varepsilon, \gamma) = \langle a, b | \mathrm{a} = 1, b^{p^\beta} = a^{p^{\alpha-\varepsilon}}, a^b = a^r \rangle,$$
where $r = p^{\alpha-\gamma} + 1$. We use the notation $[b, a] = bab^{-1}a^{-1} = a^b a^{-1}$ for the commutator of b and a.

Lemma 2.1 *Let α, β, r and ε be integers with α, β non-negative and let*
$$G \cong \langle a, b | a^{p^\alpha} = 1, b^{p^\beta} = a^{p^{\alpha-\varepsilon}}, a^b = a^r \rangle,$$
be a metacyclic p-group, where $r = p^{\alpha-\gamma} \pm 1$. If $a, b \in G$ with $a = x^i y^j$ and $b = x^s y^t$, then the following hold in G:

(i) $ab = x^{i+sr^j} y^{j+t}$,

(ii) $a^b = x^{s(1-r^j)+ir^t} y^j$,

(iii) $[a, b] = x^{i(1-r^t)+s(r^j-1)}$.

Proof. Since $x^y = x^r$, we get $x^{y^j} = x^{r^j}$ and so $x^{iy^j} = x^{ir^j}$. Hence $y^j x^i = x^{ir^j} y^j$ and the result follows. ■

Lemma 2.2 [11] *Let G be a metacyclic p-group of type*
$$G(p, \alpha, \beta, \varepsilon, \gamma) = \langle a, b | a^{p^\alpha} = 1, yb^{p^\beta} = a^{p^{\alpha-\varepsilon}}, a^b = a^r \rangle,$$
where $r = p^{\alpha-\gamma} + 1$. Then

(i) $|G| = p^{\alpha+\beta}$;

(ii) $Z(G) = \langle a^{p^\gamma}, b^{p^\gamma} \rangle$;

(iii) $|Z(G)| = p^{\alpha+\beta-2\gamma}$.

Lemma 2.3 [15, Proposition 4.10] *Let G be a metacyclic 2-group of type*
$$G(2, \alpha, \beta, \varepsilon, \gamma) = \langle a, b | a^{2^\alpha} = 1,\ b^{2^\beta} = a^{2^{\alpha-\varepsilon}}, a^b = a^t \rangle,$$
where $r = 2^{\alpha-\gamma} - 1$. Then

(i) $|G| = 2^{\alpha+\beta}$;

(ii) $Z(G) = \langle a^{2^{\alpha-1}}, b^{2^{max\{1,\gamma\}}} \rangle$;

(iii) $|Z(G)| = 2^{\beta - max\ \{1,\gamma\}+1}$.

If G is a metacyclic p-group of order $p^{\alpha+\beta}$ with relation $ba = a^r b$, then each element in the group can be written in the unique form $a^s b^t$, where all possible value for s and t is $0 \leq s < p^\alpha,\ 0 \leq t < p^\beta$.

Now we are ready to find the value of commutativity degree of split and non-split metacyclic 2-group of classes at least 3.

Lemma 2.4 [14] *Consider a group of type $G(p, \alpha, \beta, \varepsilon, \gamma)$, then*

(i) *The class of G is greater than 2 if and only if $\alpha < 2\gamma$,*

(ii) *If $\beta + \varepsilon \leq \alpha$, then G is isomorphic to a split metacyclic p-group and in particular, $G \cong G(p, \alpha, \beta, 0, \gamma)$.*

The following two corollaries show that when a classification of metacyclic 2-group is a split group, and when it has class 2 or greater than 2. For a proof, we refer to above lemma and [15].

Corollary 2.5 [14] *Let G be a group of type $G(2;\alpha,\beta,1,\gamma)$, where $t=2^{\alpha-\gamma}-1$. If $\beta=\gamma$, then G is isomorphic to a split metacyclic 2-group and in particular, $G\cong G(2;\ \alpha,\beta,0,\beta)$. Moreover, the class of G is greater than 2 if and only if $\alpha>2$.*

Corollary 2.6 [14] *Let G be a group of type $G(p;\alpha,\beta,\varepsilon,\gamma)$ where $r=p^{\alpha-\gamma}+1$. If $\beta+\varepsilon\ \geq\alpha$, then G is isomorphic to a split metacyclic p-group and in particular, $G\cong G(p;\ \alpha,\beta,0,\gamma)$. Moreover, the class of G is greater than 2 if and only if $\alpha<2\gamma$.*

In the following four theorems we give a formula for $P(G)$ in terms of α .

3. MAIN THEOREMS

Theorem 3.1 *Let G be a metacyclic 2-group of nilpotency class at least 3. If*

$$G\cong\langle a,b|a^{2^{\alpha}}=b^{2}=1,[b,a]=a^{2^{\alpha-1}-2}\rangle,$$

where $\alpha\geq 3$, then

$$P(G)=\frac{2^{\alpha-1}+3}{2^{\alpha-1}}.$$

Proof. This group is generalized quaternion group $Q_{2^{\alpha+1}}$. By Lemma 2.3 the order of G is $2^{\alpha+1}$ and $Z(G)=\langle a^{2^{\alpha-1}},\ b^{2}\rangle$ and $|Z(G)|=2$. We can write an arbitrary element of G in the unique form a^{i} and $a^{i}b$ where $1\leq i\leq 2^{\alpha}-1$. From relation $[b,a]=a^{-2}$, we have $ba=a^{-1}b$ so any element of the form $b^{j}a^{i}$ can be written as the following case:

$$b^{j}a^{i}=\begin{cases} a^{i}b^{j}, & j\text{ even}\\ a^{-i}b^{j}, & j\text{ odd.}\end{cases}$$

First we consider an element of the form $a^{i}\in G$ to conjugate by element a^{t}

$$(a^{i})^{a^{t}}=a^{t}a^{i}a^{-t}=a^{i}$$

and

$$(a^{i})^{a^{t}b}=a^{t}ba^{i}ba^{-t}=a^{t}a^{-(t+i)}b=a^{-i}.$$

Thus

$$[a^{i}]=\{a^{i},\ a^{-i}\}.$$

In this case we have $2^{\alpha}-2$ non-central elements which are partitioned into 2 element conjugacy classes of the form $[a^{i}]$. Consequently we have $(2^{\alpha}-2)/2=2^{\alpha-1}-1$ such classes. Next conjugate an element of the second form $a^{i}b$ by a^{t} and $a^{t}b$ then

$$(a^{i}b)^{a^{t}}=a^{t}a^{i}ba^{t}=a^{t+i}a^{t}b=a^{2t+i}b$$

and

$$(a^{i}b)^{a^{t}b}=a^{t}ba^{i-t}=a^{t}a^{t-i}b=a^{2t-i}b.$$

Thus

$$[a^{i}b]=\{a^{2t+i}b,a^{2t-i}b|\ 0\ \leq t\leq 2^{\alpha}-1\}.$$

Consequently,

$$[ab]=\left\{ab,a^{3}b,\dots,a^{2^{\alpha}-1}b\right\}=\{a^{i}b\mid i\text{ is odd}\}$$

and

$$[a^{2}b]=\left\{b,a^{2}b,\dots,a^{2^{\alpha}-2}b\right\}=\{a^{i}b\mid i\text{ is even}\},$$

are 2 conjugacy classes including of all elements of the form $a^{i}b$ with $1\leq i<2^{\alpha}$. Thus all of non-central elements of G are consisted in one of the classes mentioned above. In addition,

$$Z(G)=\langle a^{2^{\alpha-1}},\ b^{2}\rangle=\langle a^{2^{\alpha-1}}\rangle$$

and $|Z(G)|=2$. Therefore, the number of conjugacy classes of G is equal to $2^{\alpha-1}+3$ and then

$$P(G)=\frac{2^{\alpha-1}+3}{2^{\alpha+1}}.\qquad\blacksquare$$

Theorem 3.2 Let *G be a metacyclic group presented by $G\cong\langle a,b|a^{2^{\alpha}}=b^{2}=1,[b,a]=a^{-2}\rangle$,* *where $\alpha\geq 3$. Then*

$$P(G)=\frac{2^{\alpha-1}+3}{2^{\alpha+1}}.$$

Proof. This group is the dihedral group $D_{2^{\alpha+1}}$ of order $2^{\alpha+1}$. The group G is split extension and by Lemma 2.3, $Z(G)=\langle a^{2^{\alpha-1}}\rangle$ and $|Z(G)|=2$. From the relation $[b,a]=a^{-2}$, we have $ba=a^{-1}b$. We can write

$$G=\{a^{i}\big|\ 0\leq i\leq 2^{\alpha}-1\}\cup\{a^{i}b\big|\ 0\leq i\leq 2^{\alpha}-1\}.$$

We now conjugate elements of G by a^{i} and $a^{i}b$ to find the conjugacy classes. Thus any element of the form $b^{j}a^{i}$ can be written by $b^{j}a^{i}=a^{i(-1)^{j}}b^{j}$. Suppose $0\leq t\leq 2^{\alpha}-1$ then

$$(a^{t})^{a^{i}}=a^{i}a^{t}a^{-i}=a^{t}$$

and

$(a^t)^{a^i b} = a^i b a^t b a^{-i} = a^i b a^{t+i} b = a^{-t} b^2 = a^{-t}.$

Thus

$[a^t] = \{a^t, a^{-t}\}.$

We suppose that $t \neq 2^{\alpha-1}, 0$. Then $2^\alpha - 2$ non-central elements are partitioned into two element conjugacy classes of the form $[a^t] = \langle a \rangle$. Hence we have $(2^\alpha - 2)/2 = 2^{\alpha-1} - 1$ such classes. Next we conjugate b by a^{-j} and $a^j b$, then we have

$(b)^{a^{-j}} = a^{-j} b a^j = a^{-2j} b$

and

$(b)^{a^j b} = a^j b a^{-j} = a^{2j} b.$

Hence

$[b] = [a^{2j} b] = \{a^{2j} b \mid 0 \leq j \leq \frac{2^\alpha - 1}{2}\}.$

Here half of the elements of the form $a^i b$ are included by [b]. We should find all further conjugacy classes including elements of the form $a^i b$ establishing with $[ab]$. Suppose $0 \leq j \leq 2^\alpha - 1$ then

$(ab)^{a^{-j}} = a^{-j} a b a^j = a^{1-2j} b,$

and

$(ab)^{a^{-j} b} = a^{-j} b a b^2 a^{-j} = a^{2j-1} b.$

Since

$\langle a^{2j+1} \rangle = \{a^i : 0 \leq i < 2^\alpha \text{ and } i \text{ is odd}\},$

then

$[ab] = [a^{2j+1} b] = \{a^i b : 0 \leq i < 2^\alpha \text{ and } i \text{ is odd}\}.$

Hence $[ab]$ includes the other half of the elements of the form $a^i b$. Therefore all classes containing elements of the form $a^i b$ with $0 \leq i \leq 2^\alpha - 1$ have been found. Hence we have 2 conjugacy classes of the form $[ab]$ and $[a]$. All non-central elements of G are included in one of the classes expressed above. As mentioned before we have

$Z(G) = \langle a^{2^{\alpha-1}}, b^2 \rangle = \langle a^{2^{\alpha-1}} \rangle.$

So $Z(G)$ contains $|Z(G)| = 2$ conjugacy classes. Therefore we have

$k(G) = 2 + 2^{\alpha-1} + 1 = 2^{\alpha-1} + 3.$

Hence

$$P(G) = \frac{2^{\alpha-1}+3}{2^{\alpha+1}}. \qquad \blacksquare$$

Theorem 3.3 Let G *be a metacyclic 2-group. If*

$G \cong \langle a, b \mid a^{2^\alpha} = b^2 = 1, [b, a] = a^{2^{\alpha-1}-2} \rangle,$

where $\alpha \geq 3$, *then*

$$P(G) = \frac{2^{\alpha-1}+3}{2^{\alpha+1}}.$$

Proof. This group is semi-dihedral group $SD_{2^{\alpha+1}}$ with order $2^{\alpha+1}$. Also,

$Z(G) = \langle a^{2^{\alpha-1}}, b^2 \rangle = \langle a^{2^{\alpha-1}} \rangle$

and $|Z(G)| = 2$. Each element of the group G can be written in the unique form a^s or $a^s b$ with $0 \leq s < 2^\alpha$. We rewrite $[b, a] = a^{2^{\alpha-1}-2}$ to $ba = a^r b$ where $r = 2^{\alpha-1} - 1$. We now conjugate elements of G to find conjugacy classes. Suppose $a^i \in G$. Then conjugate a^t by a^i and $a^i b$ thus we have

$(a^t)^{a^i} = a^i a^t a^{-i} = a^t.$

We now conjugate a^t by $a^i b$. By using Lemma 2.1 and for $1 \leq t < 2^\alpha$ we have

$$\begin{aligned}(a^t)^{a^i b} &= a^i b a^t a^{-ir} b \\ &= a^i b a^{t-ir} b \\ &= a^{i+(t-ir)r} \\ &= a^{i+(t-i(2^{\alpha-1}-1))(2^{\alpha-1}-1)} \\ &= a^{t2^{\alpha-1}-t}.\end{aligned}$$

If t is even, then a^t is a central element and $[a^t] = \{a^t, a^{-t}\} = \{a^t\}$. If t is odd then

$[a^t] = \{a^t, a^{2^{\alpha-1}-t}\}$

contains $\frac{2^\alpha - 2}{2} = 2^{\alpha-1} - 1$ conjugacy classes of order two. Next we conjugate $a^t b$ by a^i and $a^i b$ respectively. Thus for $1 \leq t < 2^\alpha$

$$\begin{aligned}(a^t b)^{a^i} &= a^i a^t b a^{-i} \\ &= a^{i(1-r)+t} b \\ &= a^{2i - i2^{\alpha-1} + t} b\end{aligned}$$

and

$$\begin{aligned}(a^t b)^{a^i b} &= a^i b a^t b b^{-1} a^{-i} \\ &= a^i a^{(t-i)r} b \\ &= a^{i+(t-i)(2^{\alpha-1}-1)} b \\ &= a^{(t-i)2^{\alpha-1}+2i-t} b.\end{aligned}$$

Therefore,

$$[a^t b] = \{ a^{2i-i2^{\alpha-1}+t}b , a^{(t-i)2^{\alpha-1}+2i-t}b \mid 0 \le i \le 2^\alpha - 1\}$$
$$= \{ a^t b , a^{t2^{\alpha-1}-t}b , \dots , a^{2^\alpha-2^{2\alpha-2}+t}b$$
$$a^{(t-2^{\alpha-1})2^{\alpha-1}+2^\alpha-t}b \}.$$

For $t = 1$, we have

$$[ab] = \{ ab , a^{2^{\alpha-1}-1}b , \dots , a^{2^\alpha-2^{2\alpha-2}+1}b ,$$
$$a^{(1-2^{\alpha-1})2^{\alpha-1}+2^\alpha-1}b \}$$
$$= \{ a^k b \mid k \text{ is odd} \}.$$

For $t = 2$, we have

$$[a^2b] = \{ a^2b , a^{2^{\alpha-1}-2}b , \dots , a^{2^\alpha-2^{2\alpha-2}+1}b ,$$
$$a^{(2-2^{\alpha-1})2^{\alpha-1}+2^\alpha-2}b \}$$
$$= \{ a^k b \mid k \; is \; even\}.$$

Thus there are two conjugacy classes with $2^{\alpha-1}$ element. On the other hand, $Z(G) = \langle a^{2^{\alpha-1}}, b^2\rangle = \langle a^{2^{\alpha-1}}\rangle$ contains $|Z(G)| = 2$ conjugacy classes. Therefore we have

$$P(G) = \frac{2^{\alpha-1}+3}{2^{\alpha+1}}. \quad \blacksquare$$

Theorem 3.4 Let G *is a metacyclic* 2-*group and*

$$G \cong \langle a, b | a^{2^\alpha} = b^2 = 1, [b, a] = a^{2^{\alpha-1}}\rangle,$$

where $\alpha \ge 2$. *Then* $P(G) = \frac{5}{8}$.

Proof. This group is the quasi-dihedral group $QD_{2^{\alpha+1}}$ of order $2^{\alpha+1}$. Using Lemma 2.2, $Z(G) = \langle a^2\rangle$ and $|Z(G)| = 2^{\alpha-1}$. By Corollary 2, this group is a split group of class greater than 2. We obtain $k(G)$ by computing the number of x^G for $x \in G$. Note that an arbitrary element of G can be written uniquely in the form

$$G = \{a^i b^j \,|\, 0 \le i < 2^\alpha, 0 \le j < 2\}.$$

Also, $Z(G) = \langle a^2, b^2\rangle = \langle a^2\rangle$. Moreover, from Lemma 2.1 we have

$$(a^i b^j)^{a^s b^t} = a^s b^t a^i b^j b^{-t} a^{-s} = a^{s(1-r^j)+ir^t} b^j ,$$

where $r = 2^{\alpha-1} + 1$. Since $b^2 = 1$, it is convenient to work with two forms a^k and $a^k b$. Hence we can apply again Lemma 2.1 to find the $|x^G|$ for some $x \in G$. Thus

$$(a^i)^{a^s b} = a^s b a^i b a^{-s} = a^{s+ir-sr^2} = a^{i(2^{\alpha-1}+1)},$$

because $|a| = 2^\alpha$. Similarly $(a^i)^{a^s} = a^i$. Hence

$$[a^i] = \{a^i, a^{i(2^{\alpha-1}+1)}\}.$$

If i is even then $a^i \in Z(G)$ and $[a^i]$ is the singleton $\{a^i\}$. If i is odd then

$$[a^i] = \{a^i, a^{2^{\alpha-1}+i}\}.$$

In this case we have $\frac{2^\alpha/2}{2} = 2^{\alpha-2}$ conjugacy classes of order 2. Likewise, we have

$$(a^i b)^{a^s} = a^{s(1-r)+i} b = a^{i-s2^{\alpha-1}} b$$

and

$$(a^i)^{a^s b} = a^{s(1-r^j)+ir^t} b^j = a^{(s+i)(2^{\alpha-1})+i} b.$$

Thus

$$[a^i b] = \{a^i b, a^{(2^{\alpha-1}+i)} b\}.$$

In this case we have $2^{\alpha-1}$ conjugacy classes with 2 elements. All non-central elements of G are included in one of the classes mentioned above. Also, $Z(G) = \langle a^2\rangle$ contains $|Z(G)| = 2^{\alpha-1}$ conjugacy classes. Hence we have

$$k(G) = 2^{\alpha-2} + 2^{\alpha-1} + 2^{\alpha-1} = 2^\alpha + 2^{\alpha-2}$$

$$P(G) = \frac{2^{\alpha-1}+2^{\alpha-2}}{2^{\alpha+1}} = \frac{5}{8}. \quad \blacksquare$$

4. CONCLUSION

The commutativity degree of dihedral groups, semi-dihedral groups and quasi-dihedral groups are the same.

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REFERENCES

[1] P. Erdos and P. Turan, On Some Problems of Statistical Group Theory, Acta Math. Acad. Sci. Hung., 19 (1968), 413-435.

[2] W. H. Gustafson, What is the Probability That Two Groups Elements Commute? Amer. Math. Monthly, 80 (1973), 1031-1304.

[3] A. Erfanian and F. Russo, Isoclinism in Probability of Commuting n-Tuples, Ital. J. Pure Appl. Math., 25 (2009) 27–36.

[4] R. Kanti Nath and A. Kumar, On a Lower Bound of Commutativity Degree, Rend. Circ. Mat. Palermo, 59 (2010) 137–142.

[5] B. Huppert, Character Theory of Finite Groups. De Gruyter Press, Berlin, 1998.

[6] A. Jaikin-Zapirain, On the Number of Conjugacy Classes in Finite p-Groups. J. Lond. Math. Soc. (2), 68 (2003), 699-711.

[7] G. Sherman, A Lower Bound for the Number of Conjugacy Classes in a Finite Nilpotent Group, Pacific J. Math., 80 (1979), 253-254.
[8] R. Brandl and L.Verardi, Metacyclic p-Groups and Their Conjugacy Classes of Subgroups, Glasg. Math. J., 35 (1993), 339-334.
[9] L. Hethelyi, Characters, Conjugacy Classes and Centrally Large Subgroups of p-Groups of Small Rank. J. Algebra, 340 (2011), 199-210.
[10] H. Sim, Conjugacy Classes of Subgroups of Split Metacyclic Groups of Prime Power Order, Bull. Korean Math. Soc., 35 (1998), 719-726.
[11] K. Moradipour, N. H. Sarmin and A. Erfanian, Conjugacy Classes and Commutativity Degree of Metacyclic p-Groups, *SicenceAsia*, (Accepted on 26 February 2012).
[12] M. Bacon, L.-C. Kappe, The Nonabelian Tensor Square of a 2-Generator p-Group of Class 2, Arch. Math., 61 (1993), 508-516.
[13] L.-C. Kappe, N. Sarmin and M. Visscher, Two-Generator 2-Groups of Class 2 and Their Nonabelian Tensor Squares, Glasg. Math. J., 41 (1999), 417-430.
[14] J. R. Beuerle, An Elementary Classification of Finite Metacyclic p-Groups of Class at Least Three, Algebra Colloq., 12 (2005), 553-562.
[15] B. W. King, Presentations of Metacyclic Groups, Bull. Aust. Math. Soc., 8 (1973), 103-131.