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Interval-Valued Fuzzy Congruences on Inverse Semigroups

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Abstract We introduce a quotient semigroup $S/\overline{\delta}$ by an interval-valued fuzzy congruence relation $\overline{\delta}$ on a semigroup S, and present Homomorphism Theorems with respect to an interval-valued fuzzy congruence relation. In this paper, we introduce an idempotent-separating interval-valued fuzzy congruence, a group interval-valued fuzzy congruence on inverse semigroup, and some of their properties on inverse semigroups.

Keywords Interval-valued fuzzy congruence, Inverse semigroup, Idempotent-separating Interval-valued fuzzy congruence, Group Interval-valued fuzzy congruence.

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1 Preliminaries and Basic Results

Zadeh introduced the concept of fuzzy sets [1] in 1965, which provides a natural framework for generalizing the basic notions of algebra e.g. set theory, group theory, groupoids, real analysis etc. Several researchers get sufficient motivation to review various concepts and results from the realm of algebra in broader framework of fuzzy setting.

Mordeson et al. in [2] introduced an up to date account of fuzzy subsemigroups and fuzzy ideals of a semigroup. Although semigroups concentrate on theoretical aspects, they also include applications in error-correcting codes, control engineering, formal language, computer science and information science. Biswas [3] applied the fuzzy concept to Rosenfeld's fuzzy subgroups and studied some properties of interval valued membership function in terms of fuzzy subgroups (also see [4, 5]). Kuroki [6] characterized inverse semigroups by the properties of their fuzzy congruences. In [7], Tan studied regular semigroups in terms of fuzzy congruences and investigated some important results.

In this paper, we introduced the concept of a quotient semigroup $S/\overline{\delta}$ by an interval-valued fuzzy congruence relation $\overline{\delta}$ on a semigroup S, and present Homomorphism Theorems with respect to an interval-valued fuzzy congruence relation. We also investigate idempotent-separating interval-valued fuzzy congruence, a group interval-valued fuzzy congruence on inverse semigroup and studied some important results.

A non-empty set S together with a binary operation "$*$" defined on it is called a semigroup if it is associative. An element a of a semigroup S is called a regular element if there exists an element $x \in S$ such that $a = axa$. A semigroup S is called regular if each element of S is regular. By an idempotent element of S we mean that an element a of S such that $a^2 = a$, and a^{-1} is an inverse element of a in S if $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. By an inverse semigroup we mean that every element of S will possess a unique inverse.

A function f from a nonempty set X to the unit interval [0, 1] is called a fuzzy subset of X. Throughout this paper S will denote a semigroup and X be any nonempty set.

Definition 1 [8] *Let A be a non-empty subset of a set X. Then the characteristic function of A is the function C_A of X into $[0,1]$ defined by,*

$$C_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Definition 2 [9] *For fuzzy subsets δ and γ of X, $\delta \subseteq \gamma$ means that for all $x \in X$, $\delta(x) \leq \gamma(x)$.*

Definition 3 [9] *Let S be a semigroup. A function λ from $S \times S$ to the unit interval $[0,1]$ is called a fuzzy relation on S.*

Definition 4 *A fuzzy relation λ on S is called fuzzy reflexive if $\lambda(a,a) = 1$ for all $a \in S$, fuzzy symmetric if $\lambda(a,b) = \lambda(b,a)$ for all $a, b \in S$.*

Definition 5 [9] *Let λ and μ be two fuzzy relations on S. Then the product $\lambda \circ \mu$ of λ and μ is defined by*

$$\lambda \circ \mu(a,b) = \vee_{x \in S}(\lambda(a,x) \wedge \mu(x,b))$$

for all $a, b \in S$.

If $\lambda = \mu$, say and $\lambda \circ \lambda \subseteq \lambda$, then the fuzzy relation λ is called fuzzy transitive.

A fuzzy relation λ on S is called a fuzzy equivalence relation on S if it is fuzzy reflexive, fuzzy symmetric and fuzzy transitive.

A fuzzy relation λ on S is called compatible if

$$\lambda(ax, bx) \geq \lambda(a,b) \text{ and } \lambda(xa, xb) \geq \lambda(a,b)$$

for all $a, b, x \in S$. A fuzzy equivalence relation on a semigroup S which is compatible is called a fuzzy congruence relation on S.

Lemma 6 [10] *If S is a regular semigroup, then the following conditions are equivalent.*

(i) S has exactly one idempotent.

(ii) S is cancellative.

(iii) S is a group.

Lemma 7 [10] *A semigroup S is an inverse semigroup if and only if S is regular and idempotents of S commute.*

Throughout this paper S will denote semigroup, δ fuzzy subset and $\overline{\delta}$ an interval-valued fuzzy subset unless otherwise stated.

Definition 8 *Let Ω denote the family of all closed sub-intervals of the interval* [0, 1] *with minimal element $\overline{O} = [0,0]$ and maximal element $\overline{I} = [1,1]$. Let $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$ and $I_i = [a_i, b_i]$ be the elements of Ω. Then define*

$$\begin{aligned} I_1 \wedge I_2 &= [a_1 \wedge a_2,\ b_1 \wedge b_2]. \\ I_1 \vee I_2 &= [a_1 \vee a_2,\ b_1 \vee b_2]. \\ \wedge_i\{I_i\} &= [\wedge_i\{a_i\},\ \wedge_i\{b_2\}]. \\ \vee_i\{I_i\} &= [\vee_i\{a_i\},\ \vee_i\{b_2\}]. \end{aligned}$$

We say $I_2 \leq I_1$ if and only if $a_2 \leq a_1$ and $b_2 \leq b_1$ for all $a_i,\ b_i \in S$.

Definition 9 [4] *Let X be a non-empty set and μ_A^l, μ_A^u be two fuzzy subsets of X. That is $\mu_A^l : X \to [0,\ 1]$ and $\mu_A^u : X \to [0,\ 1]$ such that $0 \leq \mu_A^l(x) \leq \mu_A^u(x) \leq 1$ for each $x \in X$. Then $\overline{A} : X \to \Omega$ defined by $\overline{A}(x) = [\mu_A^l(x),\ \mu_A^u(x)] \subseteq [0,1]$ for each $x \in X$, is called an interval-valued fuzzy subset of X.*

2 Interval-valued Fuzzy Congruences

In this section we will introduce the concept of an interval valued fuzzy congruence relation, and investigate some of its properties. We will define an interval valued fuzzy equivalence class on S in this section.

Definition 10 [4] *Let S be a semigroup and δ^l, δ^u be two fuzzy subsets of $S \times S$, that is $\delta^l : S \times S \to [0,1]$ and $\delta^u : S \times S \to [0,1]$, such that $\delta^l(a,b) \leq \delta^u(a,b)$ for all $(a,b) \in S \times S$. Then a function $\overline{\delta} : S \times S \to \Omega$ defined by*

$$\overline{\delta}(a,b) = [\delta^l(a,b),\ \delta^u(a,b)]$$

is called an interval-valued fuzzy relation on S.

Definition 11 *An interval-valued fuzzy relation $\overline{\delta}$ of S is called an interval-valued fuzzy reflexive relation on S if $\overline{\delta}(a,a) = \overline{I} = [1,\ 1]$ for all $(a,a) \in S \times S$. $\overline{\delta}$ is an interval-valued fuzzy symmetric if $\overline{\delta}(a,b) = \overline{\delta}(b,a)$ for all $(a,b) \in S \times S$.*

Definition 12 *Let $\overline{\delta}$ and $\overline{\lambda}$ be two interval-valued fuzzy relations on S. Then the product of $\overline{\delta}$ and $\overline{\lambda}$ is defined by*

$$\overline{\delta} \circ \overline{\lambda}(a,b) = \vee_{x \in S}\{\overline{\delta}(a,x) \wedge \overline{\lambda}(x,b)\} \text{ for all } (a,b) \in S \times S.$$

If $\overline{\delta} = \overline{\lambda}$, and $\overline{\delta} \circ \overline{\delta} \subseteq \overline{\delta}$, then the interval-valued fuzzy relation $\overline{\delta}$ on S is called an interval-valued fuzzy transitive relation.

An interval-valued fuzzy relation $\overline{\delta}$ on S is called compatible if $\overline{\delta}(ax,bx) \geq \overline{\delta}(a,b)$ and $\overline{\delta}(xa,xb) \geq \overline{\delta}(a,b)$ for all $a,b,x \in S$.

Definition 13 *An interval-valued fuzzy equivalence relation of a semigroup S which is compatible is called an interval-valued fuzzy congruence relation of S.*

Definition 14 *Let S be a semigroup and R be a relation on S. Then $\overline{\delta}_R$:$S \times S \to \Omega$ defined by*

$$\overline{\delta}_R(a,b) = \begin{cases} \overline{I} = [1,\ 1] & \text{when } (a,b) \in R, \\ \overline{O} = [0,\ 0] & \text{when } (a,b) \notin R, \end{cases}$$

is called an interval-valued characteristic function of S.

Lemma 15 *Let R be a relation on S. Then R is an equivalence relation on S if and only if $\overline{\delta}_R$ is an interval-valued fuzzy equivalence relation on S.*

Proof. ($\Longleftarrow$) Let $\overline{\delta}_R$ be an interval-valued fuzzy equivalence relation on S. Then $\overline{\delta}_R(a,a) = \overline{I} = [1,\ 1]$ for all $(a,a) \in S \times S$, implies that $(a,a) \in R$ for all $(a,a) \in S \times S$. Hence R is reflexive. Let $(a,b) \in R$, then $\overline{\delta}_R(a,b) = \overline{I}$ and since $\overline{\delta}_R$ is an interval-valued fuzzy symmetric relation, so $\overline{\delta}_R(a,b) = \overline{\delta}_R(b,a)$. Hence $\overline{\delta}_R(b,a) = \overline{I} = [1,\ 1]$, so $(b,a) \in R$ means that R is symmetric. Suppose (a,b), $(b,c) \in R$. Then $\overline{\delta}_R(a,b) = \overline{I} = \overline{\delta}_R(b,c)$. Since $\overline{\delta}_R$ is an interval-valued fuzzy transitive relation, so by Definition 12, $\overline{\delta}_R \circ \overline{\delta}_R \subseteq \overline{\delta}_R$. Therefore we have

$$\begin{aligned}
\overline{\delta}_R(a,c) &\geq \overline{\delta}_R \circ \overline{\delta}_R(a,c) \\
&= \vee_{x\in S}\{\overline{\delta}_R\,(a,x) \wedge \overline{\delta}_R(x,c)\} \\
&\geq \overline{\delta}_R\,(a,b) \wedge \overline{\delta}_R\,(b,c) \\
&= \overline{I} \wedge \overline{I} \\
&= \overline{I} = [1,\ 1].
\end{aligned}$$

This implies that $\overline{\delta}_R\,(a,c) \geq [1,1]$. Hence $\overline{\delta}_R\,(a,c) = [1,1]$. Therefore $(a,c) \in R$. Thus R is transitive and consequently R is an equivalence relation.

($\Longrightarrow$) Suppose that R is an equivalence relation. So $(a,a) \in R$ for all $(a,a) \in S \times S$. Therefore $\overline{\delta}_R\,(a,a) = \overline{I} = [1,\ 1]$ for all $(a,a) \in S \times S$ which implies $\overline{\delta}_R$ is an interval-valued fuzzy reflexive. Let $(a,b) \in S \times S$. If $\overline{\delta}_R\,(a,b) = \overline{I}$, then $(a,b) \in R$. Since R is symmetric, so $(b,a) \in R$. Hence $\overline{\delta}_R\,(b,a) = \overline{I}$. If $\overline{\delta}_R\,(a,b) = \overline{O}$ then $(a,b) \notin R$ implies that $(b,a) \notin R$. Thus $\overline{\delta}_R\,(b,a) = \overline{O} = [0,0]$. Thus in any case $\overline{\delta}_R\,(a,b) = \overline{\delta}_R\,(b,a)$. Now let $(a,b) \in S \times S$. If $(a,b) \in R$ then we have $\overline{\delta}_R\,(a,b) = \overline{I} = [1,1] \geq \overline{\delta}_R \circ \overline{\delta}_R\,(a,b)$. If $(a,b) \notin R$, then by Definition 12, $\overline{\delta}_R \circ \overline{\delta}_R\,(a,b) = \vee_{x\in S}\{\overline{\delta}_R\,(a,x) \wedge \overline{\delta}_R\,(x,b)\}$. If (a,x), $(x,b) \in R$ then $(a,b) \in R$ which is a contradiction. Hence $(a,x) \notin R$ or $(x,b) \notin R$. This implies $\overline{\delta}_R\,(a,x) = \overline{O}$ or $\overline{\delta}_R\,(x,b) = \overline{O}$ which gives $\overline{\delta}_R\,(a,x) \wedge \overline{\delta}_R\,(x,b) = \overline{O}$. Therefore $\vee_{x\in S}\{\overline{\delta}_R\,(a,x) \wedge \overline{\delta}_R\,(x,b)\} = \overline{O} = [0,\ 0]$. So $\overline{\delta}_R \circ \overline{\delta}_R\,(a,b) = \overline{\delta}_R\,(a,b)$ which shows that $\overline{\delta}_R \circ \overline{\delta}_R \subseteq \overline{\delta}_R$. Hence $\overline{\delta}_R$ is an interval-valued fuzzy transitive relation on S. ■

Theorem 16 *Let R be a binary relation on a semigroup S. Then R is a congruence relation on S if and only if $\overline{\delta}_R$ is an interval-valued fuzzy congruence relation on S.*

Proof. The proof follows frome Lemma 15. ■

Definition 17 *Let $\overline{\delta}$ be an interval-valued fuzzy equivalence relation on a semigroup S. For each $a \in S$, we define an interval-valued fuzzy subset $\overline{\delta}_a$ of S as follows,*

$$\overline{\delta}_a\,(x) = \overline{\delta}\,(a,x) \text{ for all } x \in S.$$

Theorem 18 *Let $\overline{\delta}$ be an interval-valued fuzzy equivalence relation on a semigroup S. Let $a,b \in S$. Then $\overline{\delta}_a = \overline{\delta}_b$ if and only if $\overline{\delta}\,(a,b) = \overline{I} = [1,1]$.*

Proof. Suppose $\overline{\delta}_a = \overline{\delta}_b$, for some $a,b \in S$. Then

$$\overline{\delta}\,(a,b) = \overline{\delta}_a\,(b) = \overline{\delta}_b\,(b) = \overline{\delta}\,(b,b) = \overline{I} = [1,1].$$

Conversely, suppose that $\overline{\delta}(a,b)=\overline{I}$.

$$\begin{aligned}
\text{Now } \overline{\delta}_a(x) &= \overline{\delta}(a,x)\\
&\geq \overline{\delta}\circ\overline{\delta}(a,x)\\
&= \vee_{y\in S}\{\overline{\delta}(a,y)\wedge\overline{\delta}(y,x)\}\\
&\geq \overline{\delta}(a,b)\wedge\overline{\delta}(b,x)\\
&= \overline{I}\wedge\overline{\delta}(b,x)\\
&= \overline{\delta}(b,x)\\
&= \overline{\delta}_b(x) \qquad \text{for all } x\in S.
\end{aligned}$$

Thus $\overline{\delta}_a \supseteq \overline{\delta}_b$. Since $\overline{\delta}$ is an interval-valued fuzzy equivalence relation, so interval-valued fuzzy symmetric relation, thus $\overline{\delta}(a,b)=\overline{\delta}(b,a)$ for all $(a,b)\in S\times S$. Since $\overline{\delta}(a,b)=\overline{I}$, so $\overline{\delta}(b,a)=\overline{I}$.Therefore we also have $\overline{\delta}_b\supseteq\overline{\delta}_a$. Consequently $\overline{\delta}_a=\overline{\delta}_b$. ■

Definition 19 *An interval-valued fuzzy subset $\overline{\delta}_a$ of semigroup S is called an interval-valued fuzzy equivalence class of $\overline{\delta}$ containing $a\in S$. Let $\overline{\delta}$ be an interval-valued fuzzy congruence relation on S. Denote*

$$S/\overline{\delta}=\left\{\overline{\delta}_a : a\in S\right\}.$$

Let $\overline{\delta}$ and $\overline{\lambda}$ be two interval-valued fuzzy subsets of S. Then the product $\overline{\delta}\circ\overline{\lambda}$ of $\overline{\delta}$ and $\overline{\lambda}$ is defined by

$$\overline{\delta}\circ\overline{\lambda}(x)=\begin{cases}\vee_{x=yz}\left(\overline{\delta}(y)\wedge\overline{\delta}(z)\right) & \text{if } x \text{ is expressible as } x=yz,\\ \overline{O}=[0,0] & \text{otherwise.}\end{cases}$$

Note that the same symbol is used for both this product and the product of interval-valued fuzzy relation.

Let $\overline{\delta}$ be an interval-valued fuzzy congruence on S. Let $a,b\in S$ and $\overline{\delta}_a$, $\overline{\delta}_b$ be two interval-valued fuzzy congruence classes of $\overline{\delta}$, and let $x\in S$. Then,

$$\begin{aligned}
\overline{\delta}_a\circ\overline{\delta}_b(x) &= \vee_{x=yz}\left(\overline{\delta}_a(y)\wedge\overline{\delta}_b(z)\right)\\
&= \vee_{x=yz}\left(\overline{\delta}(a,y)\wedge\overline{\delta}(b,z)\right)\\
&\leq \vee_{x=yz}\left(\overline{\delta}(ab,yb)\wedge\overline{\delta}(yb,yz)\right)\\
&= \vee_{x=yz}\left(\overline{\delta}(ab,yb)\wedge\overline{\delta}(yb,x)\right)\\
&\leq \vee_{t\in S}\left(\overline{\delta}(ab,t)\wedge\overline{\delta}(t,x)\right)\\
&= \overline{\delta}\circ\overline{\delta}(ab,x)\\
&\leq \overline{\delta}(ab,x)\\
&= \overline{\delta}_{ab}(x) \qquad \text{for all } x\in S.
\end{aligned}$$

This implies $\overline{\delta}_a\circ\overline{\delta}_b\subseteq\overline{\delta}_{ab}$. Therefore we can define the binary operation "$*$" on $S/\overline{\delta}$ as follows:

$$\overline{\delta}_a*\overline{\delta}_b=\overline{\delta}_{ab} \text{ for all } \overline{\delta}_a,\ \overline{\delta}_b\in S/\overline{\delta}.$$

Lemma 20 *The binary operation "$*$" on $S/\overline{\delta}$ is well-defined.*

Proof. Assume that $\overline{\delta}_a = \overline{\delta}_b$ and $\overline{\delta}_c = \overline{\delta}_d$. Then by Theorem 18, $\overline{\delta}(a, b) = \overline{\delta}(c, d) = \overline{I}$. Thus

$$\begin{aligned}
\overline{\delta}(ac, bd) &\geq \overline{\delta} \circ \overline{\delta}(ac, bd) \\
&= \vee_{x \in S}\{\overline{\delta}(ac, x) \wedge \overline{\delta}(x, bd)\} \\
&\geq \overline{\delta}(ac, bc) \wedge \overline{\delta}(bc, bd) \\
&\geq \overline{\delta}(a, b) \wedge \overline{\delta}(c, d) \\
&= \overline{I} \wedge \overline{I} \\
&= \overline{I} = [1,\ 1].
\end{aligned}$$

This shows that $\overline{\delta}(ac, bd) \geq [1,\ 1]$ which implies that $\overline{\delta}(ac, bd) = \overline{I} = [1,\ 1]$. Hence by Theorem 18 we get $\overline{\delta}_{ac} = \overline{\delta}_{bd}$. It implies that $\overline{\delta}_a * \overline{\delta}_c = \overline{\delta}_b * \overline{\delta}_d$, which shows that the binary operation "$*$" is well-defined on $S/\overline{\delta}$. ■

Theorem 21 *Let $\overline{\delta}$ be an interval-valued fuzzy congruence on a semigroup S. Then*

$$\overline{\delta}^{-1}([1,\ 1]) = \left\{(a, b) \in S \times S \mid \overline{\delta}(a, b) = \overline{I}\right\}$$

is a congruence on S.

Proof. It is easy to see that $\overline{\delta}^{-1}([1,\ 1])$ is both reflexive and symmetric. To prove transitivity let $(a, b), (b, c) \in \overline{\delta}^{-1}([1,\ 1])$. Then, since $\overline{\delta}(a, b) = \overline{I} = \delta(b, c)$ so we have

$$\begin{aligned}
\overline{\delta}(a, c) &\geq \overline{\delta} \circ \overline{\delta}(a, c) \\
&= \vee_{x \in S}\{\overline{\delta}(a, x) \wedge \overline{\delta}(x, c)\} \\
&\geq \overline{\delta}(a, b) \wedge \overline{\delta}(b, c) \\
&= \overline{I} \wedge \overline{I} \\
&= \overline{I}.
\end{aligned}$$

So $\overline{\delta}(a, c) = \overline{I}$. Thus $(a, c) \in \overline{\delta}^{-1}([1,\ 1])$ implies that $\overline{\delta}^{-1}([1,\ 1])$ is transitive. Hence $\overline{\delta}^{-1}([1,\ 1])$ is an equivalence relation on S.

Now suppose $(a, b) \in \overline{\delta}^{-1}([1,\ 1])$ and $x \in S$. Since $\overline{\delta}$ is an interval-valued fuzzy congruence relation on S so

$$\overline{\delta}(ax, bx) \geq \overline{\delta}(a, b) = \overline{I}.$$

This implies that $\delta(ax, bx) = \overline{I}$, that is $(ax, bx) \in \overline{\delta}^{-1}([1,\ 1])$. Similarly we can show that $(xa, xb) \in \overline{\delta}^{-1}([1,\ 1])$, which implies that $\overline{\delta}^{-1}([1,\ 1])$ is compatible. Consequently $\overline{\delta}^{-1}([1,\ 1])$ is a congruence relation on S. ■

3 Homomorphism Theorems for Interval-valued Fuzzy Congruences

Homomorphism theorems for an interval valued fuzzy congruence relation will be proved and an interval valued fuzzy kernel of homomorphism will also be defined in this section.

Lemma 22 [6] *Let S and Y be two semigroups and f a homomorphism of S into Y. Then the relation,*

$$Ker(f) = \{(a, b) \in S \times S \mid f(a) = f(b)\}$$

is a congruence on S.

Definition 23 *Suppose $K(f) = \overline{\delta}_{\ker(f)}$. Then $K(f)$ is an interval-valued fuzzy congruence called the interval-valued fuzzy kernel of f. Obviously for all $(a, b) \in S \times S$,*

$$K(f)(a,b) = \begin{cases} \overline{I} = [1,\ 1] & \text{if } f(a) = f(b), \\ \overline{O} = [0,\ 0] & \text{if } f(a) \neq f(b). \end{cases}$$

Theorem 24 *If $\overline{\delta}$ is an interval-valued fuzzy congruence relation on a semigroup S. Then $S/\overline{\delta}$ is a semigroup with respect to the binary operation "$*$" (defined as $\overline{\delta}_a * \overline{\delta}_b = \overline{\delta}_{ab}$ by Definition 19). The function $\overline{\delta}^k : S \to S/\overline{\delta}$ defined by $\overline{\delta}^k(a) = \overline{\delta}_a$ for all $a \in S$ is a homomorphism. Suppose S and Y be semigroups. If $f : S \to Y$ is a homomorphism, then the interval-valued fuzzy relation $K(f)$ is an interval-valued fuzzy congruence on S, and there is a monomorphism $q : S/K(f) \to Y$ such that the diagram*

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "(k(f))^k"'] & Y \\ S/K(f) \arrow[ur, "q"'] & \end{tikzcd}$$

commutes.

Proof. By using Definition 19, $S/\overline{\delta}$ is a semigroup. Let $a, b \in S$, then by definition of $\overline{\delta}^k$ we have $\overline{\delta}^k(ab) = \overline{\delta}_{ab} = \overline{\delta}_a * \overline{\delta}_b = \overline{\delta}^k(a) * \overline{\delta}^k(b)$. Since $K(f) = \overline{\delta}_{\ker(f)}$ is an interval-valued fuzzy congruence, so $S/K(f)$ is a semigroup. Let us define $q : S/K(f) \to Y$ by $q((K(f))_a) = f(a)$ for all $a \in S$. Let $a, b \in S$ such that, $(K(f))_a = (K(f))_b$. Then $K(f)(a,b) = \overline{I} = [1, 1]$ by Theorem 18. But $K(f) = \overline{\delta}_{\ker(f)}$ so $\overline{\delta}_{\ker(f)}(a,b) = \overline{I}$. Thus $(a,b) \in \ker(f)$. Hence $f(a) = f(b)$. This implies $q((K(f))_a) = q((K(f))_b)$. Hence q is well defined. Now suppose $(K(f))_a$, $(K(f))_b \in S/K(f)$ such that $q((K(f))_a) = q((K(f))_b)$. Thus $f(a) = f(b)$. This implies that $(a,b) \in Ker(f)$ so $\overline{\delta}_{\ker(f)}(a,b) = \overline{I}$. However, since $K(f) = \overline{\delta}_{\ker(f)}$, so $K(f)(a,b) = \overline{I} = [1, 1]$. Hence by Theorem 18, $(K(f))_a = (K(f))_b$. This shows that q is one to one function. Let $(K(f))_a$, $(K(f))_b \in S/K(f)$, then

$$\begin{aligned} q((k(f))_a * (k(f))_b) &= q((K(f))_{ab}) \\ &= f(ab) \\ &= f(a) f(b) \\ &= q((K(f))_a) q((K(f))_b). \end{aligned}$$

Thus q is a homomorphism. Consequently q is a monomorphism.

Let $a \in S$, then we have

$$\begin{aligned} \left(q\left(k\left(f\right)\right)^{k}\right)(a) &= q\left(\left(k\left(f\right)\right)_{a}\right) \\ &= f\left(a\right) \end{aligned}$$

which shows that $q\left(k\left(f\right)\right)^{k} = f$. Thus the diagram commutes. ■

Theorem 25 *Let $\overline{\delta}$ and $\overline{\lambda}$ be interval-valued fuzzy congruences on a semigroup S such that $\overline{\delta} \subseteq \overline{\lambda}$. Then there is a unique homomorphism $g : S/\overline{\delta} \to S/\overline{\lambda}$ such that the diagram*

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\overline{\delta}^{k}"'] & S/\overline{\lambda} \\ S/\overline{\delta} \arrow[ur, "g"'] & \end{tikzcd}$$

commutes, and $\left(S/\overline{\delta}\right)/K\left(g\right)$ is isomorphic to $S/\overline{\lambda}$.

Proof. Sine $\overline{\delta}$ and $\overline{\lambda}$ are interval-valued fuzzy congruences on a semigroup S, therefore $S/\overline{\delta}$ and $S/\overline{\lambda}$ are semigroups with respect to the binary operation "$*$" by Theorem 24. Also $\overline{\delta}^{k} : S \to S/\overline{\delta}$ and $\overline{\lambda}^{k} : S \to S/\overline{\lambda}$ defined by $\overline{\delta}^{k}(a) = \overline{\delta}_a$ and $\overline{\lambda}^{k}(a) = \overline{\lambda}_a$ respectively for all $a \in S$ are homomorphisms. Now define $g : S/\overline{\delta} \to S/\overline{\lambda}$ by $g(\overline{\delta}_a) = \overline{\lambda}_a$ for all $a \in S$. Let $\overline{\delta}_a, \overline{\delta}_b \in S/\overline{\delta}$ such that $\overline{\delta}_a = \overline{\delta}_b$. Thus $\overline{\delta}(a,b) = \overline{I}$ by Theorem 18. Since $\overline{\delta} \subseteq \overline{\lambda}$ so $\overline{I} = \overline{\delta}(a,b) \leq \overline{\lambda}(a,b)$. Therefore $\overline{\lambda}(a,b) = \overline{I} = [1,\ 1]$, which gives $\overline{\lambda}_a = \overline{\lambda}_b$. Hence $g(\overline{\delta}_a) = g(\overline{\delta}_b)$ so g is well defined. Let $\overline{\delta}_a$, $\overline{\delta}_b \in S/\overline{\delta}$. Then

$$g(\overline{\delta}_a * \overline{\delta}_b) = g\left(\overline{\delta}_{ab}\right) = \overline{\lambda}_{ab} = \overline{\lambda}_a * \overline{\lambda}_b = g\left(\overline{\delta}_a\right) * g\left(\overline{\delta}_b\right).$$

So g is a homomorphism.

The remaining proof is left for the reader. ■

4 Idempotent-separating Interval-valued Fuzzy Congruences

In this section we will give the necessary and sufficient conditions for interval valued fuzzy congruence relation $\overline{\delta}$ to be an idempotent-separating. Also some theorems for an interval valued fuzzy congruence relation $\overline{\delta}$ on S will be proved here.

Definition 26 *An interval-valued fuzzy congruence $\overline{\delta}$ on a semigroup S is called an idempotent-separating if for all $e, f \in E\left(S\right)$, the equality $\overline{\delta}_e = \overline{\delta}_f$ implies that $e = f$ where $E\left(S\right)$ is the set of all idempotents of S.*

Theorem 27 *Let S be a regular semigroup $\overline{\delta} \in i\text{-}Con_f(S)$ (the set of all interval-valued fuzzy congruences of S), and $a \in S$. Then the following conditions are equivalent.*

(i) $\overline{\delta}_a \in E\left(S/\overline{\delta}\right)$.

(ii) $\overline{\delta}_a = \overline{\delta}_e$ for some $e \in E(S)$ and $Se \subseteq Sa$, $eS \subseteq aS$.

(iii) $\overline{\delta}_a = \overline{\delta}_e$ for some $e \in E(S)$.

Proof. First we prove that (i) implies (ii).

Let $\overline{\delta}_a \in E\left(S/\overline{\delta}\right)$ where $a \in S$. Then

$$\overline{\delta}_a = \overline{\delta}_a * \overline{\delta}_a = \overline{\delta}_{aa} = \overline{\delta}_{a^2}.$$

Let s be the inverse of a^2 in S, that is $a^2 = a^2sa^2$ and $s = sa^2s$. Now set $e = asa$ then we can write,

$$\begin{aligned} e^2 &= (asa)(asa) \\ &= as(aa)sa \\ &= asa^2sa \\ &= asa \\ &= e, \end{aligned}$$

so $e \in E(S)$. Now $\overline{\delta}_a \in E\left(S/\overline{\delta}\right)$ implies that

$$\overline{\delta}_a = \overline{\delta}_a * \overline{\delta}_a = \overline{\delta}_{aa} = \overline{\delta}_{a^2}.$$

So we have,

$$\begin{aligned} \overline{\delta}_e &= \overline{\delta}_{asa} \\ &= \overline{\delta}_a * \overline{\delta}_s * \overline{\delta}_a \\ &= \overline{\delta}_{a^2} * \overline{\delta}_s * \overline{\delta}_{a^2} \qquad \text{since } \overline{\delta}_a = \overline{\delta}_{a^2} \\ &= \overline{\delta}_{a^2sa^2} \\ &= \overline{\delta}_{a^2} \\ &= \overline{\delta}_a \end{aligned}$$

Thus $\overline{\delta}_e = \overline{\delta}_a$. On the other hand we have, $eS = (asa)S = a(saS) \subseteq aS$ which implies $eS \subseteq aS$ and also $Se = S(asa) = (Sas)a \subseteq Sa$. Therefore $Se \subseteq Sa$.

(ii) implies (iii) is clear. Next we show that (iii) implies (i). Let $\overline{\delta}_a = \overline{\delta}_e$ for some $e \in E(S)$. Then $\overline{\delta}_a * \overline{\delta}_a = \overline{\delta}_e * \overline{\delta}_e = \overline{\delta}_{e \cdot e} = \overline{\delta}_{e^2} = \overline{\delta}_e = \overline{\delta}_a$ which implies that $\overline{\delta}_a$ is an idempotent in $S/\overline{\delta}$ so $\overline{\delta}_a \in E\left(S/\overline{\delta}\right)$. ■

Lemma 28 [6] *If S is an inverse semigroup with semilattice of idempotents $E(S)$. Then the relation,*

$$\eta = \left\{(a,b) \in S \times S : a^{-1}ea = b^{-1}eb \text{ for all } e \in E(S)\right\}$$

is the greatest idempotent-separating congruence relation on S.

Theorem 29 *Let $\overline{\delta}$ be an interval-valued fuzzy congruence on an inverse semigroup S. Then $S/\overline{\delta}$ is an inverse semigroup and $\overline{\delta}\left(a^{-1}, b^{-1}\right) = \overline{\delta}\left(a, b\right)$ for all $a, b \in S$.*

Proof. Let $a \in S$. Since S is an inverse semigroup so regular by Lemma 7 implies that there exist an element $t \in S$ such that $a = ata$. Now suppose $\overline{\delta}_a \in S/\overline{\delta}$ we have, $\overline{\delta}_a = \overline{\delta}_{ata} = \overline{\delta}_a * \overline{\delta}_t * \overline{\delta}_a$ where $\overline{\delta}_t \in S/\overline{\delta}$. This shows that $\overline{\delta}_a$ is regular. Now let $\overline{\delta}_a, \overline{\delta}_b \in E\left(S/\overline{\delta}\right)$ where $a, b \in S$. Then by Theorem 27, there exist idempotents $e, f \in E\left(S\right)$ such that $\overline{\delta}_a = \overline{\delta}_e$ and $\overline{\delta}_b = \overline{\delta}_f$. Since S is an inverse semigroup so by Lemma 7, we have $ef = fe$. Therefore $\overline{\delta}_a * \overline{\delta}_b = \overline{\delta}_e * \overline{\delta}_f = \overline{\delta}_{ef} = \overline{\delta}_{fe} = \overline{\delta}_f * \overline{\delta}_e = \overline{\delta}_b * \overline{\delta}_a$. Hence $S/\overline{\delta}$ is regular, and idempotent of $S/\overline{\delta}$ commutes. So by Lemma 7, $S/\overline{\delta}$ is an inverse semigroup. Let $\overline{\delta}_{a^{-1}}, \overline{\delta}_a \in S/\overline{\delta}$, then we have $\overline{\delta}_a * \overline{\delta}_{a^{-1}} * \overline{\delta}_a = \overline{\delta}_{aa^{-1}a} = \overline{\delta}_a$ (since S is regular). Also $\overline{\delta}_{a^{-1}} * \overline{\delta}_a * \overline{\delta}_{a^{-1}} = \overline{\delta}_{a^{-1}aa^{-1}} = \overline{\delta}_{a^{-1}}$ implies $\overline{\delta}_{a^{-1}}$ is the inverse of $\overline{\delta}_a$. Hence $(\overline{\delta}_a)^{-1} = \overline{\delta}_{a^{-1}}$. Now we prove $\overline{\delta}\left(a^{-1}, b^{-1}\right) = \overline{\delta}\left(a, b\right)$ for all $a, b \in S$. Suppose $a, b \in S$ then

$$
\begin{aligned}
\overline{\delta}\left(a^{-1}, b^{-1}\right) &= \overline{\delta}_{a^{-1}}\left(b^{-1}\right) \\
&= (\overline{\delta}_a\left(b^{-1}\right))^{-1} \\
&= \left(\overline{\delta}\left(a, b^{-1}\right)\right)^{-1} \\
&= \left(\overline{\delta}\left(b^{-1}, a\right)\right)^{-1} \\
&= \left(\overline{\delta}_{b^{-1}}\left(a\right)\right)^{-1} \\
&= (\left(\overline{\delta}_b\left(a\right)\right)^{-1})^{-1} \\
&= \overline{\delta}_b\left(a\right) \\
&= \overline{\delta}\left(b, a\right) \\
&= \overline{\delta}\left(a, b\right).
\end{aligned}
$$

Hence $\overline{\delta}\left(a^{-1}, b^{-1}\right) = \overline{\delta}\left(a, b\right)$ for all $a, b \in S$. ■

Theorem 30 *Let $\overline{\delta}$ be an interval-valued fuzzy congruence on an inverse semigroup S. Then $\overline{\delta}$ is an idempotent-separating if and only if $\overline{\delta}^{-1}\left([1,\ 1]\right) \subseteq \eta$, where η is as defined in Lemma 28.*

Proof. Suppose $\overline{\delta}$ is an idempotent-separating. Let $(a, b) \in \overline{\delta}^{-1}\left([1,\ 1]\right)$. Then $\overline{\delta}\left(a, b\right) = [1,\ 1]$. Now let $e \in E\left(S\right)$. Since $\overline{\delta}$ is an interval-valued fuzzy congruence relation on S, so we have,

$$
\begin{aligned}
\overline{\delta}\left(a^{-1}ea, b^{-1}eb\right) &\geq \overline{\delta} \circ \overline{\delta}\left(a^{-1}ea, b^{-1}eb\right) \\
&= \vee_{x \in S}\{\overline{\delta}\left(a^{-1}ea, x\right) \wedge \overline{\delta}\left(x, b^{-1}eb\right)\} \\
&\geq \overline{\delta}\left(a^{-1}ea, b^{-1}ea\right) \wedge \overline{\delta}\left(b^{-1}ea, b^{-1}eb\right) \\
&\geq \overline{\delta}\left(a^{-1}, b^{-1}\right) \wedge \overline{\delta}\left(a, b\right) \\
&= \overline{\delta}\left(a, b\right) \wedge \overline{\delta}\left(a, b\right) \qquad \text{by Theorem 29} \\
&= \overline{I} \wedge \overline{I} \\
&= \overline{I} = [1,\ 1].
\end{aligned}
$$

This shows that $\overline{\delta}\left(a^{-1}ea, b^{-1}eb\right) \geq [1,\ 1]$. Hence $\overline{\delta}\left(a^{-1}ea, b^{-1}eb\right) = [1,\ 1]$, which further, by Theorem 18, implies that $\overline{\delta}_{a^{-1}ea} = \overline{\delta}_{b^{-1}eb}$. Since δ is an idempotent-separating and both $a^{-1}ea$ and $b^{-1}eb$ are idempotent, so $a^{-1}ea = b^{-1}eb$ by Definition 26. Hence $(a, b) \in \eta$. Therefore $\overline{\delta}^{-1}([1,\ 1]) \subseteq \eta$.

Conversely, assume that $\overline{\delta}^{-1}([1,\ 1]) \subseteq \eta$. Let $\overline{\delta}_e = \overline{\delta}_f$ for some $e, f \in E(S)$. Then $\overline{\delta}(e, f) = [1,\ 1]$, which implies that $(e, f) \in \overline{\delta}^{-1}([1,\ 1]) \subseteq \eta$ so $(e, f) \subseteq \eta$. Since η is idempotent-separating we have $e = f$. Thus $\overline{\delta}$ is an idempotent-separating interval-valued fuzzy congruence on S. ■

Theorem 31 *Let S be an inverse semigroup. Then an interval-valued fuzzy relation $\overline{\delta}_\eta$ is an interval-valued idempotent-separating fuzzy congruence relation on S.*

Proof. Follows from above and Theorem 16. ■

5 Group Interval-valued Fuzzy Congruences

In this last section, we will provide the necessary and sufficient conditions for $\overline{\delta}$ to be group interval valued fuzzy congruence relation.

Definition 32 *An interval-valued fuzzy congruence $\overline{\delta}$ on a semigroup S is called a group interval-valued fuzzy congruence if $S/\overline{\delta}$ is a group under the binary operation "$*$" defined in Definition 19. A group is an inverse semigroup having only one idempotent* [6].

Lemma 33 [6] *If S is an inverse semigroup with semilattice of idempotents $E(S)$. Then the relation*

$$\sigma = \{(a, b) \in S \times S : ea = eb \text{ for some } e \in E(S)\}$$

is the least group congruence relation on S.

Theorem 34 *Let S be an inverse semigroup and $\overline{\delta}$ be interval-valued fuzzy congruence relation on S. Then $\overline{\delta}$ is a group interval-valued fuzzy congruence if and only if $\sigma \subseteq \overline{\delta}^{-1}([1,\ 1])$.*

Proof. Let $\overline{\delta}$ be a group interval-valued fuzzy congruence relation on S. This implies that $S/\overline{\delta}$ is a group by Definition 32. Let $(a, b) \in \sigma$, then $ea = eb$ for some $e \in E(S)$, so we have $\overline{\delta}_a = \overline{\delta}_e * \overline{\delta}_a = \overline{\delta}_{ea} = \overline{\delta}_{eb} = \overline{\delta}_e * \overline{\delta}_b = \overline{\delta}_b$, where $\overline{\delta}_e$ is identity element of $S/\overline{\delta}$. This shows that $\overline{\delta}_a = \overline{\delta}_b$. Hence $\overline{\delta}(a, b) = [1,\ 1]$. Therefore $(a, b) \in \overline{\delta}^{-1}([1,\ 1])$. Thus $\sigma \subseteq \overline{\delta}^{-1}([1,\ 1])$.

Conversely, assume that $\sigma \subseteq \overline{\delta}^{-1}([1,\ 1])$. Let $e, f \in E(S)$. Since S is an inverse semigroup so $E(S)$ is commutative by Lemma 7, also $efe \in E(S)$ and $(efe)e = (efe)f$. Therefore by Lemma 33, we have $(e, f) \in \sigma$. But $\sigma \subseteq \overline{\delta}^{-1}([1,\ 1])$ implies that $(e, f) \in \overline{\delta}^{-1}([1,\ 1])$. Hence $\overline{\delta}(e, f) = [1,\ 1]$, which further implies that $\overline{\delta}_e = \overline{\delta}_f$. This shows that $S/\overline{\delta}$ has exactly one idempotent. Since S is an inverse semigroup, therefore $S/\overline{\delta}$ is an inverse semigroup by Theorem 29. This clears that $S/\overline{\delta}$ is regular by Lemma 7. Thus $S/\overline{\delta}$ is group by Lemma 6. Hence $\overline{\delta}$ is a group interval-valued fuzzy congruence relation on S by Definition 32. ■

Theorem 35 *Let S be an inverse semigroup. Then $\overline{\delta}_\sigma$ is a group interval-valued fuzzy congruence relation on S.*

Proof. From the above, Lemma 33 and Theorem 16 we can prove the required theorem. ■

6 Conclusion

In this research we have succeeded to introduce the concept of an interval valued fuzzy congruence relation and quotient semigroup $S/\overline{\delta}$, which satisfy some properties on inverse semigroup. We also defined an interval valued fuzzy kernel of homomorphism and successfully proved homomorphism theorems with respect to an interval valued fuzzy congruence relation $\overline{\delta}$ in this paper.

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