THE EXACT CONNECTION BETWEEN THE FEYNMAN INTEGRAL AND A COMPLETELY INTEGRABLE SYSTEM

(KAITAN TEPAT DI ANTARA KAMIRAN FEYNMAN DAN SUATU SISTEM TERKAMIR LENGKAP)

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ABSTRACT

We review recent attempts to relate the concept of Feynman integral and integrable systems. This constitutes an endeavour on our part in making the Feynman path integral into a mathematically meaningful entity. We then presents a framework which is rooted in the hypothetical relationship between the heuristic concept of Feynman integral in physics and the rigorous mathematical results derived from the theory of (physically significant) completely integrable systems. This idea originates primarily from Witten's (1991) conjecture and Kontsevich's (1992) model which conjecturally able to formulate this remarkable connection. Essentially this link refers to a generator function of intersection numbers on moduli space for stable curves (or *r*-spin curves) and the tau-function of Korteweg-de Vries (or Gelfand-Dikii) hierarchy. In order to display the calculational aspects of this deliberation, certain special models with superpotentials are examined.

ABSTRAK

Kami mengimbau kajian-kajian terkini yang mengaitkan konsep kamiran Feynman dan sistem-sistem terkamir. Hal ini merangkumi usaha kami untuk menjadikan kamiran lintasan Feynman sebagai suatu entiti matematik yang bermakna. Kami seterusnya mengemukakan suatu kerangka kerja yang berakar umbi daripada hubungan hipotetikal di antara konsep heuristik kamiran Feynman dalam fizik dan keputusan-keputusan matematik yang rapi terjana daripada teori (berkepentingan fizikal) sistem-sistem terkamir lengkap. Idea ini berasal terutamanya daripada konjektur Witten (1991) dan model Kontsevich (1992) yang berupaya secara konjektur memformulasikan kaitan hebat itu. Secara asasnya hubungan ini merujuk kepada suatu fungsi penjana bagi nombor-nombor persilangan pada ruang moduli bagi lengkung-lengkung stabil (atau lengkung-lengkung-lengkung spin-*r*) dan fungsi-tau hierarki Korteweg-de Vries (atau Gelfand-Dikii). Untuk memperlihatkan aspek-aspek pengiraan pendekatan ini, model-model khusus tertentu dengan potensisuper adalah diselidiki.

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CHAPTER 1

INTRODUCTION

Physics is considered as a natural science based on experiments and observations relating to our understanding with respect to the natural surroundings. Mathematics is normally perceived as a kind of human's intellectual diversion (see Faddeev 1990). Nonetheless, the two offshoots of human cultures are intimately intertwined. We list out this relationship by highlighting several relevant examples. The mathematical question pertaining to the Euclid's fifth postulate, which concerns the straight lines in Euclidean geometry had catalysed the discovery of the non-Euclidean geometry during the early nineteenth century by Lobachevski, Bolya and Gauss. In particular, the creation of the geometry by Riemann, and this resulted in Riemannian geometry. Generally it is well documented that in the early twentieth century, Riemannian geometry became the underlying fabric of Einstein's gravitational theory. Approximately several thousand years back, many mathematicians became apprehensive with the solutions of algebraic equations in the form of quadratures. Galois finally put this problem to rest in the early twentieth century when he discovered group theory. Currently as it is well known, group theory forms an essential component in the description of symmetry in physics. The aforementioned examples exhibit the inner developments of mathematical conundrums, which originally seen as impractical mind games, are now of profound importance with extensive applications in physics.

The research and development in mathematics and physics is a two-way street process. For example, the first observation of the 'solitary wave' was carried out by J. Scott-Russell and recorded in his article '*Report on Waves*' (1844), and this was being modelled by Korteweg and de Vries (1895) as a phenomenon associated with the Korteweg-de Vries equation (KdV). Subsequently at the Los Alamos laboratory, Fermi et al (FPU) (1955) studied this phenomenon via computer simulations of a one-dimensional lattice with a network of 64 particles of equivalent mass and together with nonlinear interaction between the close neighbours. This FPU report was validated by Kruskal and Zabusky (1965) at the Bell Telephones laboratory, and portrayed as solitary waves – a squared hyperbolic secant form, which goes through each other (dual, triple or more interactions) without change in form and unaffected except for their phases. These physical entities are called 'solitons'. Since this conception, KdV is generally well recognised both in physics and as well as in mathematics, particularly after the sterling works of Gardner et al (GGKM) (1967) on the discovery of the Inverse Scattering Technique (IST) and Lax's (1968) mathematical rigour. This industry progresses aggressively until now with broad mainstream directions. Most notably, as an important prototype of completely integrable system, KdV brought tremendous impact in both fields of physics and mathematics. Mathematically these include differential equations, algebraic geometry, Lie group theory, loop groups, differential geometry, random matrices and etc (see Palais 1997). In physics, these include quantum field theory, statistical physics, quantum gravity, string theory, non-perturbative physics and etc (see

Marshakov 2004). Currently, the results generated from this enterprise have benefited concretely various problems in the field of 'modern mathematical physics' such as string theory, conformal field theory and quantum gravity (see Bullough & Caudrey 1995). The concept of Feynman integral traces a similar historical path. Feynman (1948) heuristically introduced the renowned path integral so as to demonstrate the existence of another alternative formulation of non-relativistic quantum mechanics as to the existing formulations of Schroedinger (wave function) and Heisenberg (matrices or operators). Nevertheless, until now this concept of Feynman integral has influenced and catalysed intense research and development in diverse fields of mathematics and physics (see Johnson & Lapidus 2000). We wish to refer to a vivid comment by Profesor Gian-Carlo Rota (1997) regarding this matter:

The Feynman integral is the mathematicians' 'pons asinorum'. *Attempts to put it on a sound footing have generated more mathematics than any subject in physics since the hydrogen atom. To no avail. The mystery remains, and it will stay with us for a long time.*

CHAPTER 2

LITERATURE REVIEW

2.1 Feynman Path Integral

Before Feynman (1948), essentially there exist two formulations of non-relativistic quantum mechanics, via Schroedinger's differential equation (Schroedinger 1926) based on the particle's wave functions, and Heisenberg's algebra (Heisenberg 1925) rested on the noncommutative Heisenberg's matrices. Dirac's transformation theory made it possible for the two seemingly disparate mathematical formulations to be sysnthesized (Dirac, 1958). According to Schweber (1986), Dirac (1933, 1945) was the one who actually sowed the seeds to this third approach of Feynman's (1948) formulation of quantum mechanics by discussing quantum Lagrangian theory in the setting of well-established ideas from classical Lagrangian theory. This space-time formulation of quantum mechanics was originally constructed in 1948 when Feynman (1948) first introduced heuristically this notion of path integral. Historically the fundamentals of this conception were already developed earlier in Feynman's Doctoral thesis '*The Principle of Least Action in Quantum Mechanics'*, Princeton University, 1942.

Feynman (1948) used the quantum superposition principle as a postulate to derive the path integral formulation. Ingeniously he transformed the postulate into the terminology of the 'sum over all paths', whereby this idea can be expressed as

$$K_q(q_a,t;q_s,t_0) = \sum_{\gamma} \varphi\left[\gamma(s)\right] \qquad (1)$$

K_q is the 'complete probabilistic amplitude', 'kernel' or 'propagator' for a particle (non-relativistic) moving from an initial space-time point (q_a,t_0) to the final (q_s,t), where q represents the point's position vector on the path in n-tuple real numbers R^n generalized coordinate q_j, *j* = 1,2, …, *n*; $\sum_{\gamma}$ is the summation of the contributions from each path γ (inclusive of all probable paths in the intercession) connecting (q_a,t_0) to (q_s,t), and $\varphi\left[\gamma(s)\right]$ the 'probabilistic amplitude' of the path γ. It is customary from the point of view of classical mechanics that there is only one contributing path based on the Hamilton's principle (Goldstein, 1980), the classical path with 'minimum' action that connects the two endpoints of the particle's motion. In the domain of quantum mechanics, all the paths contribute comparably towards the amplitude of the particle's motion with differents phases. These include the paths that do not comply with Euler-Lagrange equation of motion (see Sakurai 1994, Peres 1995). Nonetheless, according to Shaharir's (1986) framework (also Zainal 2001), specifically only the classical path $\gamma \in$ classical path space would contribute effectively in both the classical and quantum domains.

The actual form of the contribution, $\varphi\,[\gamma(s)]$, from each path was proposed by Dirac (1933, 1945) (refer Schweber 1986, Gleick 1992). The observation by Dirac determined, with his unusual intuition, the expression $exp.\left(\frac{iS}{\hbar}\right)$ as an acceptable approximation of K_q, which propagates in an infinitesimal time, where $i=\sqrt{-1}$ and S the classical action function given by (S is measured in terms of $\hbar$, quantum action unit)

$$S[\gamma\,(s)] = \int_{t_0}^{t} L(\gamma\,(s),\dot{\gamma}(s))\,ds \tag{2}$$

where $\cdot$ stands for the derivative with respect to time, L is the Lagrangian of the particle's dynamical system. Subsequently a particle of mass m under the influence of a potential V and Lagrangian

$$L\,(\gamma(s),\dot{\gamma}(s)) = \frac{1}{2}m\dot{\gamma}(s)\cdot\dot{\gamma}(s) - V(\gamma(s))$$

is propagated by K_q of the form (in infinitesimal time interval ε)

$$\begin{aligned} K_q(q_s,q_a;\varepsilon) &= K_q(q_s,t+\varepsilon;\,q_a,t) \\ &= \frac{1}{A}\,exp\left(\frac{i\varepsilon}{\hbar}\left[\frac{m}{2}\left(\frac{q_s-q_a}{\varepsilon}\right)^2 - V\left(\frac{q_s+q_a}{2}\right)\right]\right) \quad , \end{aligned} \tag{3}$$

where A stands for the normalization constant.

Consequently the particle's path is being discretised by dividing the time interval $[t_0,t]$ to N subinterval $[t_{i-1},t_i]$, each of width ε and identifying each by $t_s = t_N$, $t_N - t_0 = N\varepsilon$ and $q(t_i) = q_i$, where q_{i_j}, $j=1,2,\ldots,n;$ as the generalized coordinates.

By employing this novel geometrical idea, Feynman was able to reexpressed the propagator (1) as

$$K_N(q_N,t_N;q_0,t_0) = \underbrace{\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}}_{all\ \ paths}\varphi(q_0,q_1,\ldots,q_{N-1})\,dq_1\ldots dq_{N-1} \tag{4}$$

where q_0, q_N are fixed points.

According to the quantum superposition principle and time slicing prescription, Feynman obtained the propagator (4) of 'path integral' form, which later named as the Feynman integral (polygonal or discrete definition)

$$K(q_s,t;q_a,t_0) = \lim_{N\to\infty}\frac{1}{A}\int_{\substack{all\\paths}} exp\left[\frac{i}{\hbar}\sum_{i=1}^{N} S(q_i,q_{i-1};\varepsilon)\right]\prod_{i=1}^{N-1}\frac{dq_i}{A} \tag{5}$$

where $S(q_i,q_{i-1};\varepsilon)$ is the action function on an infinitesimal time interval $\varepsilon = t_i - t_{i-1}$.

Feynman’s arguments in deriving the expression (5) are primarily based only on analogy, intuitions and heuristics. As a result, the outcome (exclusive of general potentials *V*, refer Inomata (1988)) suggested the presence of a ‘functional integral’, which is normally used in physics literatures (Roepstorff, 1994), that is the Feynman path integral

$$K(q_s, t; q_a, t_0) = \int_{\substack{all \\ paths}} exp\left[\frac{i}{\hbar} S[\gamma(s)]\right] D[\gamma(s)] \quad , \tag{6}$$

where the symbol $D[\gamma(s)]$ refers to ‘Feynman measure’ or ‘path differential measure’. Let us consider the case where the solution of Schroedinger equation is in the form of Feynman integral (6), then the formal path integral representation of the wave function ψ at time *t* is

$$\psi(q_a, t) = M\int_{R^n} \int_{\substack{all \\ paths}} exp\left[\frac{-i}{\hbar}\int_{t_0}^{t} \frac{m\dot{\gamma}(s)\cdot\dot{\gamma}(s)}{2} + V(\gamma(s))ds\right] \phi(q_s)\, D[\gamma(s)] dq_s \tag{7}$$

with the initial condition $\psi(q_a, t) = \phi(q_a)$.

The expression (7) is conventionally interpreted and widely used in the physics literatures in accordance with the polygonal definition (5), which resulted in the form (Khandekar et al, 1993)

$$\psi(q_a, t) = \lim_{N\to\infty} \int_{N^n} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} exp\left[\frac{-mi}{\hbar} \sum_{i=1}^{N} \frac{(q_i - q_{i-1})^2}{2\varepsilon} - \frac{i}{\hbar}\sum_{i=1}^{N-1} V(q_i)\varepsilon\right] \phi(q_s) \prod_{i=0}^{N-1} \frac{dq_i}{\sqrt{\frac{2\pi i \varepsilon \hbar}{m}}}\, dq_s \tag{8}$$

If we follow Kac (1949) who tried to comprehend Feynman’s works (1948) via the methods and tools available in the probability theory, then the construction (7) can be construed as a ‘functional integral’

$$\psi(q_a, t) = M\int_{R^n} \int_{Path[0,t]} exp\left[\frac{-i}{\hbar}\int_{t_0}^{t} V(\gamma(s))ds\right] \phi(q_s)\, dF^{t}_{q_a, q_s}(\gamma(s)) dq_s \tag{9}$$

where $Path[0,t]$ is the continuous path space γ on the interval [0, *t*], and $dF^{t}_{q_a, q_s}(\gamma)$ is the conditional ‘Feynman measure’.

Various names are being associated with the entity $dF^{t}_{q_a, q_s}(\gamma)$, certain well-known examples are the quasi-measure terminology following Skorohod (1974), Dalecky & Fomin (1991) and the language of pro-distribution by Dewitt-Morette et al (1979). Whatever the descriptions being imparted to $dF^{t}_{q_a, q_s}(\gamma)$, it is definitively not a *bona fide* measure and as a result the relation (9) is yet to be genuinely accepted as an integral.

Consequently the Feynman path integrals (5) and (6) with the propagators K, are generally non-existence and currently difficult to verify their validity respectably via the standard mathematical rigourousness. Nonetheless, the intuitive conception (precisely the use of (5)) is found to be very successful indeed in the explicit path integral calculation in various specific fields such as quantum & satistical mechanics, particle field theory, condensed matter physics, polymer physics, quantum chemistry and particle physics theory : quantum electrodynamics, quantum field theory and string theory (Schulman 1981, Wiegel 1986, Lundquist et al 1988, Salam 1988, Kleinert 1990, Khandekar et al 1993, Polyakov 1987, Ramond 1981, Feher et al 1992, Polchinski 1998). The remarkable achievements bestowed upon the Feynman integral in the above-mentioned fields have attracted relentless efforts by the mathematicians to answer the question on the existence of Feynman integral and propounding various integral theories to suit this 'elusive' integral.

2.2 Completely Integrable Systems

Approximately several decades back, two huge themes dominated the development in the theory of dynamical systems. On the one hand, we have remarkable and rapid development in the Chaos theory. The driving question here is how a deterministic system can portray unpredictable and varying behaviours. The other class of system creates the same state of perplexity, though with contradicting reasons behind it. For this 'integrable system', the challenge is to explain the unusual properties of predictability, regularity and the almost periodic nature being shown by the special class of the system's 'soliton'-form of solution. Soliton exhibits 'particle-like' characteristics, for example soliton posseses geometrical forms that display high degree of resistance under various conditions, which normally destroy these characteristics. The preservation of these geometrical features are intimately associated with the symmetrical concepts.This system does not show clearly these symmetries, thus making it difficult to take into account other exceptional conservation laws.

The implication of Noether's theorem (Arnold, 1980; Cantrijn & Sarlet 1981) states that the first integrals of a symmetrical Hamiltonian dynamical system are determined by one parameter symmetry groups; and if the system allows a bigger symmetry group, then correspondingly there exist several other integrals. Examples include the correspondence between linear and angular momentums with translational and rotational invariances respectively. This is observed from the definition that a system's symmetry is in fact a group action and this action does not change the Hamiltonian function and equation of motions. Nonetheless, not all first integrals of the dynamical system's equations can be physically interpreted nor outlined by obvious symmetries in a particular problem. Such a system is normally referred to as a system with hidden symmetries (Arnold 1980).

Influential works by Arnold and Moser in the sixties and seventies of last century (refer Guillermin & Sternberg 1984) stimulated the idea that complete integrability of most of the classical integrable systems can be explained by the existence of the hidden symmetries. We define this system as follows.

Definition: *Completely Integrable System*

An infinite dimensional Hamiltonian system, (P,ω,X_H), with a set $\left(X_{I_j}(x)\middle| x\in P, j=1,2,...\right)$ forms a basis for T_xP and modelled in a Banach space H_T^S, is said to be completely integrable when there exists a sequence of independent functional numeration $I_j[\varphi]$ on $R_{x_{I_j}} \subset P;\varphi \in H_T^S$, which are involutive with respect to the Poisson brackets $\{\cdot,\cdot\}$, that is $\{I_j, I_k\}=0,\quad \forall\, j,k\in Z_+$. (P,ω) is a symplectic manifold with *P* a Banach manifold S^∞ and ω a 2-form on *P*, X_H is the Hamiltonian vector function. □

This stimulation is meaningfully so for the Korteweg-de Vries (KdV) dynamical system since there exist integrals I_1, I_3 and I_5 corresponding to preservation with respect to the Poisson structure or with respect to physical laws i.e. mass, momentum and energy conservation laws respectively, whereas $I_{2j+1}\ (j>2)$ which are infinite in number are yet to be conformed to any physical laws. Consequently KdV is said to be completely integrable with hidden symmetries (eg. Zainal1998). It was conjectured (refer Abraham & Marsden 1978) that the hidden symmetries be associated with an infinite dimensional Lie group action. This is reasonable for Souriau (1970) had shown via moment mappings that Lie groups symmetry of a Hamiltonian function are related to the Lie algebras of the corresponding conserved quantities. The interesting question is: What are the structure form of the Lie group ***G*** which acts on the symplectic manifold (P,ω) and the corresponding Lie algebra ***g*** **,** satisfying the KdV equation and are most harmonized to the infinite dimensional background?

This research project will not claim to answer the question above rigorously, but instead it would throw some light intuitively and depending hugely on the methods related to the theory of coadjoint orbit of Kirillov-Konstant-Souriau (K-K-S) and the properties of Hamiltonian system on Lie groups. This perspective was successfully carried out by Ebin and Marsden (1970) by showing that the solutions of Euler equation for ideal fluids are equivalent to geodesic equations on the group of volume preserving diffeomorphisms. Our framework is largely based on works done by Adler (1979), Lebedev & Manin (1980) and Berezin & Perelomov (1980) and it would portray two important observations. Firstly the relevant symplectic structure is the symplectic structure of the K-K-S orbit and secondly the integrability of the generalised KdV dynamical system and its Lax representation are closely related to the Lie algebra splitting.

Extensive studies on group representations and the foundations of mechanics of recent years have found that the dual space $\boldsymbol{g}^*$ to the Lie algebra $\boldsymbol{g}$ of a Lie group ***G***, supports a natural Poisson structure (refer Souriau 1970, Konstant 1970, Kirillov 1976, Abraham dan Marsden 1978, Arnold 1980, Weinstein 1983, Marle 1983, Palais 1997, Schimd 2004). We wish to sketch that the Poisson structure form of K-K-S represents the symplectic structure of the generalised KdV Hamiltonian system which is obtainable from $\boldsymbol{g}^*$ of an infinite dimensional Lie algebra $\boldsymbol{g}$ - algebra of formal pseudo differential operator of negative degree (Kohn dan Nirenberg 1965, Hormander 1965, Boos dan Bleecker 1985). Subsequently the generalised KdV equation acts as a Hamiltonian system on a coadjoint orbit of the Lie algebra dual space of the pseudo differential operator. Recent research works (refer Date et al 1983, Flashka et al 1983, Drinfeld & Sokolov 1985, Sato 1988, Frenkel 2002) based on similar arguments have successfully shown that intrinsically the Zakharov-Shabat-Ablowitz-Kaup-Newell-Segur (ZS-AKNS) system acts as a Hamiltonian system on a coadjoint orbit of the Kac-Moody algebra dual space (graded infinite-dimensional Lie algebra).

Interestingly, within the framework of Lax approach, Gardner-Faddeev-Zakharov and Gelfand-Dikii programmes, the Lax pairs, $L_t = [B_i, L]$, represent the Hamiltonian system and thus the pairs can reside and operate in $\boldsymbol{g}$. Strictly speaking, the K-K-S formulation exhibits a coherent connection between Lax representation and mechanics on the coadjoint orbit space of a Lie group action $\boldsymbol{G}$. This manifestation would naturally explain the possible proposition to exhibit the complete integrability of the (generalised) Korteweg-de Vries equation (KdV) via the Kirillov-Konstant-Souriau (K-K-S) group theoretical approach.

Remarks: This research had tried to rigorously and convincingly show the remarkable ‘mating’ of two entirely different subjects of Feynman integral and integrable systems, and that it would truly result in a better understanding of the advances and applications of Feynman Path Integral in Physics.

2.3 Objectives

We have been carrying out research on a new framework which involves an approach that clearly established the relationship between ‘Feynmannian path integral’ with the Feynman integral and real/ordinary integral without the use of limiting procedures in a generalized space for a general class of potentials (refer Shaharir 1986, Shaharir 1995, Shaharir & Zainal 1995, 1996a & b, Zainal 2001). We too are looking back into this framework (for consolidating purposes) in the context of connecting it with various basic and current concepts in (physically significant) completely integrable systems (refer Zainal 1998, 2004a, b, c). Accordingly we are listing down four mainstream issues in this field that we had successfully carried out research, currently pursuing and planning to explore further. For the purpose of our proposed SAGA grant application, we would be largely concentrating our efforts to unravel the fourth issue as stated below, which is being researched intensively since the end of last century (refer Dubrovin, 1996).

I) Research on a generalized technique such as Inomata’s scheme (Ho & Inomata 1982), so as to implement an explicit calculation of the Feynman integral for a wider class of potentials *V*. This programme aims to obtain the quantization of the particle’s dynamical system via this calculational-based formulation of path integral (for example, Shaharir & Zainal 1995, 1996a).

II) Research on a generalized form of the real/ordinary integral solution in terms of the classical path for the complex diffusion equation. This is then used to relate the generalized analytical solution and the solution of the complex diffusion equation with a generalized class of potentials in terms of functional integral (in particular, the case involving classical diffusion and Schroedinger equation respectively with the solutions of the form of Feynman-Kac formula and Feynman integral). The real integral solution is strictly transformed into (using certain transformation) a functional integral form in a classical path space. This mathematical formulation aims to consolidate the existence of Feynman integral in the *n*-dimensional Euclidean space and function space by laying down detailed propositions on ideas relating to new realizations of Feynman integral and measure, the stochastic process underlying the particle's dynamics and the concept of unification of classical and quantum domains (for example, Shaharir & Zainal 1996b).

III) To extend the notion of Feynmannian integral as suggested by Shaharir (1986) in classical path space and as well as the Riemannian manifold. The purpose here is to validate and strengthen this framework so as to formulate further the functional integral form in both mentioned spaces. This exercise subsequently would display the particle's quantization procedure in a unitary space, which is in agreement with the notion of Feynman path integral. Our framework possesses several important novel aspects that assuredly allow the formal existence of Feynman integral (for example, Shaharir 1995, Zainal 2001, Shaharir & Nik Rusdi 2002):

a. the original concepts of Feynman integral (Feynman 1948) are embedded, specifically the notion of path integral on the classical path (i.e. the traversed path of particle is in compliance with the laws of classical mechanics), without the use of either limiting procedures nor other *ad hoc* assumptions, particularly with regards to the 'Feynman measure',
b. naturally provides an alternative approach towards the idea of unification between the classical domain (classical diffusion equation) and quantum domain (Schroedinger equation) via the complex diffusion equation and it's solutions (with general potentials). This at once characterises the universality of Feynman integral,
c. increasing insights into the problem of identifying the stochastic process which underlies the corresponding complex diffusion process and probabilistic distribution/measure, particularly for the formidable quantum diffusion case. Indeed, this perspective brings further to light the form of that elusive Feynman measure.

IV) We hypothesise that Witten's (1991) conjecture and Kontsevich's (1992) model can be used to formulate an interesting connection between the two entities, the 'elusive' Feynman integral and the 'predictable' completely integrable systems, that originally seems disparate in nature. Technically, this refers to a generator function of intersection numbers on moduli space for stable curves and the τ-function of the KdV (Korteweg-de Vries) hierarchy. All at once this aims to verify the formal existence of the entity: Feynman path integral, as suggested from a statement made by the eminent mathematician Prof. Michael Atiyah (1994):

"*The Feynman integrals will have been given precise meanings, not by analysis, but by a mixture of combinatorial and algebraic techniques*".

This conjecture is derived fundamentally from two approaches with respect to two-dimensional quantum gravity. Essentially, the correlator of the two-dimensional quantum gravity is the Feynman integral with respect to the 'metric space' of the two-dimensional topological real space. One of the methods of evaluation of the path integral involves the topological field theoretic technique and finally reduces to the integration with respect to the moduli curve spaces. Another method considers an approximation to the metric space with piecewise flat metric and subsequently taking the appropriate continuous limit. In the former approach, the free energy becomes a (tau) function that is defined geometrically as

$$\tau^{\,pt}(t_0, t_1, \ldots) = exp\left[\sum_{g=0}^{\infty} \eta^{g-1} F_g^{\,pt}(t_0, t_1, \ldots)\right] ,$$

where $F_g^{\,pt}(t)$ is the generator function (integral form) of the intersection numbers on the moduli space of stable curves of genus g

$$F_g(t_0, t_1, \ldots) := \sum_{\mathrm{n}} \frac{1}{n!} \int_{\overline{M}_{g,n}} \prod_{i=1}^{n} \left(\sum_k t_k \, \psi_i^{\,k} \right)$$

In precise and conjecturally, we believe that this approach would result in an alternative better concept of Feynman Path Integral: a well-defined mathematical entity with better calculational aspects for a broader class of potentials.

Remarks: Our main objective is to unravel the interesting connection between Feynman Integral and certain basic concepts in Integrable Systems as stated in mainstream issue IV) above. Our past research in issues I), II) and III) were used considerably to address this objective. Specifically we expect to formulate in the near future the Feynman integral's existence, an alternative formal definition of it and to demonstrate its calculational capability in Physics.

2.4 Justification

The main question that captivated many researchers until now (refer Klauder 1986, Shaharir 1988, Khandekar et al 1993, Roepstorff 1994, Johnson & Lapidus 2000, Jefferies 2004) is, what are the important and meaningful features, mathematically and as well as physically, that can be associated with the Feynman integral form of solutions (7), (8), (9)? Currently the existence of Feynman integral is essentially questionable, particularly in the context of mathematical rigour as compared to its remarkable effectiveness in physics (see Johnson & Lapidus 2000, Kolotkosov 2001). We list down the crucial problems that have been engulfing the expression of Feynman integral of the form (7), (8), (9).

Problem A: Tarski (1975) stated that (7) is not rigorously defined. Apart from the problem originating from the fact that the 'flat measure' $D[\gamma(s)]$ is not a *bona fide* measure (based on the definition of a standard measure, for example in Halmos 1974), the normalization constant *M* diverges and $\dot{\gamma}(s)$ also diverges everywhere.

Problem B: According to Roepstorff (1994), even for finite *N*, the right hand side of relation (8) as yet forms an 'improper' integral due to the integrand does not possess the absolute integrability (refer Kingman & Taylor 1966). This is followed by another more puzzling problem, that is the approximate sequence of complex measures (Halmos 1974), $\prod_{i=0}^{N-1} \frac{dq_i}{\sqrt{2\pi i\varepsilon}}$, fail to approximate a limiting measure *dq* on the integration space. In addition, neither in dq_i and nor even in the phase $exp\left(\frac{iS_i}{\eta}\right)$ of (8), there exist any element to avoid a path γ from digression, which causes the integrand of (8) to oscillate wildly. Strictly speaking, the Feynman integral (8) is clearly without any built-in absorbing mechanism.

Problem C: The more sophisticated expression (9) is also problematical. According to Klauder (1986), the complex 'measure' $dF^{t}_{q_a,q_s}(\gamma)$ in (9) is only finitely additive, and thus does not enjoy the special attributes of a countably additive measure (for example, satisfying Fatou Lemma, Lebesgue Theorem and etc., Halmos 1974). The fact that $dF^{t}_{q_a,q_s}(\gamma)$ is not defined as a countably additive measure, in the context of 'standard' measure theory, is due to the unbounded nature of the variations $\int |\, dF^{t}_{q_a,q_s}(\gamma)\,| = \infty$, eventhough the whole measure space is finite, i.e. $|\int dF^{t}_{q_a,q_s}(\gamma)\,| < \infty$, (see Cameron 1960). Thus strictly speaking, the expression (9) is as yet unacceptable as a functional integral.

Problem D: Kac (1949) had proven that the standard heat equation (classical diffusion) with diffusion, diffusive via the potential *V* (a piecewise continuous and non-negative function, which is defined on the interval $-\infty < q < \infty$) possesses a solution that can be expressed as a Wiener integral (Wiener 1930, Dalecky & Fomin 1991)

$$H(q,t) = \int_{C[0,t]} exp\left[\beta \int_0^t V(f(s))ds\right] dW^{t}_{q_a,q_s}(f) \qquad (10)$$

where $dW^{t}_{q_a,q_s}(f)$ stands for the conditional Wiener measure on *C*[0, *t*], a set of continuous function *f* on the interval [0, *t*], $0 \le s \le t$.
Refering to Simon (1979), Glimm & Jaffe (1981), we can write $dW^{t}_{q_a,q_s}(f) = \delta(f(t) - q_s)d\mu_W(f)$, μ_W forming a probabilistic and normalised measure which is positive and countably additive on the function space *C*[0, *t*]. Precisely

μ_W is a *bona fide* measure (according to the 'standard' measure theory). The expression (10) can be shown to be equivalent to (see Klauder 1986)

$$H(q,t) = \mathrm{N}\int exp\left[-\int_0^t \left(\frac{m\dot{f}^2(s)}{2} + V\left(f(s)\right)ds\right)\right]\prod_s df(s), \qquad (11)$$

$m = 1/2D$.

The expression (11) is a version of equivalent form to the Wiener integral (10) when the symbols $\mathrm{N} = \lim_{N\to\infty}(4\pi D\varepsilon)^{-Nn/2}$, $exp\left(-\frac{1}{2}\int \dot{f}^2\, ds\right)$, $\prod_s df(s)$ are grouped together to become a valid measure, and definitely this represents $dW^t_{q_a,q_s}(f)$.

According to this method the solution of Schroedinger equation in the Feynman integral representation can only be formally obtained via complexification of the diffusion coefficient *D*. Nevertheless this method also fails to achieve a valid result due to non-existence of 'Feynman measure' $dF^t_{q_a,q_s}(\gamma)$. Another more basic question is the lack of straightforward arguments which in turn would direct to (11) in the Feynman integral representation and the absence of a built-in 'classical path' concept in this integral.

Problem E: Until now, the present situation lacks more reliable arguments in order to obtain the solutions of heat and Schroedinger equations in the obvious forms which would suggest that there is such a Feynman path integral (and at once would reliably generate the Green function).

Problem F: Definitively and calculationally, Feynman integral must exist for a wider class of potentials *V*, particularly it must exist for the standard examples via the standard integral theory. Currently, this situation is far from satisfactory.

Remarks: Conjecturally, we believe that by untangling the mainstream issue IV) and fulfilling our main objective above-mentioned, many of the problems A-F listed above can be overcome.

CHAPTER 3

RESEARCH METHODOLOGY

The theoretical method that was developed and implemented in this research is rooted in the hypothetical relationship between the heuristic concept of Feynman integral in physics and the precise and rigorous mathematical results derived from the theory of (physically significant) completely integrable systems. Conjecturally, we believe and expect that this approach would result in an alternative better concept of Feynman Path Integral: a well-defined mathematical entity with better calculational aspects (refer Zainal 2004a, b, c).

In concise the proposed method begins from Witten's (1991) conjecture and Kontsevich's (1992) model, which we hipothesise can be used to formulate the interesting connection between the two entities, the 'elusive' Feynman integral and the 'predictable' completely integrable systems, that originally seems disparate in nature. Technically, this refers to a generator function of intersection numbers on moduli space for stable curves and the τ-function of the KdV (Korteweg-de Vries) hierarchy. This conjecture is derived fundamentally from two approaches with respect to two-dimensional quantum gravity. Essentially, the correlator of the two-dimensional quantum gravity is the Feynman integral with respect to the 'metric space' of the two-dimensional topological real space. The key point of the connection between topology of the moduli spaces of algebraic curves and the 'matrix' integrals (of the form of Feynman integral) is the asymptotic expansion of these integrals in terms of the Feynman diagrams. The technique of Feynman diagram expansion was invented by Feynman (see Ramond 1981) for reducing the infinite-dimensional Feynman integral appeared in quantum electrodynamics (QED) to an infinite-series of finite-dimensional integrals. The infinite-series is a summation over all graphs with certain properties representing the physical process, such as the collision pattern of elementary particles.

One of the methods of evaluation of the path integral involves the topological field theoretic technique and finally reduces to the integration with respect to the moduli curve spaces. Another method considers an approximation to the metric space with piecewise flat metric and subsequently taking the appropriate continuous limit. In the former approach, the free energy becomes a (tau) function that is defined geometrically as

$$\tau^{pt}(t_0, t_1, \ldots) = exp\left[\sum_{g=0}^{\infty} \eta^{g-1} F_g^{pt}(t_0, t_1, \ldots)\right] ,$$

where $F_g^{pt}(t)$ is the generator function (integral form) of the intersection numbers on the moduli space of stable curves of genus g with n marked points, $\overline{M}_{g,n}$, or the genus g descendent potential for a compact symplectic manifold $X \equiv pt$:

$$F_g(t_0, t_1, \ldots) := \sum_{n} \frac{1}{n!} \int_{\overline{M}_{g,n}} \prod_{i=1}^{n} \left(\sum_{k} t_k \psi_i^k \right),$$

ψ_i^k are the formal variables (summation convention on *k* applies).

In the latter approach, the generator function in the double scaling limit would result in a τ-function of the KdV hierarchy. We observe the statement of Witten's hypothesis that τ^{pt} is the τ-function (Virasoro's invariant solution) of the KdV hierarchy is justified on the fact that, there must necessarily exist only one quantum gravity. In addition, with reference to the moduli space geometry, we can deduce that τ^{pt} satifies the following string equation (summation convention on ν applies)

$$\frac{\partial F_0(t)}{\partial t_1^1} = \frac{1}{2}(t_0, t_0) + \sum_{n=0}^{\infty} \sum_{\upsilon} t_{n+1}^{\upsilon} \frac{\partial F_0(t)}{\partial t_n^{\upsilon}}$$

It is a basic fact in KdV hierarchy theory (or generally Kamdotsev-Patviashvili (KP)) that uniquely the string theory would determine one of the τ-functions of the KdV hierarchy from all the τ-functions that are being parametrized by Sato grassmanian (refer Date et al 1983).

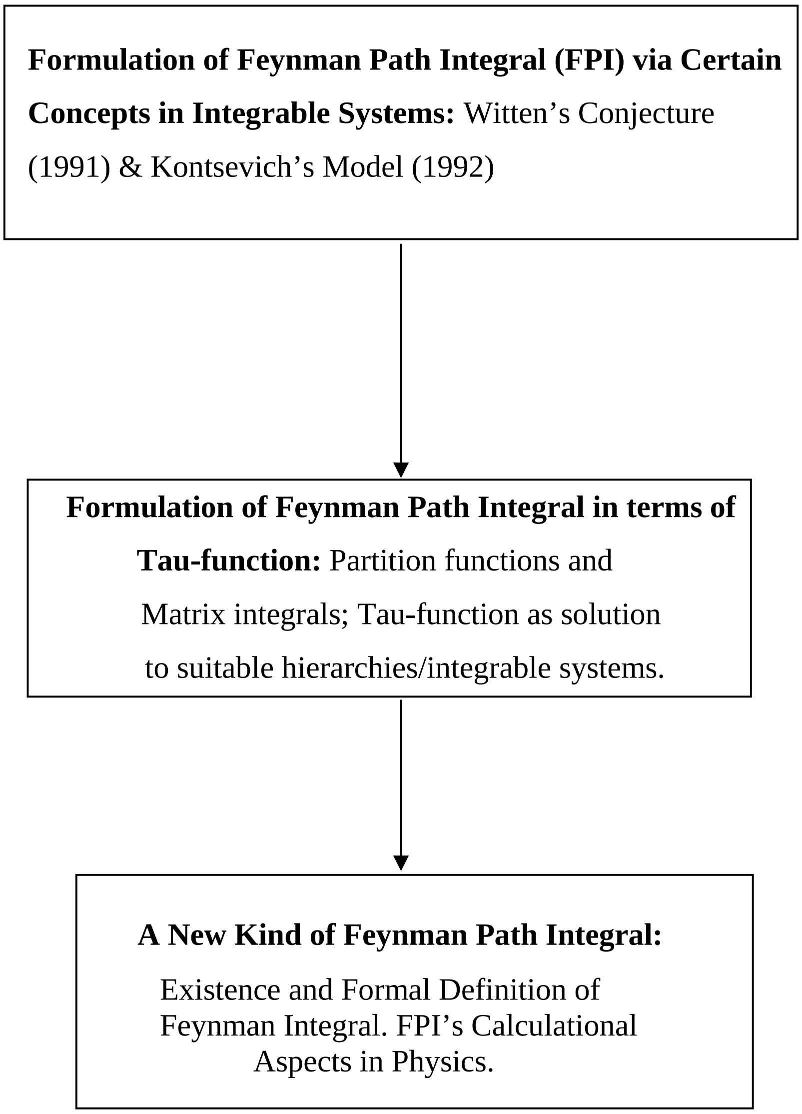
Finally our method will develop further by reformulating a generalised conjecture of Witten (1993): analogously the generator function τ^{r-spin} of the intersection numbers on the moduli space of *r*-spin curves should be identified as the τ-function of the Gelfand-Dikii hierarchy (*r*-hierarchy or generalized KdV). When *r* =2, the conjecture is reduced to the original conjecture, i.e. 2-KdV or KdV equation. This special case has been proven by Kontsevich (1992) and recently there is a new proof proposed by Okounkov-Pandharipande (2001). Nevertheless, we are of the opinion that up to now, the generalised conjecture is still an open problem and thus need further deep research. Finally, to display the calculational aspects of this approach, we would examine some special models with cubic and quartic tree level superpotentials for adjoint chiral superfield for certain gauge theories in this new background.

The research methodology flowcharts are shown below.

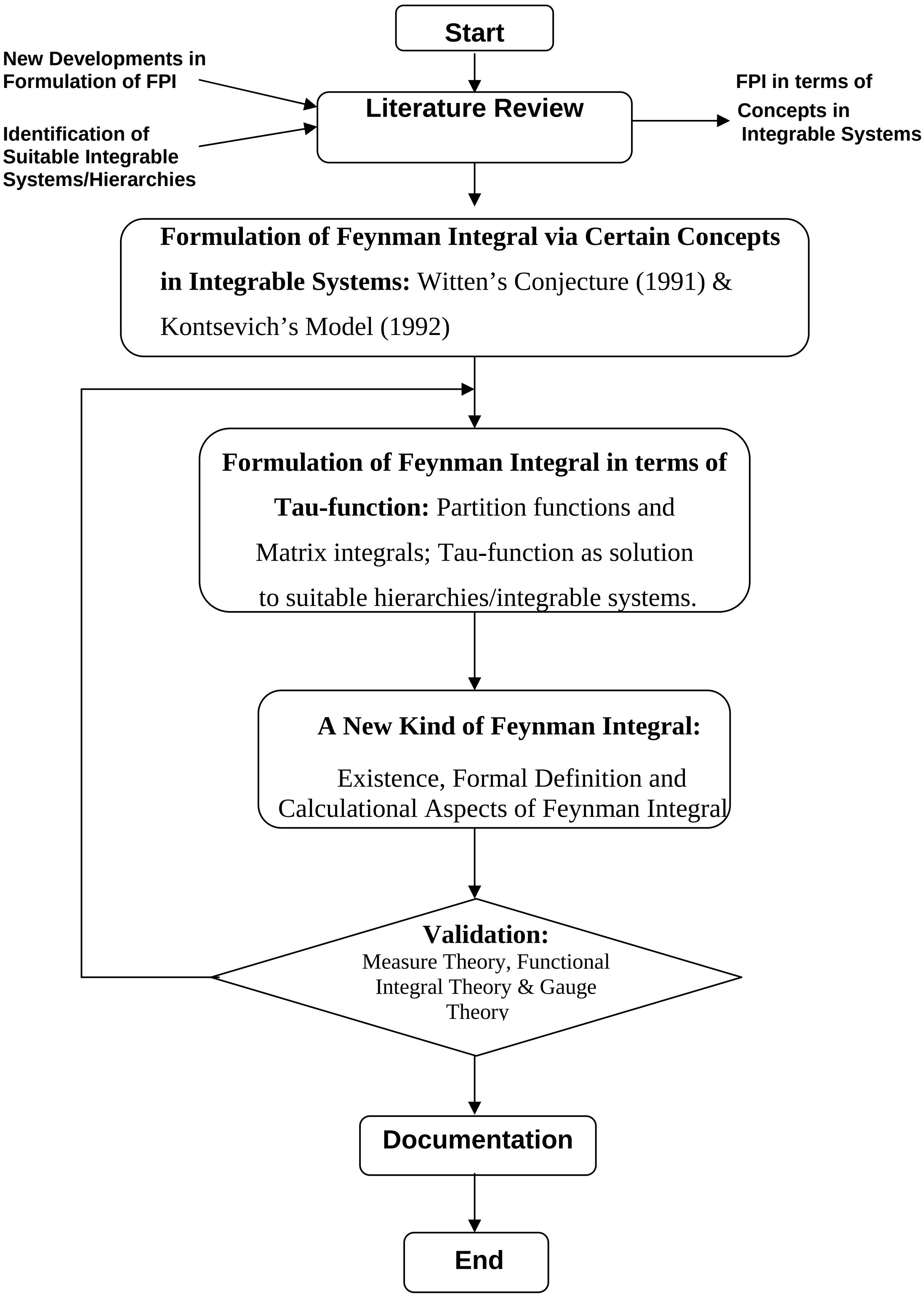
Methodology/Flowcharts

<u>Theoretical Framework</u>

Researcher's conceptual framework is shown in figure below:



Operational Framework



CHAPTER 4

DISCUSSIONS AND CONCLUSIONS

From the above proposition, it is very obvious that this research would have meaningful contributions in the realm of modern mathematical physics and theoretical physics and furthermore would have deep impact with long lasting implications and traces in the domain of fundamental researches. Nonetheless, at the frontier level of this fundamental research, it is an understatement to say that this research is already in the midst of achieving better understanding of its meaning, importance and benefits. Truly we think the current path is still haphazard, particularly to grasp the rigorous mathematical realization of Feynman integral theory & applications and the classification of physically significant integrable systems, and understandably more difficult to establish their relationship in the hypothetical domain of string theory or even M theory. There is as if a consensus amongst theoretical physicists and mathematical physicists that the real antidode to the above-mentioned problems (particularly the problem of unification between classical and quantum domains) is to rethink more meaningfully the concept of 'space-time', both from the geometrical as well as physical viewpoints (for example, probably by implementing appropriate extensions or modifications of Connes (1994) 'non-commutative geometry', or a more radical suggestion is to completely neglect the concept of 'space-time', Manin (1983)). Nevertheless, our point of view follows closely the suggestions of Atiyah (1994). This certainly rests largely upon the background of Witten's conjectures and Kontsevich's model. In precise and conjecturally, we believe that this approach would result in an alternative better concept of Feynman Path Integral: a well-defined mathematical entity with better calculational aspects. In our opinion, an acceptable existence theorem and a formal definition of a universal Feynman Path Integral would considerably dissolve this solid issue.

The discussion above essentially relates one of the surprises of modern mathematical physics, that is the appearance of the remarkable KdV equation in the organization of new invariants of quantum cohomology X (or simply the symplectic manifolds X), and in the process becomes tie up with the 'elusive' Feynman integral. In other words, certain special differential equations (a subset of those known as integrable) have surprisingly appeared predominantly in topological conformal field theory. The appearance of these equations in quantum cohomology is further reflected in the well-known "Virasoro Conjecture", which asserts that the quantum cohomological invariants are fixed points of symmetries consisting of half a Virasoro algebra. These algebras are known to act on many mathematical structures, in particular on the solutions sets of most integrable equations. As our concluding remarks, we are determined in this research to vindicate this 'wild' speculation or conjecture by rigorously and convincingly show the remarkable mating of two entirely different subjects of integrable systems and topological invariants in terms of the Feynman integral, and that it would truly result in a better understanding of the advances and applications of Feynman Path Integral in Physics.

CHAPTER 5

RECOMMENDATIONS

From the above discussion, it is very obvious that this recommended framework would have meaningful contributions in the realm of modern mathematical physics and theoretical physics and furthermore would have deep impact with long lasting implications and traces in the domain of fundamental researches. Nonetheless, at the frontier level of this fundamental research, it is an understatement to say that this framework is already in the midst of achieving better understanding of its meaning, importance and benefits. Truly we observe that the current path is still haphazard, particularly to grasp the rigorous mathematical realization of Feynman integral theory & applications via remarkable concepts emanating from the classification of physically significant integrable systems. This is understandably more difficult to establish their relationship in the hypothetical domain of string theory or even M/Matrix theory. However, via the exceptional G-W theory, it is to be expected that the intersection theory of $\overline{M}_{g,n}(X)$ will again govern by matrix models and their associated integrable hierarchies. There is as if a consensus amongst physicists that the real antidode to the above-mentioned problems (particularly the problem of unification between classical and quantum domains) is to rethink more meaningfully the concept of 'space-time', both from the geometrical as well as physical viewpoints (for example, probably by implementing appropriate extensions or modifications of Connes (1994) 'non-commutative geometry', or a more radical suggestion is to completely neglect the concept of 'space-time', Manin (1983)). Nevertheless, our point of view (Zainal (2004b)) follows closely the deep suggestions of Atiyah (1994). This largely rests upon the background of Witten's conjectures and Kontsevich's model. In precise and conjecturally, in the long run, we forsee and recommend that this approach would result in an alternative better concept of Feynman Path Integral: a well-defined mathematical entity with better calculational aspects. In our opinion, an acceptable existence theorem and a formal definition of a universal Feynman Path Integral would considerably unravel this difficult issue.

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