KADOMTSEV-PETVIASHVILI (KP)

NONLINEAR WAVES IDENTIFICATION

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We declare that this report entitled "KADOMTSEV-PETVIASHVILI (KP) NONLINEAR WAVES IDENTIFICATION " is the result of our own research except as cited in references. The report has not been accepted for any publication and is not concurrently submitted in candidature of any degree.

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ABSTRACT

By observing the periodic hexagonal pattern of surface waves in a large basin namely the MOB (Manoevering Ocean Basin) various solitons interactions patterns were observed due to the repetition of the interaction patterns of two Kadomtsev-Petviashvili (KP) solitons. This research is a systematic and comprehensive study on the Kadomtsev-Petviashvili (KP) equation. In particular the KP equation is the two dimensional form of the Korteweg-de Vries (KdV) equation. Soliton solutions of the KP equation using Hirota Bilinear method was adopted in this research. Two-soliton solutions of the KP equation can produce a triad, quadruplet and a non-resonance structures. In three-soliton solutions of the KP equation, many other interaction patterns can be observed. For example, a triad with a soliton and a quadruplet with a soliton. A computer program, KPPRO was developed using Microsoft Visual C++ to simulate various interactions patterns.

ABSTRAK

Dengan memerhatikan bentuk gelombang permukaan dalam sebuah tangki lautan (MOB: "Manoevering Ocean Basin") berbagai bentuk interaksi soliton telah diperhatikan serupa dengan corak ulangan interaksi dua soliton Kadomtsev-Petviashvili (KP). Penyelidikan ini adalah kajian yang sistematik dan menyeluruh mengenai persamaan Kadomtsev-Petviashvili (KP). Secara umumnya, persamaan KP merupakan sejenis persamaan dua dimensi Korteweg-de Vries (KdV). Penyelesaian persamaan KP yang menggunakan kaedah Bilinear Hirota dipilih dalam kajian ini. Penyelesaian dua soliton persamaan KP akan menghasilkan struktur-struktur berbentuk "triad", kuadruplet dan struktur tak beresonan dalam interaksi soliton. Dalam penyelesaian tiga soliton persamaan KP, banyak lagi struktur interaksi soliton dapat diperhatikan. Contohnya antara "triad" dengan satu soliton dan antara kuadruplet dengan satu soliton. Satu program komputer yang dinamakan sebagai KPPRO telah dibangunkan dengan menggunakan Microsoft Visual C++ supaya pelbagai struktur interaksi soliton dapat dihasilkan.

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CHAPTER I

INTRODUCTION

1.1 Preface

More than 170 years ago, the phenomenon of the solitary wave, which was discovered by the famous British scientist, John Scott Russell as early in 1834, has been greatly concerned with the development of physics and mathematics. Interest in it is growing constantly-now it has been proved that a large number of the nonlinear evolution equations have soliton solutions by using numerical calculations and the theoretical analysis.

Solitary waves have the striking property since they can keep the shape of the wave unchanged even after interaction. This is similar to the colliding property of particles. So Kruskal and Zabusky (1965) named them "solitons" due to the recurrence property. The solitary waves not only have been observed in nature, some of them recently have also been produced in laboratories.

The theory of soliton is closely related to modern physics. On one hand, this theory is also applied to explain a lot of physical problems and on the other hand, the theory of soliton is continuously progressing and developing. We will give a brief account of the history of soliton theory so as to understand this relationship more clearly in the literature review section.

1.2 Background of The Problem

There are many examples of resonance in physics. However, resonance in soliton interaction is an interesting phenomena. In this paper, we will use the Kadomtsev-Petviashvili (KP) equation to model the two-soliton interactions,Freeman, 1978. In particular, the KP equation is a two-dimensional of the Korteweg-de Vries (KdV) equation. Miles (1977),discovered that in the interaction of two solitons, the interaction region between the incident solitons and the centered-shifted solitons after interaction is essentially itself a single soliton leads to a very simple conceptual picture of the interaction process. This interaction soliton is the *resonant soliton* associated with the two incident solitons.

1.3 Statement of The Problem

A wide range of one-dimensional, nonlinear waves in weakly dispersing media such as waves in shallow water, ion acoustic and magneto acoustic waves in plasma, etc. are described by the Korteweg-de Vries equation. This equation can be equally well applied to media with negative or positive dispersion.

The extension of this equation to motions in more than one dimension were given by Kadomtsev and Petviashvili (1970), who generalized the dispersion relation to give an extra term in the equation due to the extra dimension and Satsuma (1976) had solved the Kadomtsev-Petviashvili (KP) equation by using Bilinear method while Ong (1993) had studied the solution given by Satsuma.

Earlier studies have indicated that motion of solitons of classical nonlinear evolution equations for example, the Korteweg-de Vries (KdV) equation and the Kadomtsev-

Petviashvili (KP) equation can exhibit resonance. This occurs if certain constraints on the wave numbers and frequencies of the nonlinear waves are satisfied (Chow, 1997). Since Satsuma (1976) had solved the KP equation by using Hirota Bilinear method, therefore it is possible to investigate the interactions patterns for two, three and four KP solitons resonating among themselves.

1.4 Objective of The Study

The main objectives of this research are to:

1. Study the interactions patterns of two KP solitons.
2. Study the interactions patterns of three KP solitons.

1.5 Importance of The Study

Since we cannot spend so much fund to set up an MOB that is very costly to maintain, thus this research will provide another avenue to solve KP equation using KPPRO which is a numerical solver to produce the simulation of interaction patterns of two and three KP solitons. This research will be at par with the most recent developments in nonlinear fields especially in the research of soliton. The outcomes of this research will be published in international journals and talks presented in local and international colloquiums, conferences and seminars.

1.6 Scope of The Research

In this research, we will only consider the positive dispersion of the KP equation which is shown as below.

$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} = 0. \tag{1.1}$$

We wish to observe the interactions patterns of two, three and four-soliton solutions of the above KP equation. Various interactions patterns involving a soliton, a triad or a quadruplet will be studied.

1.7 Methodology of The Research

This research adopted the analytic solution given by Satsuma (1976) using Hirota Bilinear method as

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2} \ln f \tag{1.2}$$

By using this method, we will study the interactions patterns for two, three and four KP solitons solutions. To solve Equation (1.1), we will develop a computer program namely KPPRO by using Microsoft Visual C++ Professional Edition to automatically solve Equation (1.1). Later we will plot the 2D and 3D graphs of soliton interactions.

1.8 History Of Soliton

Solitons made their first appearance in the world of science with the beautiful report on waves, presented by J. Scott Russell in 1844 at the British Association for the Advancement of Science. In his 'Report on Waves', he vividly wrote (Newell, 1985):

"I believe I shall best introduce the phenomenon by describing the circumstances of my first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped-not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some thirty feet long and a foot to a foot and half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation, a name which it now greatly bears".

This is the unique phenomenon observed by Russell on the Edinburgh-Glasgow canal in 1834. Moreover, he thought it to be a stable solution of fluid motion and coined the word "solitary waves" to name it. However, Russell could not prove his conclusion or made physicists believe it at that time. Since then, the problems about solitary waves has caused a wide range of arguments among physicists of the time.

Until 1895, after sixty years, the famous Dutch mathematician Korteweg and his student de Vries began to study the equation for the motion of the shallow water waves along a direction using the long wave approximation and the small amplitude assumption,

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{l}}\frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{\sigma}{3}\frac{\partial^2 \eta}{\partial x^2}\right) \tag{1.3}$$

where η is the height of the peak, l is the depth of the water, g is the gravitational acceleration, α and σ are physical constants (Guo, 1995). They made complete analysis on the solitary wave phenomenon and from the above equation, they obtain the pulse like solution of the solitary wave whose shape is not changeable, and is consistent with Russell's descriptions about the solitary wave. So the existence of solitary waves was confirmed by the theory.

In 1965, the famous American physicist and the member of the American Academy of Science, Kruskal and the physicist Zabusky investigated and analyzed the nonlinear simulation in detail and obtained more complete and rich results, which confirmed the conclusion that the solitary wave do not change the shapes after the interaction. Since the solitary waves have the unchangeable property like the collision of particles, they named them "solitons".

Kruskal and Zabusky's work was an important milestone in the history of the soliton theory. The concept of the "soliton" introduced by them correctly revealed the substance of solitary waves and had been accepted in general. Hence forward, the study of soliton theory had been developed more vigorously and caused a world wide study. Besides the study of solitons in the fields such as fluid dynamics, elementary particle physics, plasma physics, etc., the solitons were also found one after another in condensed matter physics, superconductivity physics, laser physics, biophysics etc.. Up to now, a rather complete mathematical and physical theory of soliton has been formed.

1.8.1 The Korteweg-de Vries (KdV) Equation

Here is a little biography of Korteweg and de Vries. Diederik Johannes Korteweg (1848-1941) was a student of J. D. van der Waals and received the first doctoral degree of the University of Amsterdam in 1878 for his dissertation on the motion of a viscous fluid in an elastic tube, with application to arterial blood flow. He occupied the chair of

Mathematics and Mechanics at the same university from 1881 to 1918. His biographical memoir does not mention any of his work on water waves nor does it cite his 1895 paper with de Vries.

Korteweg appears to have believe that the paradox posed by the solitary wave, *vis-à-vis* the prediction of Airy's shallow-water theory that long waves in a rectangular canal must necessarily change their form as they advance, becoming steeper in front and less steep behind and he suggested the problem of long waves to his student Gustav de Vries. Biographical data on Gustav de Vries are difficult to obtain (he is not to be confused with the Dutch mathematician H. de Vries), but it is known that he was a member of the Wiskundig Genootschap which is Dutch Mathematical Society from 1892, defended his thesis in 1894 and subsequently taught at the Gymnasiums in Alkmaar and Haarlem. He had published two papers on cyclones in the *Verhandlingen* of the Royal Dutch Academy of Arts and Sciences in 1896 and 1897. The 1895 paper of Korteweg & de Vries was excerpted and translated from de Vries's 1894 thesis (Miles, 1981; Kox, 1995; Bullough and Caudrey, 1995).

The equation they used to study long water waves in a rectangular canal was named after them. The KdV equation is a nonlinear partial differential equation given by

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.4}$$

where subscripts denote partial differentiations. In general the KdV equation describes the unidirectional propagation of small but finite amplitude waves in a nonlinear dispersive medium.

Historically, Korteweg and de Vries set out to settle the question: If friction is neglected, do long water waves necessarily continue to steepen in front and become less steep behind? Their answer was no; in particular they showed that Equation (1.4) has steady progressing wave solutions, namely the solitary wave

$$u(x,t) = \frac{1}{2}a^2\operatorname{sech}^2\left[\frac{1}{2}a\left(x - x_0 - c^2t\right)\right] \tag{1.5}$$

and the periodic cnoidal wave which can be written in terms of Jacobi elliptic functions. The solitary waves form a one parameter family of pulse-shaped solutions, aside from the trivial translation in x, where the velocity c^2 is proportional to the amplitude, $\frac{1}{2}a^2$ and the width $\frac{1}{a}$ is inversely proportional to the square root of the amplitude (see Figure 1.1). Therefore, taller solitary wave travel faster and are narrower than the shorter ones (Miura, 1978).

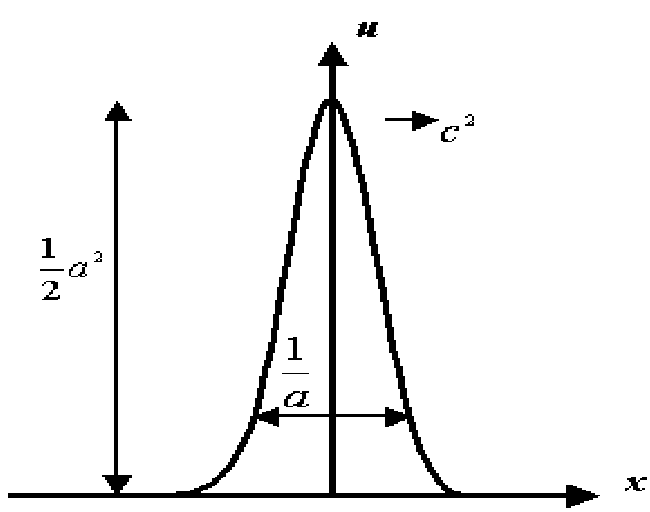

Figure 1.1: Solitary Wave

Although many nonlinear dynamical systems have solitary waves associated with them, the solitary waves of the KdV equation and some other nonlinear evolution equations has distinguished property. Consider the initial-value problem where two solitary waves of distinct amplitudes are placed on the real line with the taller one to the left of the shorter one. They should be spaced enough apart so that only their exponentially small tails overlap (see Figure 1.1).

This initial condition is then evolved in time according to the KdV equation and because the taller solitary wave is to the left, it will travel faster to the right, catch up with the shorter one and they will undergo a nonlinear interaction according to the KdV equation.

Surprisingly, they emerge from the interaction unchanged in waveform and amplitude, but slightly shifted from where they would have been had no interaction occurred (see Figure 1.2). It was this particle like properties of the solitary waves which are:

1. steady progressing pulse like solution, and
2. the preservation of their shape and speeds after interaction

which led Zabusky and Kruskal to call them solitons. Thus a single soliton is a solitary wave but solitary waves are solitons only if they have the above described properties.

The proof that two solitons emerge from the interaction unchanged was first given by Lax and the general case of N solitons is obtained using the inverse scattering method by Gardner, Greene, Kruskal and Miura (1967).

1.8.2 Properties of Solitons

Since the first discovery of solitary waves by Scott Russell, many happenings about solitary waves were discovered experimentally or theoretically. In 1965, Zabusky and Kruskal reported the celebrated numerical computation of solutions of the KdV equation and revealed remarkable stability of the solitary waves, each of which behaved like a "particle". After an interaction of two solitary waves, each wave restore its original shape and continues its course of travel. This is an example of the recurrence process. Thus, the wave behaves like "particle". Because of this behavior, the solitary waves are called "solitons". Solitary waves have the following properties:

1. These localized waves are bell-shaped and travel with permanent form and constant speed.

2. Speed of soliton is proportional to its amplitude, which means, that taller solitary waves travel faster than shorter ones.

3. The width of a soliton (at half the height) is inversely proportional to the square root of its amplitude, which mean, that a taller solitary wave is much thinner compare to the shorter ones.

4. Three fundamental physical quantities namely mass, momentum and energy of solitons were always conserved. In fact there are infinitely many conserved quantities satisfied by solitons.

5. Solitons can interact with each other without change of shape and will eventually emerge as it is after the interaction. The only indication that a linear interaction has not occurred is that the two waves are phase-shifted that is they do not in the positions after interaction which would be anticipated if each were to move at a constant speed throughout the collision. The 3D plot of this phenomena is given in Figure 1.2.

6. The taller one, therefore, appears to overtake the shorter one and continue on its way intact and undistorted. This, of course, is what we would expect if the two waves were to satisfy the linear superposition principle. But they certainly

do not and this suggests that we have a special type of nonlinear process at work here.

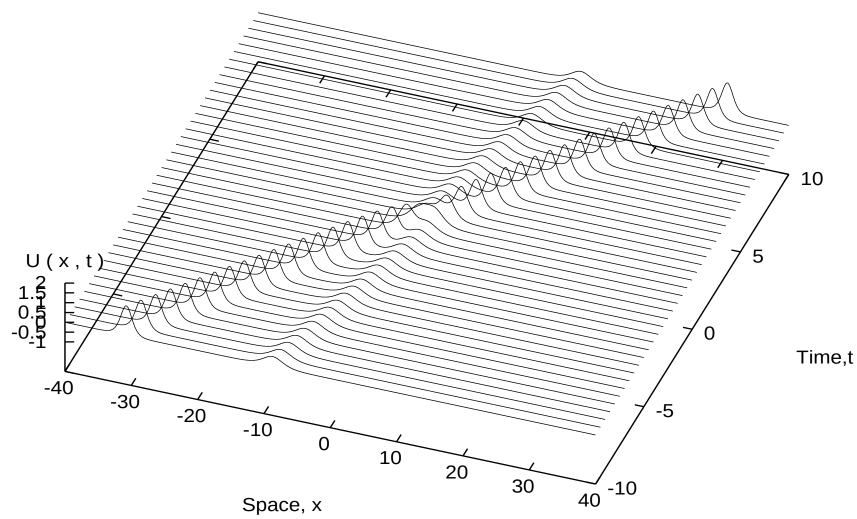

Figure 1.2: 3D plot of two-soliton interactions.

7. The sequence of interactions shows clearly the recurrence phenomena as described by Zabusky and Kruskal. Figures 1.3 and 1.5 show the numerical simulations of two solitons interactions at time $t = -10$ (before interactions) and at time $t = +10$ (after interactions) respectively. Figure 1.4 shows the full interaction of two solitons at time $t = 0$. This is the famous "collision test" of Zabusky and Kruskal.

8. The taller soliton can "chase" after a shorter soliton and eventually "overtake" it after the two solitons interacts and all the solitons retained its shape, mass, momentum and energy. This behavior is exactly like a particle-like character which seem to retain their identities in a collision. Figures 1.3 and 1.5 are exactly symmetrical and this proves that it has successfully gone through the Zabusky and Kruskal collision test.

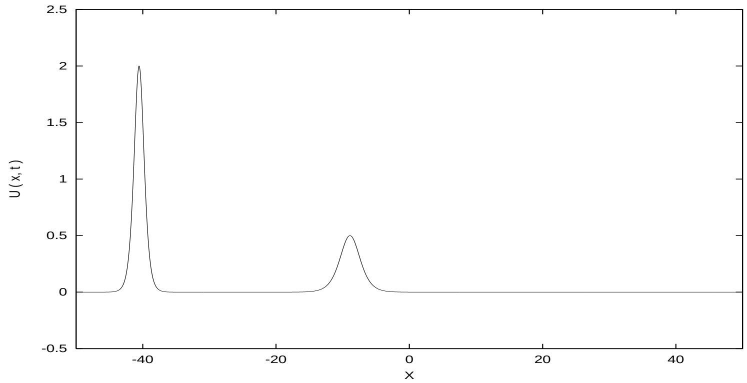

Figure 1.3: Two-soliton interactions at $t = -10$.

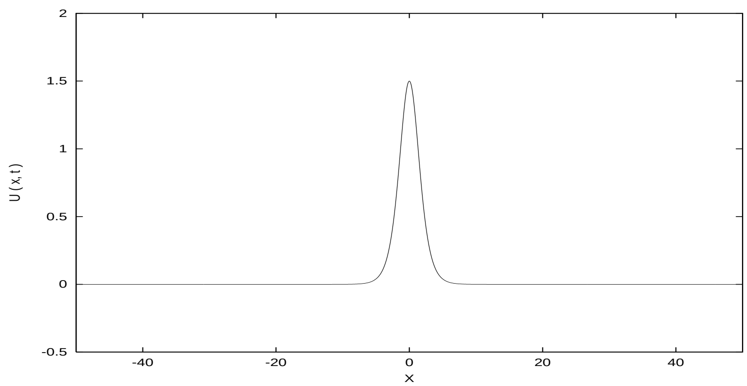

Figure 1.4: Two-soliton interactions at $t = 0$.

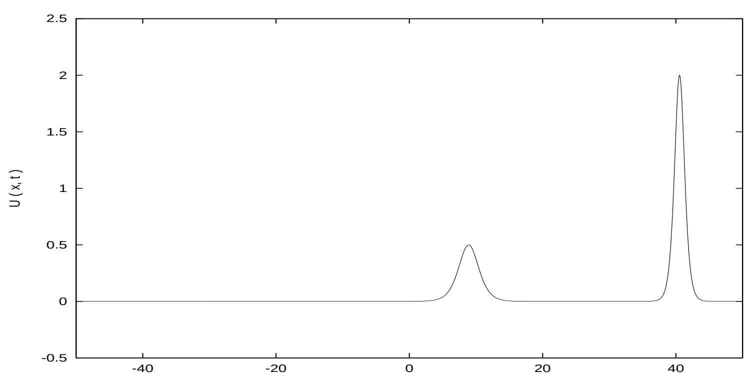

Figure 1.5: Two-soliton interactions at $t = +10$.

1.8.3 The Kadomtsev-Petviashvili (KP) Equation

In the early 1970s, attempts have been made to look for other sort of interactions apart from one dimensional approximation. Kadomtsev and Petviashvili (1970) used the idea of the parabolic Leontovich equation and the KdV equation to derive an equation describing the propagating of weakly non one dimensional acoustic waves in a dispersive medium. That equation has the same degree of universal applicability as the KdV equation. Thus, Kadomtsev and Petviashvili proposed a generalization of KdV equation to two space dimensions. In the differential form the KP equation looks like

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)+3\sigma^2 u_{yy} \quad = \quad 0, \tag{1.6}$$

where $u(x,y,t)$ is a scalar functions and $\sigma^2=\pm 1$. We can simply write the KP equation as

$$(u_t+6uu_x+u_{xxx})_x \pm 3u_{yy}=0, \tag{1.7}$$

where subscripts denote the derivatives of the corresponding variables. The upper plus sign in Equation (1.7) pertains to a medium with negative dispersion and the lower minus sign to positive dispersion (Tajiri and Murakami, 1989).

1.9 Outline Of Report RMC Vot 75023

This report examines the interactions patterns produced by the KP equation. We will use the analytic solution given by Satsuma (1976) by using Hirota Bilinear method. Our main interest is the interactions patterns produced by solitons interaction in the KP equation.

Chapter 2 will discuss about the interactions of two KP solitons. In this chapter we will look for the general solution of two KP soliton. After that, we will build a computer program to generate the wave structure so that we can produce interaction patterns. Besides that, we also discuss about the condition for resonance to happen and provide condition for three types of resonances which are full resonance, partially resonance and non resonance to happen. Each of these resonances will produce structures like *triad, quadruplet* and a *cross*. Since we have these resonance structures, we can discuss more about the interactions of three KP soliton in Chapter 3 that involves three KP soliton interactions. We will discuss about two types of interactions which are interactions between triad and a soliton and also interactions between quadruplet and a soliton. Moreover, in the interactions of triad and a soliton, we have three versions of interactions where we use the same parametric values but with different sequences. To have interactions between quadruplet and a soliton, we need to have general solution for three KP solitons. Hence we show this process of derivation in this chapter. On the other hand, we also show that the solution for some specific case can be transformed into Wronskian determinant. The conclusion and summary about our discussion can be found in the last chapter and we also propose some other research area that we can do in future.

1.10 Conclusion

This chapter gives an overview of what is going to be discussed in the following chapters. A good reason on why we do this study is stated in the background of the problem as well as in the problem statement. The objective of the study was also given in details. However, there is some limitation in this study and is stated in the scope of the study. How we carry out the study is mentioned in the methodology of the study and the outlines of the report is given at the end of this chapter.

CHAPTER II

INTERACTIONS OF TWO SOLITONS

2.1 Introduction

This chapter will discuss about the Kadomtsev-Petviashvili (KP) equation and the derivation of the general solution for the two KP solitons. Moreover we will look at the soliton interaction patterns. All illustrations of the interaction patterns were included in the computer simulation section. Besides that, we will also discuss more about the condition for resonances to occur.

2.2 The Kadomtsev-Petviashvili (KP) Equation

An extension of the Korteweg-de Vries (KdV) equation to the two-dimensional case was given by Kadomtsev and Petviashvili in order to discuss the stability of the one dimensional soliton in weakly dispersive media. It is now called the Kadomtsev-Petviashvili (KP) equation. This equation describes a slow variations in y direction of the waves propagating along the x direction. The KP equation is the most studied of nonlinear integrable equations in three independent variables x, y and t (Konopelchenko, 1993).

The KP equation looks like

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)+3\sigma^2\frac{\partial^2 u}{\partial y^2}=0, \tag{2.1}$$

where $u(x, y, t)$ is a scalar functions and $\sigma^2 = \pm 1$. We can simply write the KP equation as

$$(u_t+6uu_x+u_{xxx})_x \pm 3u_{yy}=0, \tag{2.2}$$

where subscripts denote the derivatives of the corresponding variables. The upper plus sign in Equation (2.2) pertains to a medium with negative dispersion and the lower minus sign to positive dispersion (Tajiri and Murakami, 1989). In this dissertation, we will only treat the positive dispersion case. Therefore the equation we will consider is

$$(u_t+6uu_x+u_{xxx})_x - 3u_{yy}=0. \tag{2.3}$$

Now, consider the linearized form of the Equation (2.3) which is

$$(u_t+u_{xxx})_x - 3u_{yy}=0, \tag{2.4}$$

then we have plane-wave solutions where phase variables $kx + my - \omega t$ satisfied the dispersion relation

$$\omega=-\left(\frac{3m^2}{k}+k^2\right). \tag{2.5}$$

To show this, substitute $u = e^{i(kx+my-\omega t)}$ into Equation (2.4). Thus we have

$$\begin{aligned}
\left(-i\omega e^{i(kx+my-\omega t)}-ik^3e^{i(kx+my-\omega t)}\right)_x+3m^2e^{i(kx+my-\omega t)} &= 0,\\
k\omega e^{i(kx+my-\omega t)}+k^4e^{i(kx+my-\omega t)}+3m^2e^{i(kx+my-\omega t)} &= 0.
\end{aligned}$$

After we cancel the $e^{i(kx+my-\omega t)}$ term, we have

$$\begin{aligned}
k\omega+k^4+3m^2 &= 0,\\
k\omega &= -3m^2-k^4,\\
\omega &= -\left(\frac{3m^2}{k}+k^2\right).
\end{aligned}$$

A more convenient way to parameterize this relation is to write $k = l + n$ and $m = n^2 - l^2$ whence

$$\begin{aligned} \omega &= -\left(\frac{3(n^2 - l^2)^2}{l+n} + (l+n)^2 \right), \\ &= -4(l^3 + n^3). \end{aligned} \qquad (2.6)$$

N-soliton solution for Equation (2.3) had been solved by Satsuma (1976) by using Hirota Bilinear method as

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2} \ln f, \qquad (2.7)$$

$$= 2\left[\frac{f_{xx} f - f_x^2}{f^2} \right], \qquad (2.8)$$

with the function $f(x,y,t)$ given by

$$f = \left| \delta_{ij} + \frac{a_i}{l_i + n_j} \exp(\eta_i) \right|, \qquad (2.9)$$

where a_i is a constant and

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

and

$$\eta_i = k_i x + m_i y - \omega_i t$$

with

$$\begin{aligned} k_i &= l_i + n_i, \\ m_i &= n_i^2 - l_i^2, \\ \omega_i &= -4(l_i^3 + n_i^3), \qquad i,j = 1,2,3,...,N. \end{aligned} \qquad (2.10)$$

In this chapter we only consider $N=2$, therefore we have

$$f = \begin{vmatrix} 1+\dfrac{a_1}{l_1+n_1}\exp(\eta_1) & \dfrac{a_1}{l_1+n_2}\exp(\eta_1) \\ \\ \dfrac{a_2}{l_2+n_1}\exp(\eta_2) & 1+\dfrac{a_2}{l_2+n_2}\exp(\eta_2) \end{vmatrix}, \tag{2.11}$$

$$\begin{aligned} f &= 1+\frac{a_1}{l_1+n_1}\exp(\eta_1)+\frac{a_2}{l_2+n_2}\exp(\eta_2)+\frac{a_1}{l_1+n_1}\frac{a_2}{l_2+n_2}\exp(\eta_1+\eta_2) \\ &\quad -\frac{a_1}{l_1+n_2}\frac{a_2}{l_2+n_1}\exp(\eta_1+\eta_2), \\ f &= 1+\frac{a_1}{l_1+n_1}\exp(\eta_1)+\frac{a_2}{l_2+n_2}\exp(\eta_2) \\ &\quad +a_1a_2\left(\frac{(l_1+n_2)(l_2+n_1)-(l_1+n_1)(l_2+n_2)}{(l_1+n_2)(l_2+n_1)(l_1+n_1)(l_2+n_2)}\right)\exp(\eta_1+\eta_2), \\ f &= 1+\frac{a_1}{l_1+n_1}\exp(\eta_1)+\frac{a_2}{l_2+n_2}\exp(\eta_2) \\ &\quad +a_1a_2\left(\frac{(l_1-l_2)(n_1-n_1)}{(l_1+n_2)(l_2+n_1)(l_1+n_1)(l_2+n_2)}\right)\exp(\eta_1+\eta_2). \end{aligned}$$

Suppose that $\varepsilon_i = \dfrac{a_i}{l_i+n_i}$, then

$$f = 1+\varepsilon_1\exp(\eta_1)+\varepsilon_2\exp(\eta_2)+\left(\frac{(l_1-l_2)(n_1-n_2)}{(l_1+n_2)(l_2+n_1)}\right)\varepsilon_1\varepsilon_2\exp(\eta_1+\eta_2). \tag{2.12}$$

Thus we have

$$f = 1+\varepsilon_1\exp(\eta_1)+\varepsilon_2\exp(\eta_2)+A_{12}\varepsilon_1\varepsilon_2\exp(\eta_1+\eta_2), \tag{2.13}$$

where

$$A_{12} = \frac{(l_1-l_2)(n_1-n_2)}{(l_1+n_2)(l_2+n_1)}. \tag{2.14}$$

2.3 Interaction Of Two Solitons

We will use the function $f(x,y,t)$ which is given in Equation (2.13) to study the position of each soliton while interacting. Hence we have the following 4 cases (Ong, 1993).

2.3.1 Case 1: η_1 Fixed; η_2 Tends To $+\infty$

In this case, we can observe that the value of $1+\varepsilon_1\exp(\eta_1)$ is very small compared to the value of $\varepsilon_2\exp(\eta_2)+A_{12}\varepsilon_1\varepsilon_2\exp(\eta_1+\eta_2)$. Therefore Equation (2.13) will produce

$$\begin{aligned} f &\approx \varepsilon_2\exp(\eta_2) + A_{12}\varepsilon_1\varepsilon_2\exp(\eta_1+\eta_2), \\ &\approx \varepsilon_2\exp(\eta_2)\left[1 + A_{12}\varepsilon_1\exp(\eta_1)\right]. \end{aligned} \tag{2.15}$$

We know that η_2 is a linear function of x, therefore it will become zero after we differentiate it twice with respect to x. To explain this, let say we take

$$f = \exp(\eta)(Q(x)). \tag{2.16}$$

Substitute the above equation into Equation (2.7), then we have

$$\begin{aligned} u &= 2\frac{\partial^2}{\partial x^2}\ln(\exp(\eta)Q(x)), \\ &= 2\frac{\partial^2}{\partial x^2}(\ln\exp(\eta) + \ln Q(x)), \\ &= 2\frac{\partial^2}{\partial x^2}(\eta) + 2\frac{\partial^2}{\partial x^2}(\ln Q(x)), \\ &= 0 + 2\frac{\partial^2}{\partial x^2}(\ln Q(x)). \end{aligned}$$

From the above result, it is shown that we can cancel the $\exp(\eta)$ term without affecting the value of u. Thus we can use the same concept on Equation (2.15).

$$\begin{aligned} u &= 2\frac{\partial^2}{\partial x^2}\ln(\varepsilon_2\exp(\eta_2)\left[1 + A_{12}\varepsilon_1\exp(\eta_1)\right]), \\ &= 2\frac{\partial^2}{\partial x^2}(\ln\varepsilon_2 + \ln\exp(\eta_2) + \ln\left[1 + A_{12}\varepsilon_1\exp(\eta_1)\right]), \\ &= 2\frac{\partial^2}{\partial x^2}(\ln\varepsilon_2) + 2\frac{\partial^2}{\partial x^2}(\eta_2) + 2\frac{\partial^2}{\partial x^2}(\ln\left[1 + A_{12}\varepsilon_1\exp(\eta_1)\right]), \\ &= 0 + 0 + 2\frac{\partial^2}{\partial x^2}(\ln\left[1 + A_{12}\varepsilon_1\exp(\eta_1)\right]). \end{aligned}$$

Hence our function $f(x, y, t)$ will becomes

$$\begin{aligned} f &\approx 1 + A_{12}\varepsilon_1 \exp(\eta_1), \\ &\approx 1 + \varepsilon_1 \exp(\eta_1 + \ln A_{12}). \end{aligned} \tag{2.17}$$

From Equation (2.15) and Equation (2.17), we noticed that they are different equation but still produced same soliton. We named the soliton by Equation (2.17) as soliton 1* and denoted by S_1^*. This soliton is centered at

$$\eta_1 + \ln A_{12} = 0. \tag{2.18}$$

There is a phase shift of $\ln A_{12}$.

2.3.2 Case 2: η_1 Fixed; η_2 Tends To $-\infty$

Since $\exp(\eta_2)$ and $A_{12}\varepsilon_1\varepsilon_2 \exp(\eta_1 + \eta_2)$ tends to zero, thus the function $f(x, y, t)$ in Equation (2.13) becomes

$$f \approx 1 + \varepsilon_1 \exp(\eta_1) \tag{2.19}$$

Soliton solution produced by Equation (2.19) has the same characteristic with soliton by Equation (2.17). The only difference is their position. Thus, we named the soliton produced by Equation (2.19) as soliton 1 and denoted by S_1. This soliton has center at $\eta_1 = 0$ and we also noticed that there is no phase shift of $\ln A_{12}$.

2.3.3 Case 3: η_2 Fixed; η_1 Tends To $+\infty$

With the above condition, the function $f(x, y, t)$ in Equation (2.13) becomes

$$f \approx \varepsilon_1 \exp(\eta_1) + A_{12}\varepsilon_1\varepsilon_2 \exp(\eta_1 + \eta_2) \tag{2.20}$$

because the value of $\varepsilon_1 \exp(\eta_1)$ and $A_{12}\varepsilon_1\varepsilon_2 \exp(\eta_1+\eta_2)$ are larger that 1 and $\varepsilon_2 \exp(\eta_2)$. Thus from Equation (2.20)

$$f = \varepsilon_1 \exp(\eta_1) \left[1 + A_{12}\varepsilon_2 \exp(\eta_2)\right]. \tag{2.21}$$

By using the same concept when deriving Equation (2.17), we obtained

$$\begin{aligned} f &= 1 + A_{12}\varepsilon_2 \exp(\eta_2), \\ &= 1 + \varepsilon_2 \exp(\eta_2 + \ln A_{12}). \end{aligned} \tag{2.22}$$

We called this soliton as soliton 2*, denoted by S_2^* and centered at $\eta_2 + \ln A_{12} = 0$. In this case there is a phase shift of $\ln A_{12}$.

2.3.4 Case 4: η_2 Fixed; η_1 Tends To $-\infty$

In this case, we found that the value $\varepsilon_1 \exp(\eta_1)$ and $A_{12}\varepsilon_1\varepsilon_2 \exp(\eta_1 + \eta_2)$ become very small compared to 1 and $\varepsilon_1 \exp(\eta_2)$ and thus Equation (2.13) becomes

$$f \approx 1 + \varepsilon_2 \exp(\eta_2). \tag{2.23}$$

The soliton given by Equation (2.23) is named as soliton 2 and denoted by S_2. This soliton is centered at $\eta_2 = 0$ and there is no phase shift.

By using the result from the above 4 cases, we have an illustration as given in Figure 2.1.

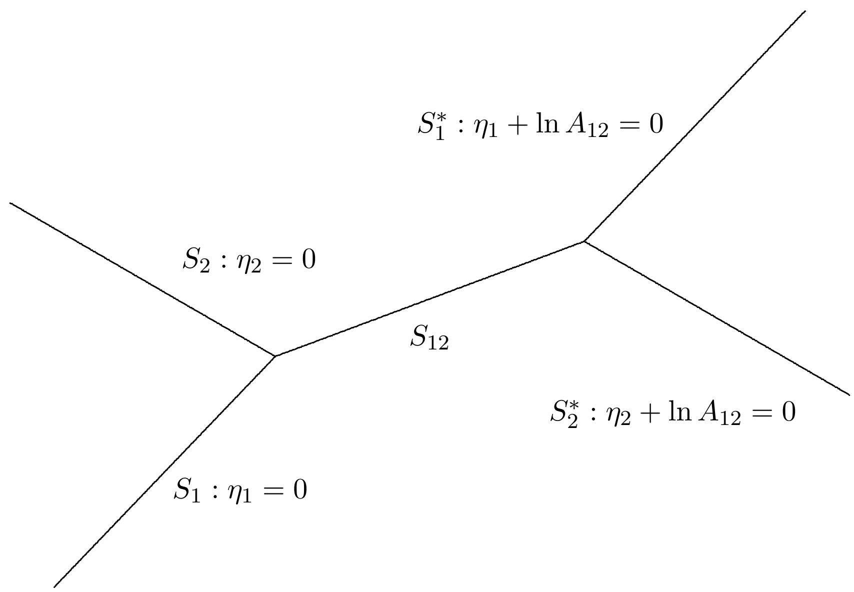

Figure 2.1: Contour plot for two-soliton interaction

2.4 Condition For Resonances

From Figure 2.1, it can be observed that the length of S_{12} depends on $\ln A_{12}$. The bigger the value of $\ln A_{12}$, the longer S_{12} will be. Therefore we can conclude that resonance will only occurs when

$$\ln A_{12} \quad \rightarrow \quad -\infty, \tag{2.24}$$

or

$$A_{12} \quad \rightarrow \quad 0, \tag{2.25}$$

In this dissertation, we will only consider A_{12} tends to zero as a condition for resonance.

2.5 Resonances In The KP Equation

Miles (1977b) in his study discovered that when two solitons interact, they will form one soliton only which is shown in Figure 2.1 (Anker and Freeman, 1978). This phenomenon is called resonance. There are three types of resonances in the KP solitons interaction which are full resonance, partial resonance and non-resonance. Resonance will only occurs when the value of A_{12} is approaching zero. Therefore the values of n_1, n_2, l_1 and l_2 will determined the resonant structure. If we fix the values of l_1 and l_2, then n_1 and n_2 will determined the value of A_{12}.

2.5.1 Full Resonance: A Triad

As mentioned above, we should fix the value of A_{12} close to zero in order for resonance to occur. From Equation (2.14), it can be observed that we have to fix the value of $n_1 = n_2$ or $l_1 = l_2$ in order to make the value of A_{12} to be close to zero. If we fix the value of l_1 and l_2 with real number but $l_1 \neq l_2$, then the resonances occurrence will be determined by the value of n_1 and n_2 or the other way round. For the full resonance case, as $A_{12} = 0$ ($n_1 = n_2$), Equation (2.13) will become

$$f = \underbrace{1}_{(1)} + \underbrace{\varepsilon_1 \exp(\eta_1)}_{(2)} + \underbrace{\varepsilon_2 \exp(\eta_2)}_{(3)}. \tag{2.26}$$

Any combination of (1), (2) and (3) from Equation (2.26) will form another soliton which is the first soliton, S_1, the second soliton, S_2 and the resonant soliton, S_{12} and they can be represented by the combination of (12), (13) and (23) respectively.

$$\text{Soliton } S_1,\ (12) \quad : \quad f = 1 + \varepsilon_1 \exp(\eta_1), \tag{2.27}$$

$$\text{Soliton } S_2,\ (13) \quad : \quad f = 1 + \varepsilon_2 \exp(\eta_2), \tag{2.28}$$

$$\text{Soliton } S_{12},\ (23) \quad : \quad f = \varepsilon_1 \exp(\eta_1) + \varepsilon_2 \exp(\eta_2). \tag{2.29}$$

By referring to Figure 2.1, when the value of A_{12} tends to zero, $\ln A_{12}$ will approach infinity. Thus the length of the resonant soliton will tends to be very long and hence will form a *triad* as in Figure 2.2.

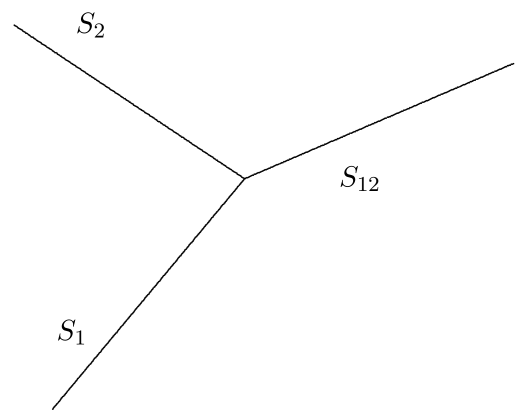

Figure 2.2: A triad

From Figure 2.2, we can observe that there are two incident solitons which are S_1 and S_2 with different amplitude interact and form a new soliton, S_{12}. We called this new soliton as resonant soliton and the whole structure is a triad.

2.5.2 Partially Resonance: A Quadruplet

In this case, we will consider the case of $n_1 \approx n_2$ so that the values of A_{12} will be so close to zero. Thus we have Equation (2.13) again with $A_{12} \approx 0$. The value of A_{12} will determined the length of resonant soliton S_{12}. In this case we will have an interaction pattern called *quadruplet* as shown in Figure 2.3 which represent the partially resonance.

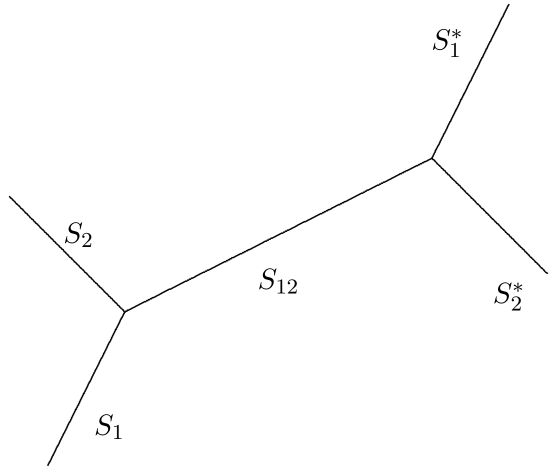

Figure 2.3: A quadruplet

From Figure 2.3, when S_1 and S_2 come into interaction it will produce a resonant soliton S_{12} and later break up again to form S_1^* and S_2^* which is actually soliton S_1 and S_2 respectively but of course with some phase-shift.

2.5.3 Non-Resonance: A Cross

For the non-resonance case, we will take $n_1 \neq n_2$ so that $A_{12} \neq 0$. This means that the line S_{12} does not exist. This phenomena is called non-resonance interactions. In every case we had discussed in the above subsection, S_1 will be centered along the line $k_1x + m_1y = 0$ whereas S_2 will be centered along the line $k_2x + m_2y = 0$. In this case we have

$$\tan\alpha = \frac{\beta_1 - \beta_2}{1 + \beta_1\,\beta_2}, \qquad \text{where} \quad \beta_1 = \frac{m_1}{k_1}, \quad \beta_2 = \frac{m_2}{k_2}. \tag{2.30}$$

In this case α is the angle of interaction between S_1 and S_2 and β_1, β_2 are the tangents of the lines respectively.

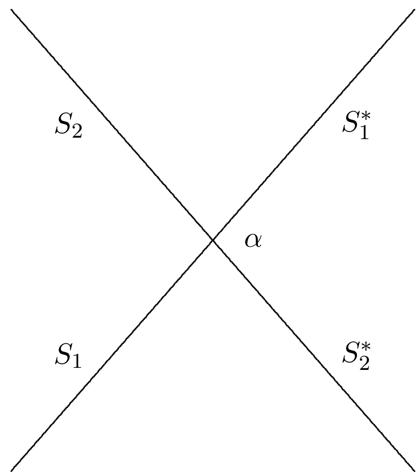

Figure 2.4: A cross

2.6 Computer Simulation

By using KPPRO which built by Microsoft Visual C++ Professional Edition, we will generate a computer programming to solve Equation (2.13). The KPPRO codes for the interaction of two solitons are in the Appendix A.

In this section, computer simulation is illustrated to verify the theory we had discussed in last the few sections. First we will look at the full resonance case, the values of n_1, n_2, l_1 and l_2 were chosen as below so that it will produce a *triad*.

$$\begin{aligned} n_1 &= 3, \quad n_2 = 3, \\ l_1 &= -2, \quad l_2 = 3, \quad (A_{12} = 0) \end{aligned} \tag{2.31}$$

By using these values, the computer simulation produces *triad* as shown in Figure 2.5.

Before this we have mentioned that each soliton will be centered along the line

$$k_i x + m_i y = 0, \tag{2.32}$$

or

$$(n_i + l_i)x + (n_i^2 - l_i^2)y = 0. \tag{2.33}$$

Therefore we have

$$\begin{aligned} S_1 &: \quad x + 5y = 0, \qquad y = -\frac{1}{5}x \\ S_2 &: \quad 6x + 0 = 0, \qquad x = 0 \end{aligned} \tag{2.34}$$

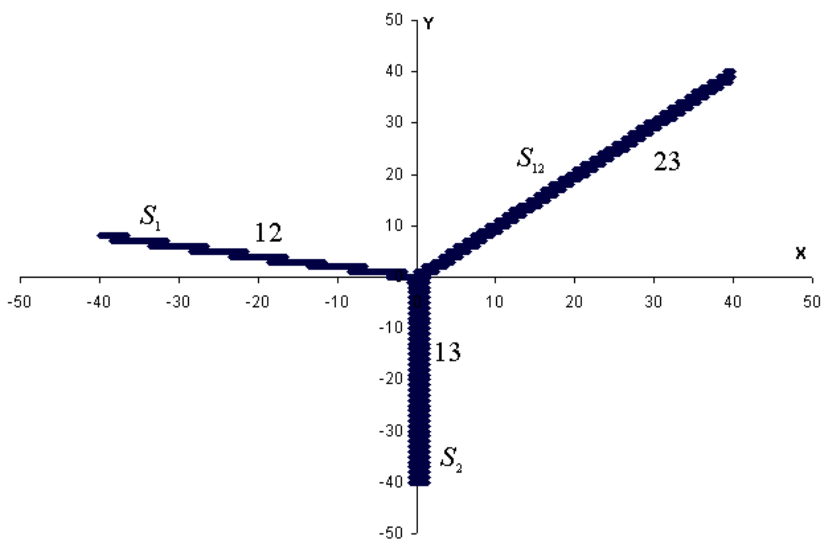

Figure 2.5: A triad with $n_1 = n_2$

From Figure 2.5, it is clearly shown that soliton 1, S_1 interacts with soliton 2, S_2 and produce the resonant soliton S_{12}. S_1 is centered along the line $y = \frac{1}{5}x$ while S_2 is centered along the line $x = 0$. The subscript terms (12), (13) and (23) are to explain the combination of the terms in Equation (2.26).

From the previous section, we noticed that either $n_1 = n_2$ or $l_1 = l_2$ will produce the same resonance structure. To verify it, we tried it on full resonance case only and is shown in Figure 2.6. In this case the values are chosen as below:

$$\begin{aligned} l_1 &= 3, \quad l_2 = 3, \\ n_1 &= -2, \quad n_2 = 3, \quad (A_{12} = 0) \end{aligned} \tag{2.35}$$

So, when $t = 0$

$$\begin{aligned} S_1 &: \ x - 5y = 0 \qquad y = \frac{1}{5}x \\ S_2 &: \ 6x + 0 = 0 \qquad x = 0 \end{aligned} \tag{2.36}$$

From Figure 2.6, we can see clearly that S_1 interacts with S_2 and also produce the resonant soliton S_{12}. The difference between both figures are the position of each soliton and the direction of each soliton when it moves.

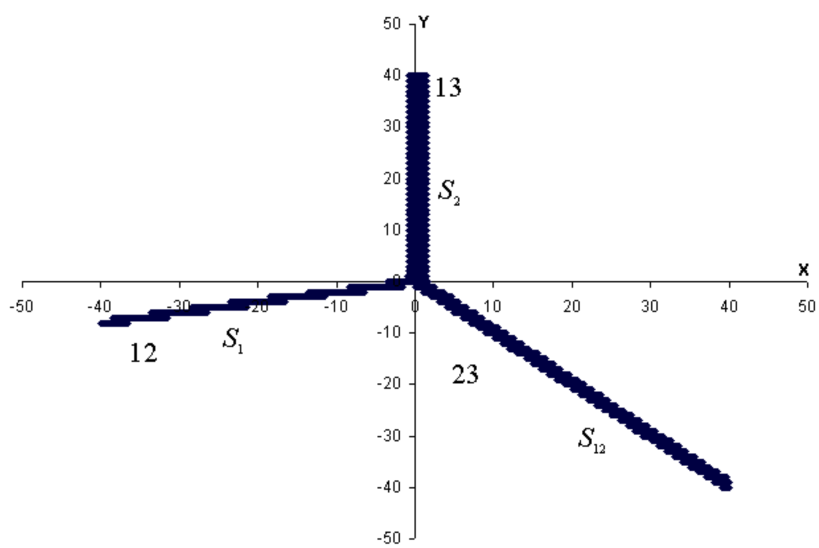

Figure 2.6: A triad with $l_1 = l_2$

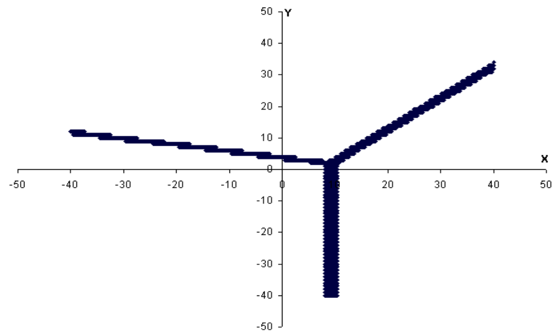

Figure 2.7a: $t = -0.25$

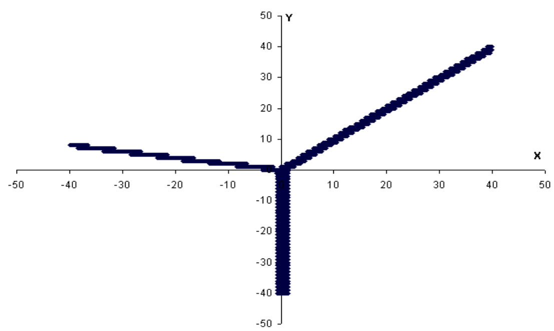

Figure 2.7b: $t = 0$

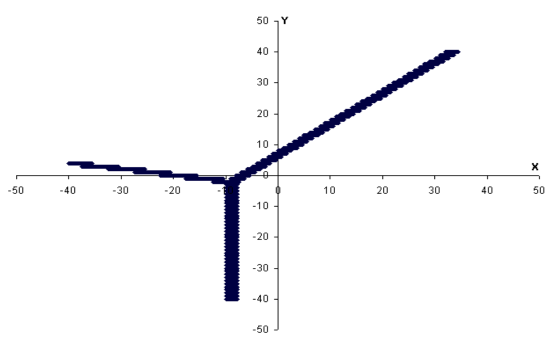

Figure 2.7c: $t = -0.25$

Figure 2.7: Movement of a triad

A triad will move as one entity as t changes. This is shown in Figure 2.7. The only difference is the phase shift. We can notice that Figure 2.7a is symmetry with Figure 2.7c.

On the other hand, we choose

$$\begin{aligned} n_1 &= 3, \quad n_2 = 3 + 10^{-p}, \\ l_1 &= -2, \quad l_2 = 3, \quad (A_{12} \approx 0) \end{aligned} \tag{2.37}$$

with $p = 10$ for the partial resonance case, hence we will have a *quadruplet*. The bigger the value of p in n_2, the longer will be the resonant soliton S_{12}. This has been shown in Figure 2.9. For this case, every soliton is centered along

$$\begin{aligned} S_1 &: \ x - 5y = 0, \qquad y = \frac{1}{5}x \\ S_2 &: \ (6 + 10^{-10})x + 0 = 0, \qquad x \approx 0 \end{aligned} \tag{2.38}$$

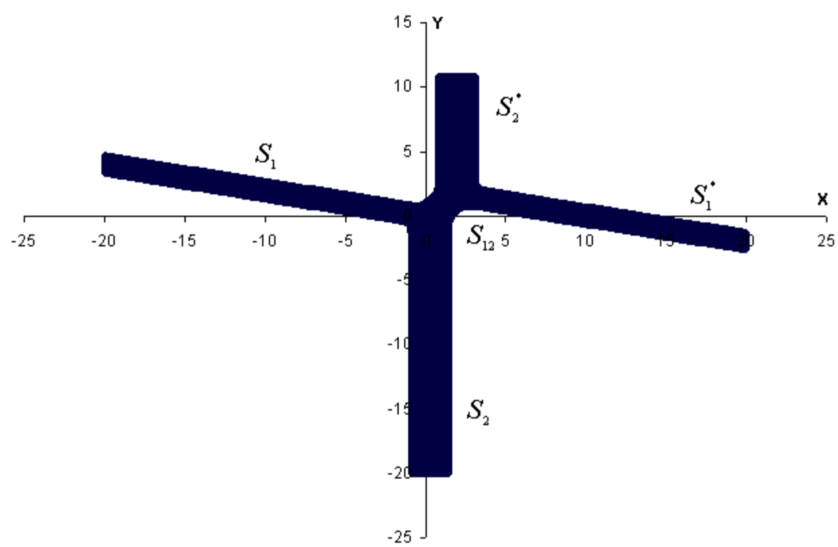

Figure 2.8: A quadruplet

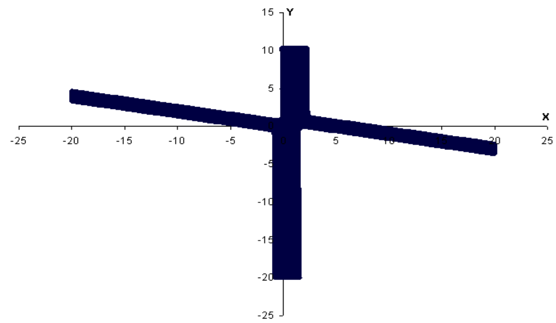

Figure 2.9a: with $p = 5$

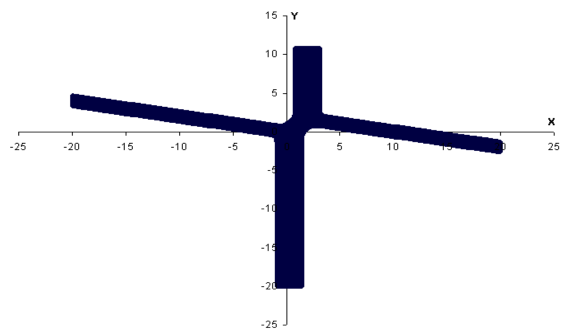

Figure 2.9b: with $p = 10$

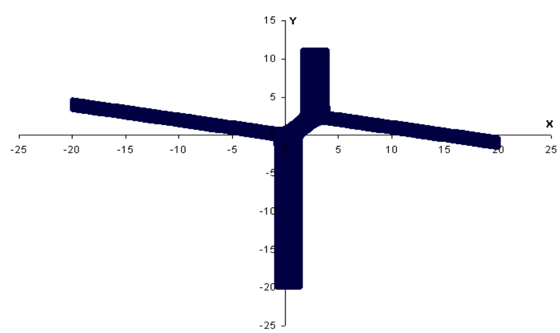

Figure 2.9c: with $p = 15$

Figure 2.9: The length of resonant soliton with different p

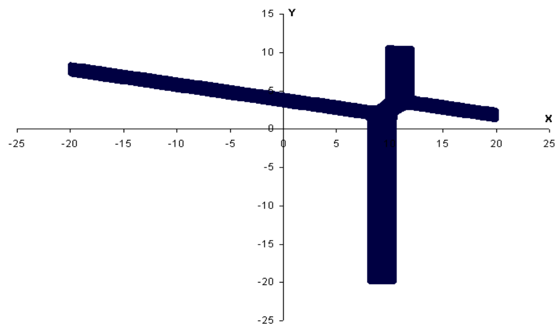

Figure 2.10a: with $t = -0.25$, $p = 10$

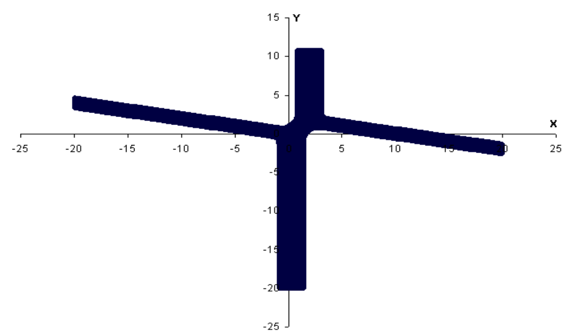

Figure 2.10b: with $t = 0$, $p = 10$

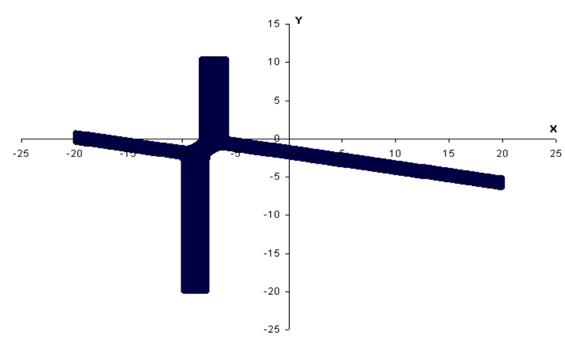

Figure 2.10c: with $t = 0.25$, $p = 10$

Figure 2.10: Movement of a quadruplet with $p = 10$

The same behavior was observed as the movement of a triad, a quadruplet also moves as one entity with some phase shift but does not have any changes as t changes. This is shown in Figure 2.10. The symmetry pattern was observed in Figure 2.10a and

Figure 2.10c

While for the non-resonance case, we have

$$\begin{aligned} n_1 &= 1, \quad n_2 = 3, \\ l_1 &= 2, \quad l_2 = 3, \quad (A_{12} \neq 0) \end{aligned} \tag{2.39}$$

and thus each soliton centered along

$$\begin{aligned} S_1 &: \; 3x - 3y = 0, \qquad y = x \\ S_2 &: \; 6x + 0 = 0, \qquad x = 0 \end{aligned} \tag{2.40}$$

The values of l_1 and l_2 are specifically chosen to ensure that the amplitude of solitons are positive. The amplitude of S_1 and S_2 were determined by $\frac{1}{2}k_i^2$ with $i = 1, 2$ and k_i as defined in (2.10). The value of k_i must be positive to produce positive amplitude of the solitons. With this conditions, we will have a *cross* as shown in Figure 2.11.

From Figure 2.11 and Figure 2.12, we can clearly observed that S_1 just cross over S_2 without interacting with it and we called this form a *cross*. Figure 2.12 is the 3D plot for Figure 2.11. Actually we intend to plot all the interaction patterns with 3D plot, however we cannot see clearly some solitons due to the big difference between the highest amplitude of soliton with lowest amplitude of soliton. As a result, we decide to use 2D plot for all interactions patterns to give a better picture of interaction. In the case of the movement of a cross, it also moved as one entity as time, t changed. This is shown in Figure 2.13.

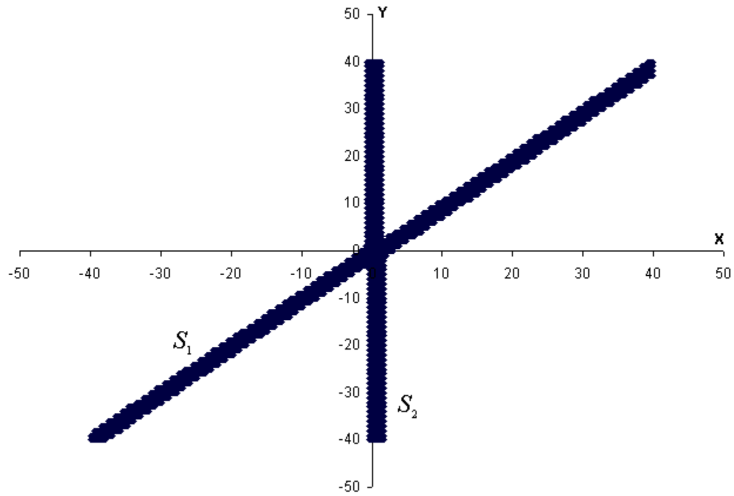

Figure 2.11: 2D non-resonant soliton

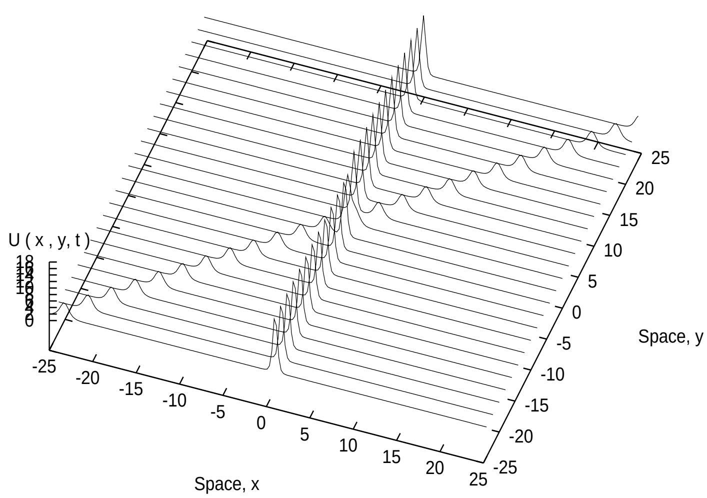

Figure 2.12: 3D non-resonant soliton

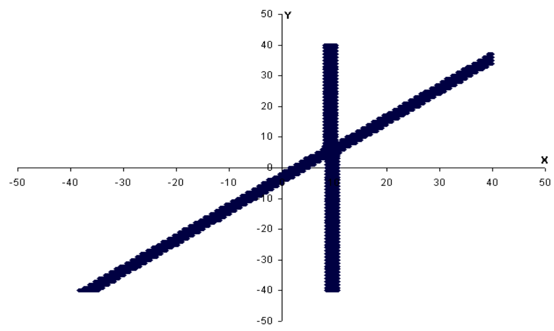

Figure 2.13a: with $t = -0.25$

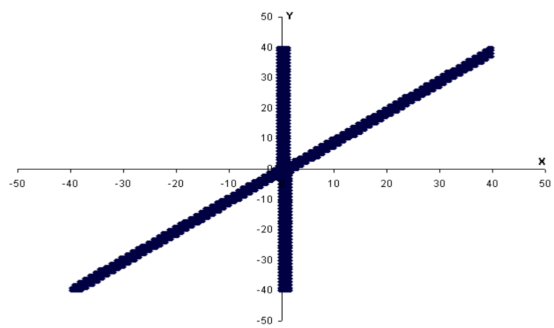

Figure 2.13b: with $t = 0$

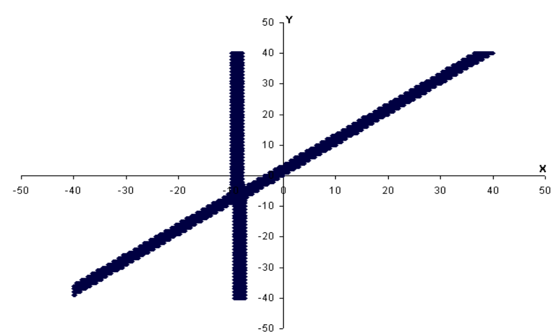

Figure 2.13c: with $t = 0.25$

Figure 2.13: Movement of a cross

2.7 Conclusion

In this chapter, we have shown that resonance in two KP soliton interactions will form interaction patterns which are triad, quadruplet and cross. These basic patterns will lead us to see more about resonances in KP N-soliton interactions such as three-soliton interactions and four-soliton interactions in the following chapter. All the movement of triad, quadruplet and cross move as an entity. This is again another evidence that these structures are indeed soliton in nature.

CHAPTER III

INTERACTIONS OF THREE SOLITONS

3.1 Introduction

In this chapter, we will discuss more about three KP solitons' interactions. First we will look at the derivation of general solution for three KP solitons' interactions. Then we can observe the interactions between a triad with a soliton and the interaction between a quadruplet and a soliton. Besides that, we also can show that the solution of interactions between triad and a soliton is Wronskian determinant.

Then, we showed that in some cases we can simplify the process of searching the function $f(x, y, t)$. We will also look at the illustration of the interaction of three solitons. Computer simulation provides clear evidences and able to verify it accurately.

3.2 General Solution For Three Solitons

In this section, the derivation of the general soliton solution for three-soliton interactions are shown. The KP equation that we consider is

$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} = 0. \tag{3.1}$$

This equation had been solved by Satsuma (1976) by using Hirota Bilinear method as

$$f = \left|\delta_{ij} + \frac{a_i}{l_i + n_j}\exp(\eta_i)\right|. \qquad (3.2)$$

We will consider $N = 3$, thus we have

$$f = \begin{vmatrix} 1 + \frac{a_1}{l_1+n_1}\exp(\eta_1) & \frac{a_1}{l_1+n_2}\exp(\eta_1) & \frac{a_1}{l_1+n_3}\exp(\eta_1) \\ \frac{a_2}{l_2+n_1}\exp(\eta_2) & 1 + \frac{a_2}{l_2+n_2}\exp(\eta_2) & \frac{a_2}{l_2+n_3}\exp(\eta_2) \\ \frac{a_3}{l_3+n_1}\exp(\eta_3) & \frac{a_3}{l_3+n_2}\exp(\eta_3) & 1 + \frac{a_3}{l_3+n_3}\exp(\eta_3) \end{vmatrix},$$

$$\begin{aligned} f &= \left(1 + \frac{a_1}{l_1+n_1}\exp(\eta_1)\right)\begin{vmatrix} 1 + \frac{a_2}{l_2+n_2}\exp(\eta_2) & \frac{a_2}{l_2+n_3}\exp(\eta_2) \\ \frac{a_3}{l_3+n_2}\exp(\eta_3) & 1 + \frac{a_3}{l_3+n_3}\exp(\eta_3) \end{vmatrix} \\ &\quad - \frac{a_1}{l_1+n_2}\exp(\eta_1)\begin{vmatrix} \frac{a_2}{l_2+n_1}\exp(\eta_2) & \frac{a_2}{l_2+n_3}\exp(\eta_2) \\ \frac{a_3}{l_3+n_1}\exp(\eta_3) & 1 + \frac{a_3}{l_3+n_3}\exp(\eta_3) \end{vmatrix} \\ &\quad + \frac{a_1}{l_1+n_3}\exp(\eta_1)\begin{vmatrix} \frac{a_2}{l_2+n_1}\exp(\eta_2) & 1 + \frac{a_2}{l_2+n_2}\exp(\eta_2) \\ \frac{a_3}{l_3+n_1}\exp(\eta_3) & \frac{a_3}{l_3+n_2}\exp(\eta_3) \end{vmatrix}, \end{aligned}$$

$$\begin{aligned} f &= \Bigg[\left(1 + \frac{a_1}{l_1+n_1}\exp(\eta_1)\right)\Bigg(1 + \frac{a_2}{l_2+n_2}\exp(\eta_2) + \frac{a_3}{l_3+n_3}\exp(\eta_3) \\ &\quad + \frac{a_2a_3}{(l_3+n_3)(l_2+n_2)}\exp(\eta_2+\eta_3) - \frac{a_2a_3}{(l_3+n_2)(l_2+n_3)}\exp(\eta_2+\eta_3)\Bigg)\Bigg] \\ &\quad - \Bigg[\left(\frac{a_1}{l_1+n_1}\exp(\eta_1)\right)\Bigg(\frac{a_2}{l_2+n_1}\exp(\eta_2) + \frac{a_2a_3}{(l_2+n_1)(l_3+n_3)}\exp(\eta_2+\eta_3) \\ &\quad - \frac{a_2a_3}{(l_3+n_1)(l_2+n_3)}\exp(\eta_2+\eta_3)\Bigg)\Bigg] \\ &\quad + \Bigg[\left(\frac{a_1}{l_1+n_3}\exp(\eta_1)\right)\Bigg(\frac{a_2a_3}{(l_2+n_1)(l_3+n_2)}\exp(\eta_2+\eta_3) \\ &\quad - \frac{a_3}{l_3+n_1}\exp(\eta_3) - \frac{a_2a_3}{(l_3+n_1)(l_2+n_2)}\exp(\eta_2+\eta_3)\Bigg)\Bigg], \end{aligned}$$

$$
\begin{aligned}
f &= 1+\frac{a_1}{l_1+n_1}\exp(\eta_1)+\frac{a_2}{l_2+n_2}\exp(\eta_2)+\frac{a_3}{l_3+n_3}\exp(\eta_3)\\
&+\left[\frac{a_1a_2}{(l_1+n_1)(l_2+n_2)}-\frac{a_1a_2}{(l_1+n_2)(l_2+n_1)}\right]\exp(\eta_1+\eta_2)\\
&+\left[\frac{a_1a_3}{(l_1+n_1)(l_3+n_3)}-\frac{a_1a_3}{(l_1+n_3)(l_3+n_1)}\right]\exp(\eta_1+\eta_3)\\
&+\left[\frac{a_2a_3}{(l_2+n_2)(l_3+n_3)}-\frac{a_2a_3}{(l_2+n_3)(l_3+n_2)}\right]\exp(\eta_2+\eta_3)\\
&+\left[\frac{a_1a_2a_3}{(l_1+n_1)(l_2+n_2)(l_3+n_3)}-\frac{a_1a_2a_3}{(l_1+n_1)(l_3+n_2)(l_2+n_3)}\right.\\
&-\frac{a_1a_2a_3}{(l_1+n_2)(l_2+n_1)(l_3+n_3)}+\frac{a_1a_2a_3}{(l_1+n_2)(l_3+n_2)(l_2+n_3)}\\
&\left.+\frac{a_1a_2a_3}{(l_1+n_3)(l_2+n_1)(l_3+n_2)}-\frac{a_1a_2a_3}{(l_1+n_3)(l_3+n_1)(l_2+n_2)}\right]\exp(\eta_1+\eta_2+\eta_3),\\
f &= 1+\varepsilon_1\exp(\eta_1)+\varepsilon_2\exp(\eta_2)+\varepsilon_3\exp(\eta_3)\\
&+\left(\frac{(l_1-l_2)(n_1-n_2)}{(l_1+n_2)(l_2+n_1)}\right)\varepsilon_1\varepsilon_2\exp(\eta_1+\eta_2)\\
&+\left(\frac{(l_1-l_3)(n_1-n_3)}{(l_1+n_3)(l_3+n_1)}\right)\varepsilon_1\varepsilon_3\exp(\eta_1+\eta_3)\\
&+\left(\frac{(l_2-l_3)(n_2-n_3)}{(l_2+n_3)(l_3+n_2)}\right)\varepsilon_2\varepsilon_3\exp(\eta_2+\eta_3)\\
&+\left[\frac{a_1a_2a_3}{(l_1+n_1)(l_2+n_2)(l_3+n_3)}-\frac{a_1a_2a_3}{(l_1+n_1)(l_3+n_2)(l_2+n_3)}\right.\\
&-\frac{a_1a_2a_3}{(l_1+n_2)(l_2+n_1)(l_3+n_3)}+\frac{a_1a_2a_3}{(l_1+n_2)(l_3+n_2)(l_2+n_3)}\\
&\left.+\frac{a_1a_2a_3}{(l_1+n_3)(l_2+n_1)(l_3+n_2)}-\frac{a_1a_2a_3}{(l_1+n_3)(l_3+n_1)(l_2+n_2)}\right]\exp(\eta_1+\eta_2+\eta_3).
\end{aligned}
$$

Suppose that

$$A_{12} = \frac{(l_1-l_2)(n_1-n_2)}{(l_1+n_2)(l_2+n_1)}, \tag{3.3}$$

$$A_{13} = \frac{(l_1-l_3)(n_1-n_3)}{(l_1+n_3)(l_3+n_1)}, \tag{3.4}$$

$$A_{23} = \frac{(l_2-l_3)(n_2-n_3)}{(l_2+n_3)(l_3+n_2)}, \tag{3.5}$$

$$
\begin{aligned}
A_{123} &= \left[\frac{a_1a_2a_3}{(l_1+n_1)(l_2+n_2)(l_3+n_3)}-\frac{a_1a_2a_3}{(l_1+n_1)(l_3+n_2)(l_2+n_3)}\right.\\
&-\frac{a_1a_2a_3}{(l_1+n_2)(l_2+n_1)(l_3+n_3)}+\frac{a_1a_2a_3}{(l_1+n_2)(l_3+n_2)(l_2+n_3)}\\
&\left.+\frac{a_1a_2a_3}{(l_1+n_3)(l_2+n_1)(l_3+n_2)}-\frac{a_1a_2a_3}{(l_1+n_3)(l_3+n_1)(l_2+n_2)}\right],
\end{aligned}
\tag{3.6}
$$

therefore $f(x,y,t)$ becomes

$$\begin{aligned} f &= 1+\varepsilon_1\exp(\eta_1)+\varepsilon_2\exp(\eta_2)+\varepsilon_3\exp(\eta_3) \\ &\quad +A_{12}\varepsilon_1\varepsilon_2\exp(\eta_1+\eta_2)+A_{13}\varepsilon_1\varepsilon_3\exp(\eta_1+\eta_3) \\ &\quad +A_{23}\varepsilon_2\varepsilon_3\exp(\eta_2+\eta_3)+A_{123}\varepsilon_1\varepsilon_2\varepsilon_3\exp(\eta_1+\eta_2+\eta_3). \end{aligned} \tag{3.7}$$

From this derivations, it can be observed that the derivation of the general solution for three solitons interactions is complicated. It will be more complicated as N increases because we are looking for the determinant of a matrix. After the general solution had been obtained, we can use this solution to produce a sequence of interactions. For example, we can observe the interaction when a triad interacts with a soliton, a quadruplet interacts with a soliton and so on. However, for some cases we can simplify the process of searching the function of $f(x,y,t)$. For example, the process to get the function $f(x,y,t)$ in the interaction between a triad and a soliton can be simplify into Wronskian determinant which will be discussed in the following section.

3.3 The Wronskian Techniques

A Wronskian of the n functions $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$,...,$\phi_n(x)$ is defined as determinant

$$W=\begin{vmatrix} \phi_1(x) & \phi_2(x) & . & . & . & \phi_n(x) \\ \phi_1^{(1)}(x) & \phi_2^{(1)}(x) & . & . & . & \phi_n^{(1)}(x) \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ \phi_1^{(n-1)}(x) & \phi_2^{(n-1)}(x) & . & . & . & \phi_n^{(n-1)}(x) \end{vmatrix}.$$

This is often written as $W(\phi_1(x), \phi_2(x), ..., \phi_n(x))$ (Freeman, 1984; Mukheta, 1989). For our discussion, the KP equation we considered is Equation (3.1). It is well known that this equation has soliton solutions which can best be expressed in the form

$$u = 2\frac{\partial^2}{\partial x^2}\ln f. \tag{3.8}$$

The N-soliton solution of this equation has been obtained by Satsuma (1976) in Equation (3.2) and can be written in the form

$$f = \left|\delta_{ij} + \frac{a_i}{l_i + n_j}\exp(\theta_i + \gamma_j)\right| \tag{3.9}$$

where

$$\theta_i = l_i x - l_i^2 y + 4l_i^3 t, \tag{3.10}$$

$$\gamma_j = n_j x + n_j^2 y + 4n_j^3 t \tag{3.11}$$

with δ_{ij} is Kronecker delta, l_i, n_i and a_i are real constants and $i, j = 1, 2, ..., N$. Freeman and Nimmo (1983) had showed that the above determinant which is Equation (3.2) or Equation (3.9) is a Wronskian determinant.

3.4 Solution For Two Solitons In Full Resonance

Before we start discussing about the three-soliton interaction in Wronskian, this section will show that even the process of looking the function $f(x, y, t)$ for a triad can be simplified by using the Equation (3.9). For N=2, $f(x, y, t)$ can be written as

$$f = \begin{vmatrix} 1 + \frac{a_1}{l_1 + n_1}\exp(\theta_1 + \gamma_1) & \frac{a_1}{l_1 + n_2}\exp(\theta_1 + \gamma_2) \\ \frac{a_2}{l_2 + n_1}\exp(\theta_2 + \gamma_1) & 1 + \frac{a_2}{l_2 + n_2}\exp(\theta_2 + \gamma_2) \end{vmatrix}.$$

In this case, we are looking for the triad, thus we need a condition where $A_{12} = 0$ which is $n_1 = n_2$ or $l_1 = l_2$ so that the first soliton, S_1 interacts with the second soliton, S_2 to produce the resonant soliton, S_{12}. Hence, we choose $n_1 = n_2$ and our function $f(x, y, t)$ becomes

$$f = \begin{vmatrix} 1 + \dfrac{a_1}{l_1 + n_1} \exp(\theta_1 + \gamma_1) & \dfrac{a_1}{l_1 + n_1} \exp(\theta_1 + \gamma_1) \\ \dfrac{a_2}{l_2 + n_1} \exp(\theta_2 + \gamma_1) & 1 + \dfrac{a_2}{l_2 + n_1} \exp(\theta_2 + \gamma_1) \end{vmatrix}.$$

Let

$$X_{ij} = \frac{a_1}{l_1 + n_j} \exp(\theta_i + \gamma_j), \tag{3.12}$$

then

$$f = \begin{vmatrix} 1 + X_{11} & X_{11} \\ X_{21} & 1 + X_{21} \end{vmatrix}.$$

After row operation $(C_2 - C_1)$, $f(x, y, t)$ can be transformed into

$$f = \begin{vmatrix} 1 + X_{11} & -1 \\ X_{21} & 1 \end{vmatrix}.$$

Next, $f(x, y, t)$ can be written as

$$f = \begin{vmatrix} 1 + X_{11} + X_{21} & 0 \\ X_{21} & 1 \end{vmatrix} \tag{3.13}$$

after row operation $(R_1 + R_2)$. Equation (3.13) can be simplified as

$$\begin{aligned} f &= 1 + X_{11} + X_{21}, & (3.14) \\ &= 1 + \frac{a_1}{l_1 + n_1} \exp(\theta_1 + \gamma_1) + \frac{a_2}{l_2 + n_2} \exp(\theta_2 + \gamma_1), & (3.15) \\ &= 1 + \varepsilon_1 \exp(\eta_1) + \varepsilon_2 \exp(\eta_2). & (3.16) \end{aligned}$$

Here, C_i and R_j represent ith column and jth row. Equation (3.16) illustrates a triad which we had shown in Chapter 2. From Equation (3.16), it can be observed that the process of searching function $f(x, y, t)$ in some cases can be simplified instead of using the general equation.

3.5 Solution For Interaction Between A Triad And A Soliton

We cannot reduced the function $f(x, y, t)$ into Wronskian type in all cases. In three-soliton interactions, we only can consider the case where a triad interacts with a soliton. To produce a triad, we have to set a condition which is $n_1 = n_2 \neq n_3$ so that soliton 1 interacts with soliton 2 in full resonance and produce a triad. Generally the function $f(x, y, t)$ can be written as

$$f = |\, I \quad + \quad X \,| \tag{3.17}$$

where I is a identity matrix and X can be written as

$$X = \begin{bmatrix} \frac{a_1}{l_1 + n_1} e^{\theta_1 + \gamma_1} & \frac{a_1}{l_1 + n_2} e^{\theta_1 + \gamma_2} & \frac{a_1}{l_1 + n_3} e^{\theta_1 + \gamma_3} \\ \frac{a_2}{l_2 + n_1} e^{\theta_2 + \gamma_1} & \frac{a_2}{l_2 + n_2} e^{\theta_2 + \gamma_2} & \frac{a_2}{l_2 + n_3} e^{\theta_2 + \gamma_3} \\ \frac{a_3}{l_3 + n_1} e^{\theta_3 + \gamma_1} & \frac{a_3}{l_3 + n_2} e^{\theta_3 + \gamma_2} & \frac{a_3}{l_3 + n_3} e^{\theta_3 + \gamma_3} \end{bmatrix} \tag{3.18}$$

After substitution by Equation (3.12), Equation (3.18) becomes

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}. \tag{3.19}$$

If we put the condition $n_1 = n_2 \neq n_3$ or $\gamma_1 = \gamma_2 \neq \gamma_3$, the function $f(x, y, t)$ can be transformed into

$$f = \begin{vmatrix} 1 + X_{11} & X_{11} & X_{13} \\ X_{21} & 1 + X_{21} & X_{23} \\ X_{31} & X_{31} & 1 + X_{33} \end{vmatrix}. \tag{3.20}$$

By using row operation $(C_2 - C_1)$, Equation (3.20) becomes

$$f = \begin{vmatrix} 1 + X_{11} & -1 & X_{13} \\ X_{21} & 1 & X_{23} \\ X_{31} & 0 & 1 + X_{33} \end{vmatrix}. \tag{3.21}$$

Next with row operation $(R_1 + R_2)$, Equation (3.21) can be written as

$$
\begin{aligned}
f &= \begin{vmatrix} 1 + X_{11} + X_{21} & 0 & X_{13} + X_{23} \\ X_{21} & 1 & X_{23} \\ X_{31} & 0 & 1 + X_{33} \end{vmatrix}, \\
&= \begin{vmatrix} 1 + X_{11} + X_{21} & X_{13} + x_{23} \\ X_{31} & 1 + X_{33} \end{vmatrix}. \qquad (3.22)
\end{aligned}
$$

So far we had reduced 3×3 matrix into 2×2 matrix successfully. Next we transpose the function $f(x, y, t)$ and make some arrangement,

$$
f = \begin{vmatrix} 1 + X_{11} + X_{21} & X_{31} \\ X_{13} + x_{23} & 1 + X_{33} \end{vmatrix}, \qquad (3.23)
$$

$$
= \begin{vmatrix} 1 + \dfrac{a_1}{l_1 + n_1} e^{\theta_1 + \gamma_1} + \dfrac{a_2}{l_2 + n_1} e^{\theta_2 + \gamma_1} & \dfrac{a_3}{l_3 + n_1} e^{\theta_3 + \gamma_1} \\ \dfrac{a_1}{l_1 + n_3} e^{\theta_1 + \gamma_3} + \dfrac{a_2}{l_2 + n_3} e^{\theta_2 + \gamma_3} & 1 + \dfrac{a_3}{l_3 + n_3} e^{\theta_3 + \gamma_3} \end{vmatrix}, \qquad (3.24)
$$

$$
= \begin{vmatrix} e^{\gamma_1} \left[e^{-\gamma_1} + \dfrac{a_1}{l_1 + n_1} e^{\theta_1} + \dfrac{a_2}{l_2 + n_1} e^{\theta_2} \right] & e^{\gamma_1} \left[\dfrac{a_3}{l_3 + n_1} e^{\theta_3} \right] \\ e^{\gamma_3} \left[\dfrac{a_1}{l_1 + n_3} e^{\theta_1} + \dfrac{a_2}{l_2 + n_3} e^{\theta_2} \right] & e^{\gamma_3} \left[e^{-\gamma_3} + \dfrac{a_3}{l_3 + n_3} e^{\theta_3} \right] \end{vmatrix}. \qquad (3.25)
$$

Next, we factor out the $e^{\gamma_1 + \gamma_3}$ term. Hence, $f(x, y, t)$ can be written as

$$
f = e^{\gamma_1 + \gamma_3} \begin{vmatrix} \dfrac{a_1}{l_1 + n_1} e^{\theta_1} + \dfrac{a_2}{l_2 + n_1} e^{\theta_2} + e^{-\gamma_1} & \dfrac{a_3}{l_3 + n_1} e^{\theta_3} \\ \dfrac{a_1}{l_1 + n_3} e^{\theta_1} + \dfrac{a_2}{l_2 + n_3} e^{\theta_2} & \dfrac{a_3}{l_3 + n_3} e^{\theta_3} + e^{-\gamma_3} \end{vmatrix}.
$$

We know that $\exp(\gamma_1+\gamma_3)$ term is linear in x. Therefore this term will become zero after we differentiate it twice with respect to x. So, $f(x,y,t)$ becomes

$$
\begin{aligned}
f &= \begin{vmatrix} \frac{a_1}{l_1+n_1}e^{\theta_1}+\frac{a_2}{l_2+n_1}e^{\theta_2}+e^{-\gamma_1} & \frac{a_3}{l_3+n_1}e^{\theta_3} \\ \frac{a_1}{l_1+n_3}e^{\theta_1}+\frac{a_2}{l_2+n_3}e^{\theta_2} & \frac{a_3}{l_3+n_3}e^{\theta_3}+e^{-\gamma_3} \end{vmatrix}, \\
&= \left| \begin{bmatrix} e^{-\gamma_1} & 0 \\ 0 & e^{-\gamma_3} \end{bmatrix} + \begin{bmatrix} \frac{a_1}{l_1+n_1}e^{\theta_1} & \frac{a_3}{l_3+n_1}e^{\theta_3} \\ \frac{a_1}{l_1+n_3}e^{\theta_1} & \frac{a_3}{l_3+n_3}e^{\theta_3} \end{bmatrix} + \begin{bmatrix} \frac{a_2}{l_2+n_1}e^{\theta_2} & 0 \\ \frac{a_2}{l_2+n_3}e^{\theta_2} & 0 \end{bmatrix} \right|.
\end{aligned}
$$

The above equation can be written in a more simplified form

$$
f = \left| E_N^{-1} + E_{L_1} + E_{L_2} \right| \tag{3.26}
$$

where

$$
E_N^{-1} = \begin{bmatrix} e^{-\gamma_1} & 0 \\ 0 & e^{-\gamma_3} \end{bmatrix}, \tag{3.27}
$$

$$
E_{L_1} = \begin{bmatrix} \frac{a_1}{l_1+n_1}e^{\theta_1} & \frac{a_3}{l_3+n_1}e^{\theta_3} \\ \frac{a_1}{l_1+n_3}e^{\theta_1} & \frac{a_3}{l_3+n_3}e^{\theta_3} \end{bmatrix}, \tag{3.28}
$$

$$
E_{L_2} = \begin{bmatrix} \frac{a_2}{l_2+n_1}e^{\theta_2} & 0 \\ \frac{a_2}{l_2+n_3}e^{\theta_2} & 0 \end{bmatrix}. \tag{3.29}
$$

Next, we can rewrite Equation (3.28) in the form of

$$E_{L_1} = \begin{bmatrix} \frac{a_1}{l_1+n_1}e^{\theta_1} & \frac{a_3}{l_3+n_1}e^{\theta_3} \\ \frac{a_1}{l_1+n_3}e^{\theta_1} & \frac{a_3}{l_3+n_3}e^{\theta_3} \end{bmatrix}, \tag{3.30}$$

$$= \begin{bmatrix} \frac{1}{l_1+n_1} & \frac{1}{l_3+n_1} \\ \frac{1}{l_1+n_3} & \frac{1}{l_3+n_3} \end{bmatrix} \begin{bmatrix} a_1 e^{\theta_1} & 0 \\ 0 & a_3 e^{\theta_3} \end{bmatrix}, \tag{3.31}$$

$$= \begin{bmatrix} \frac{1}{l_1+n_1} & \frac{1}{l_3+n_1} \\ \frac{1}{l_1+n_3} & \frac{1}{l_3+n_3} \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & a_3 \end{bmatrix} \begin{bmatrix} e^{\theta_1} & 0 \\ 0 & e^{\theta_3} \end{bmatrix}, \tag{3.32}$$

$$= M_1\, A_1\, E_{l_1}, \tag{3.33}$$

with

$$M_1 = \begin{bmatrix} \frac{1}{l_1+n_1} & \frac{1}{l_3+n_1} \\ \frac{1}{l_1+n_3} & \frac{1}{l_3+n_3} \end{bmatrix}, \tag{3.34}$$

$$A_1 = \begin{bmatrix} a_1 & 0 \\ 0 & a_3 \end{bmatrix}, \tag{3.35}$$

$$E_{l_1} = \begin{bmatrix} e^{\theta_1} & 0 \\ 0 & e^{\theta_3} \end{bmatrix}. \tag{3.36}$$

Equation (3.29) also can be rewritten as

$$
\begin{aligned}
E_{L_2} &= \begin{bmatrix} \dfrac{a_2}{l_2+n_1} e^{\theta_2} & 0 \\ \\ \dfrac{a_2}{l_2+n_3} e^{\theta_2} & 0 \end{bmatrix}, \\
&= \begin{bmatrix} \dfrac{1}{l_2+n_1} & \dfrac{1}{l_2+n_1} \\ \\ \dfrac{1}{l_2+n_3} & \dfrac{1}{l_2+n_3} \end{bmatrix} \begin{bmatrix} a_2 e^{\theta_2} & 0 \\ \\ 0 & 0 \end{bmatrix}, && (3.37) \\
&= \begin{bmatrix} \dfrac{1}{l_2+n_1} & \dfrac{1}{l_2+n_1} \\ \\ \dfrac{1}{l_2+n_3} & \dfrac{1}{l_2+n_3} \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e^{\theta_2} & 0 \\ \\ 0 & 0 \end{bmatrix}, && (3.38) \\
&= M_2\, A_2\, E_{l_2}, && (3.39)
\end{aligned}
$$

where

$$
\begin{aligned}
M_2 &= \begin{bmatrix} \dfrac{1}{l_2+n_1} & \dfrac{1}{l_2+n_1} \\ \\ \dfrac{1}{l_2+n_3} & \dfrac{1}{l_2+n_3} \end{bmatrix}, && (3.40) \\
A_2 &= \begin{bmatrix} a_2 & 0 \\ \\ 0 & 0 \end{bmatrix}, && (3.41) \\
E_{l_2} &= \begin{bmatrix} e^{\theta_2} & 0 \\ \\ 0 & 0 \end{bmatrix}. && (3.42)
\end{aligned}
$$

It can be shown that $M = \left[\dfrac{1}{l_a + n_b}\right]$ may be related to the Van der Monde matrices (Freeman, 1984) which are

$$
V = \left[(-n_b)^{i-1}\right] \qquad (3.43)
$$

and

$$W \;=\; \left[(-1)^{j-1}(l_a)^{i-1}\right]. \tag{3.44}$$

Next, we introduced two diagonal matrices which are

$$P \;=\; \left[\delta_{ij}\prod_{p,p\neq b}(n_p - n_b)\right] \tag{3.45}$$

and

$$Q \;=\; \left[\delta_{ij}(-1)^{i-1}\prod_{a}(l_a + n_b)\right] \tag{3.46}$$

and we have

$$V^{-1}\,W \;=\; P^{-1}\,M\,Q \tag{3.47}$$

so that

$$M \;=\; P\,V^{-1}\,W\,Q^{-1} \tag{3.48}$$

Now, we want to show that the function $f(x, y, t)$ that had been transformed is a determinant of Wronskian. Thus we rewrite the function $f(x, y, t)$ as:

$$f = \left|\, E_N^{-1} \quad + \quad M_1\,A_1\,E_{l_1} \quad + \quad M_2\,A_2\,E_{l_2} \,\right|. \tag{3.49}$$

From Equation (3.48), Freeman (1984) had shown that

$$M_1 \;=\; P\,V^{-1}\,W_1\,Q_1^{-1}, \tag{3.50}$$

$$M_2 \;=\; P\,V^{-1}\,W_2\,Q_2^{-1}, \tag{3.51}$$

where V and W are in Van der Monde matrix form. From Equation (3.43) and Equation

(3.44), we know that

$$V = \left[(-n_b)^{i-1}\right], \tag{3.52}$$

$$= \begin{bmatrix} (-n_1)^{1-1} & (-n_3)^{1-1} \\ (-n_1)^{2-1} & (-n_3)^{2-1} \end{bmatrix}, \tag{3.53}$$

$$= \begin{bmatrix} 1 & 1 \\ -n_1 & -n_3 \end{bmatrix}, \tag{3.54}$$

and

$$W_1 = \left[(-1)^{j-1}(l_a)^{i-1}\right], \tag{3.55}$$

$$= \begin{bmatrix} (-1)^{1-1}(l_1)^{1-1} & (-1)^{2-1}(l_3)^{1-1} \\ (-1)^{1-1}(l_1)^{2-1} & (-1)^{2-1}(l_3)^{2-1} \end{bmatrix}, \tag{3.56}$$

$$= \begin{bmatrix} 1 & -1 \\ l_1 & -l_3 \end{bmatrix}, \tag{3.57}$$

and

$$W_2 = \left[(-1)^{j-1}(l_a)^{i-1}\right], \tag{3.58}$$

$$= \begin{bmatrix} (-1)^{1-1}(l_2)^{1-1} & (-1)^{2-1}(l_2)^{1-1} \\ (-1)^{1-1}(l_2)^{2-1} & (-1)^{2-1}(l_2)^{2-1} \end{bmatrix}, \tag{3.59}$$

$$= \begin{bmatrix} 1 & -1 \\ l_2 & -l_2 \end{bmatrix}. \tag{3.60}$$

While matrix P and Q are

$$P = \left[\delta_{ij} \prod_{p, p \neq b} (n_p - n_b) \right], \tag{3.61}$$

$$= \begin{bmatrix} n_3 - n_1 & 0 \\ & \\ 0 & n_1 - n_3 \end{bmatrix}, \tag{3.62}$$

and

$$Q_1 = \left[\delta_{ij} (-1)^{i-1} \prod_a (l_a + n_b) \right], \tag{3.63}$$

$$= \begin{bmatrix} (-1)^{1-1}(l_1 + n_1)(l_1 + n_3) & 0 \\ & \\ 0 & (-1)^{2-1}(l_3 + n_1)(l_3 + n_3) \end{bmatrix}, \tag{3.64}$$

$$= \begin{bmatrix} (l_1 + n_1)(l_1 + n_3) & 0 \\ & \\ 0 & -(l_3 + n_1)(l_3 + n_3) \end{bmatrix}, \tag{3.65}$$

and

$$Q_2 = \begin{bmatrix} (l_2 + n_1)(l_2 + n_3) & 0 \\ & \\ 0 & -(l_2 + n_1)(l_2 + n_3) \end{bmatrix}. \tag{3.66}$$

When we applied this result to the function $f(x, y, t)$, it becomes

$$f = \left| E_N^{-1} + P\, V^{-1}\, W_1\, Q_1^{-1}\, A_1\, E_{l_1} + P\, V^{-1}\, W_2\, Q_2^{-1}\, A_2\, E_{l_2} \right|, \tag{3.67}$$

$$= \left| P\, V^{-1} \right| \left| V\, P^{-1}\, E_N^{-1} + W_1\, Q_1^{-1}\, A_1\, E_{l_1} + W_2\, Q_2^{-1}\, A_2\, E_{l_2} \right|. \tag{3.68}$$

Since $P\, V^{-1}$ is not a function in x, thus we can omit this term and hence

$$f = \left| V\, P^{-1}\, E_N^{-1} + W_1\, Q_1^{-1}\, A_1\, E_{l_1} + W_2\, Q_2^{-1}\, A_2\, E_{l_2} \right|. \tag{3.69}$$

Now we will look at each term in the function $f(x, y, t)$. To get the form of each term faster, Maple software was used here to save our time.

$$V\,P^{-1}\,E_N^{-1} = \begin{bmatrix} -\dfrac{e^{-\gamma_1}}{n_1-n_3} & \dfrac{e^{-\gamma_3}}{n_1-n_3} \\ \\ \dfrac{n_1e^{-\gamma_1}}{n_1-n_3} & \dfrac{-n_3e^{-\gamma_3}}{n_1-n_3} \end{bmatrix}, \tag{3.70}$$

$$= \begin{bmatrix} \dfrac{e^{-\gamma_1}}{n_3-n_1} & \dfrac{e^{-\gamma_3}}{n_1-n_3} \\ \\ -\dfrac{n_1e^{-\gamma_1}}{n_1-n_3} & -\dfrac{n_3e^{-\gamma_3}}{n_1-n_3} \end{bmatrix}, \tag{3.71}$$

$$W_1\,Q_1^{-1}\,A_1\,E_{l_1} = \begin{bmatrix} \dfrac{a_1e^{\theta_1}}{(l_1+n_1)(l_1+n_3)} & \dfrac{a_3e^{\theta_3}}{(l_3+n_1)(l_3+n_3)} \\ \\ \dfrac{l_1\,a_1\,e^{\theta_1}}{(l_1+n_1)(l_1+n_3)} & \dfrac{l_3\,a_3e^{\theta_3}}{(l_3+n_1)(l_3+n_3)} \end{bmatrix}, \tag{3.72}$$

$$W_2\,Q_2^{-1}\,A_2\,E_{l_2} = \begin{bmatrix} \dfrac{a_2e^{\theta_2}}{(l_2+n_1)(l_2+n_3)} & 0 \\ \\ \dfrac{l_2\,a_2\,e^{\theta_2}}{(l_2+n_1)(l_2+n_3)} & 0 \end{bmatrix}. \tag{3.73}$$

By substitution Equation (3.71), (3.72) and (3.73), function $f(x, y, t)$ has been transformed into Wronskian determinant which is in the form of

$$f = \begin{vmatrix} \phi_1 & \phi_2 \\ \\ \dfrac{\partial\phi_1}{\partial x} & \dfrac{\partial\phi_2}{\partial x} \end{vmatrix}, \tag{3.74}$$

where

$$\phi_1 = \frac{e^{-\gamma_1}}{n_3-n_1} + \frac{a_1e^{\theta_1}}{(l_1+n_1)(l_1+n_3)} + \frac{a_2e^{\theta_2}}{(l_2+n_1)(l_2+n_3)}, \tag{3.75}$$

and

$$\phi_2 = \frac{e^{-\gamma_3}}{n_1-n_3} + \frac{a_3e^{\theta_3}}{(l_3+n_1)(l_3+n_3)}. \tag{3.76}$$

3.6 Computer Simulation

In this section, illustration of the interactions of three solitons are shown. The codes of KPPRO which is the computer program to simulate the three solitons interactions are included in the Appendix A.

3.6.1 A Triad Interacts With A Soliton

We have three types of interactions for the interactions between a triad with a soliton which are $n_1 = n_2 \approx n_3$, $n_2 = n_3 \approx n_1$ and $n_1 = n_3 \approx n_2$ respectively.

3.6.1.1 Type 1: $n_1 = n_2 \approx n_3$

From the previous section, we have shown that the function $f(x, y, t)$ for interaction of a triad with a soliton can be simplified into Wronskian determinant. Thus, before we discuss more about the illustration of the interactions, let us look at the function $f(x, y, t)$ which will govern the interaction. From Equation (3.74), (3.75) and (3.76), we know that

$$f = \begin{vmatrix} \dfrac{e^{-\gamma_1}}{n_3 - n_1} + \dfrac{a_1 e^{\theta_1}}{(l_1 + n_1)(l_1 + n_3)} + \dfrac{a_2 e^{\theta_2}}{(l_2 + n_1)(l_2 + n_3)} & \dfrac{e^{\gamma_3}}{n_1 - n_3} + \dfrac{a_3 e^{\theta_3}}{(l_3 + n_1)(l_3 + n_3)} \\ \dfrac{-n_1 e^{-\gamma_1}}{n_3 - n_1} + \dfrac{a_1 l_1 e^{\theta_1}}{(l_1 + n_1)(l_1 + n_3)} + \dfrac{a_2 l_2 e^{\theta_2}}{(l_2 + n_1)(l_2 + n_3)} & \dfrac{-n_3 e^{-\gamma_3}}{n_1 - n_3} + \dfrac{a_3 l_3 e^{\theta_3}}{(l_3 + n_1)(l_3 + n_3)} \end{vmatrix},$$

$$f = \Bigg[\left(\frac{e^{-\gamma_1}}{n_3 - n_1} + \frac{a_1 e^{\theta_1}}{(l_1+n_1)(l_1+n_3)} + \frac{a_2 e^{\theta_2}}{(l_2+n_1)(l_2+n_3)} \right)$$
$$\left(\frac{-n_3 e^{-\gamma_3}}{n_1 - n_3} + \frac{a_3 l_3 e^{\theta_3}}{(l_3+n_1)(l_3+n_3)} \right)$$
$$- \left(\frac{-n_1 e^{-\gamma_1}}{n_3 - n_1} + \frac{a_1 l_1 e^{\theta_1}}{(l_1+n_1)(l_1+n_3)} + \frac{a_2 l_2 e^{\theta_2}}{(l_2+n_1)(l_2+n_3)} \right)$$
$$\left(\frac{e^{\gamma_3}}{n_1 - n_3} + \frac{a_3 e^{\theta_3}}{(l_3+n_1)(l_3+n_3)} \right) \Bigg],$$

$$\begin{aligned} f = {} & \frac{(-n_3+n_1)e^{-\gamma_1-\gamma_3}}{(n_3-n_1)(n_1-n_3)} + \frac{(l_3+n_1)a_3 e^{-\gamma_1+\theta_3}}{(n_3-n_1)(l_3+n_1)(l_3+n_3)} \\ & - \frac{(n_3+l_1)a_1 e^{-\gamma_3+\theta_1}}{(n_1-n_3)(l_1+n_1)(l_1+n_3)} + \frac{(l_3-l_1)a_1 a_3 e^{\theta_1+\theta_3}}{(l_1+n_1)(l_1+n_3)(l_3+n_1)(l_3+n_3)} \\ & - \frac{(n_3+l_2)a_2 e^{-\gamma_3+\theta_2}}{(l_2+n_1)(l_2+n_3)(n_1-n_3)} + \frac{(l_3+l_2)a_2 a_3 e^{\theta_2+\theta_3}}{(l_2+n_1)(l_2+n_3)(l_3+n_1)(l_3+n_3)}. \end{aligned}$$

Let $\varepsilon_i = \dfrac{a_i}{l_i + n_i}$ and make some simplification, the above equation becomes

$$\begin{aligned} f \;=\; & \frac{e^{-\gamma_1-\gamma_3}}{(n_3-n_1)} + \frac{\varepsilon_3 e^{\gamma_1+\theta_3}}{(n_3-n_1)} - \frac{\varepsilon_1 e^{-\gamma_3+\theta_1}}{(n_1-n_3)} + \frac{(l_3-l_1)\varepsilon_1\varepsilon_3 e^{\theta_1+\theta_3}}{(l_1+n_3)(l_3+n_1)} \\ & - \frac{a_2 e^{-\gamma_3+\theta_2}}{(l_2+n_1)(n_1-n_3)} + \frac{(l_3-l_2)a_2\varepsilon_3 e^{\theta_2+\theta_3}}{(l_2+n_1)(l_2+n_3)(l_3+n_1)}. \end{aligned} \tag{3.77}$$

Multiply Equation (3.77) with $(n_3 - n_1)$, then $f(x, y, t)$ becomes

$$\begin{aligned} f \;=\; & e^{-\gamma_1-\gamma_3} + \varepsilon_3 e^{\gamma_1+\theta_3} + \varepsilon_1 e^{-\gamma_3+\theta_1} + \frac{(l_3-l_1)(n_3-n_1)\varepsilon_1\varepsilon_3 e^{\theta_1+\theta_3}}{(l_1+n_3)(l_3+n_1)} \\ & + \frac{a_2 e^{-\gamma_3+\theta_2}}{(l_2+n_1)} + \frac{(l_3+l_2)(n_3-n_1)a_2\varepsilon_3 e^{\theta_2+\theta_3}}{(l_2+n_1)(l_2+n_3)(l_3+n_1)}. \end{aligned} \tag{3.78}$$

Next, we multiply the function $f(x, y, t)$ with e^{γ_1}

$$\begin{aligned} f \;=\; & e^{-\gamma_3} + \varepsilon_3 e^{\theta_3} + \varepsilon_1 e^{\gamma_1-\gamma_3+\theta_1} + \frac{(l_3-l_1)(n_3-n_1)\varepsilon_1\varepsilon_3 e^{\gamma_1+\theta_1+\theta_3}}{(l_1+n_3)(l_3+n_1)} \\ & + \frac{a_2 e^{\gamma_1-\gamma_3+\theta_2}}{(l_2+n_1)} + \frac{(l_3+l_2)(n_3-n_1)a_2\varepsilon_3 e^{\gamma_1+\theta_2+\theta_3}}{(l_2+n_1)(l_2+n_3)(l_3+n_1)}. \end{aligned} \tag{3.79}$$

After that, multiply again Equation (3.79) with e^{γ_3}

$$\begin{aligned} f \;=\; & 1 + \varepsilon_3 e^{\gamma_3+\theta_3} + \varepsilon_1 e^{\gamma_1+\theta_1} + \frac{(l_3-l_1)(n_3-n_1)\varepsilon_1\varepsilon_3 e^{\gamma_1+\theta_1+\gamma_3+\theta_3}}{(l_1+n_3)(l_3+n_1)} \\ & + \frac{a_2 e^{\gamma_1+\theta_2}}{(l_2+n_1)} + \frac{(l_3+l_2)(n_3-n_1)a_2\varepsilon_3 e^{\gamma_1+\theta_2+\gamma_3+\theta_3}}{(l_2+n_1)(l_2+n_3)(l_3+n_1)}. \end{aligned} \tag{3.80}$$

From Equation (3.9), we had stated that $\eta_i = \theta_i + \gamma_i$, thus Equation (3.80) can be rewritten as

$$\begin{aligned} f &= 1 + \varepsilon_3 e^{\eta_3} + \varepsilon_1 e^{\eta_1} + \frac{(l_3 - l_1)(n_3 - n_1)\varepsilon_1\varepsilon_3 e^{\eta_1+\eta_3}}{(l_1+n_3)(l_3+n_1)} \\ &\quad + \frac{a_2 e^{\gamma_1+\theta_2}}{(l_2+n_1)} + \frac{(l_3+l_2)(n_3-n_1)a_2\varepsilon_3 e^{\gamma_1+\theta_2+\eta_3}}{(l_2+n_1)(l_2+n_3)(l_3+n_1)}. \end{aligned} \tag{3.81}$$

For this case, we had set the condition which is $n_1 = n_2$ or $\gamma_1 = \gamma_2$, therefore Equation (3.81) becomes

$$\begin{aligned} f &= 1 + \varepsilon_3 e^{\eta_3} + \varepsilon_1 e^{\eta_1} + \frac{(l_3 - l_1)(n_3 - n_1)\varepsilon_1\varepsilon_3 e^{\eta_1+\eta_3}}{(l_1+n_3)(l_3+n_1)} \\ &\quad + \varepsilon_2 e^{\eta_2} + \frac{(l_3-l_2)(n_3-n_2)\varepsilon_2\varepsilon_3 e^{\eta_2+\eta_3}}{(l_2+n_3)(l_3+n_2)}, \end{aligned} \tag{3.82}$$

$$f = 1 + \varepsilon_1 e^{\eta_1} + \varepsilon_2 e^{\eta_2} + \varepsilon_3 e^{\eta_3} + A_{13}\varepsilon_1\varepsilon_3 e^{\eta_1+\eta_3} + A_{23}\varepsilon_2\varepsilon_3 e^{\eta_2+\eta_3} \tag{3.83}$$

where

$$A_{13} = \frac{(l_1+l_3)(n_1+n_3)}{l_1+n_3)(l_3+n_1)}, \tag{3.84}$$

$$A_{23} = \frac{(l_2+l_3)(n_2+n_3)}{l_2+n_3)(l_3+n_2)}, \tag{3.85}$$

or we can conclude that

$$A_{ij} = \frac{(l_i+l_j)(n_i+n_j)}{l_i+n_j)(l_j+n_i)}. \tag{3.86}$$

Now we want to discuss the interaction between a triad and a soliton. For this purpose we will use KPPRO. The following is the series of picture about the interactions of three KP solitons, where a triad interacts with a soliton.

For this case the values of parameter were chosen as follow where soliton 1, S_1 and soliton 2, S_2 will be in full resonance ($n_1 = n_2 = 3$) and we also choose n_3 to be very close to n_1, n_2 so that it will then go into resonance process with the other two solitons.

$$
\begin{aligned}
n_1 &= 3 \qquad n_2 = 3 \qquad n_3 = 3 + 10^{-11} \\
l_1 &= -2 \qquad l_2 = 3 \qquad l_3 = 2
\end{aligned} \tag{3.87}
$$

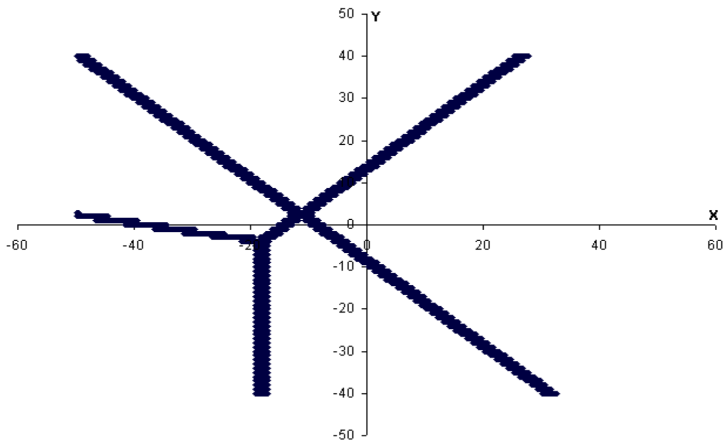

Figure 3.1: A triad interacts with a soliton at $t = 0.5s$

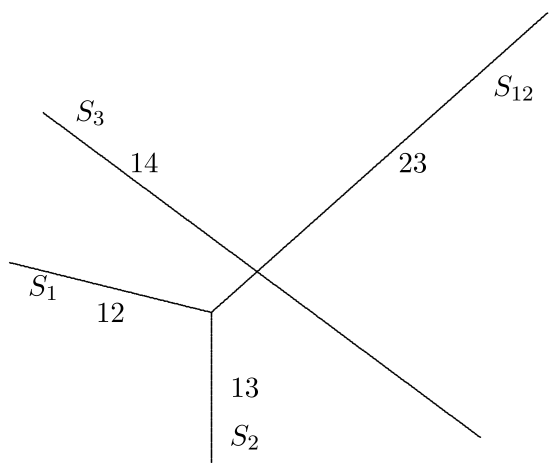

Figure 3.2: Geometrical representation of a triad interacts with a soliton at $t = 0.5s$

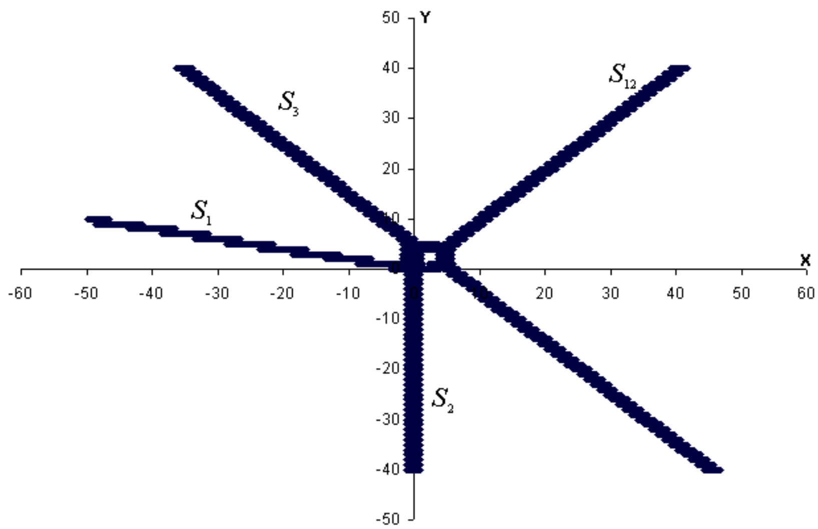

Figure 3.3: A triad interacts with a soliton at $t = 0s$

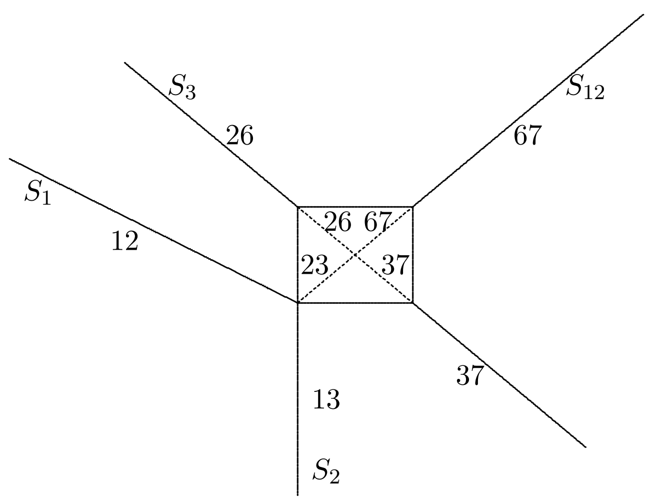

Figure 3.4: Geometrical representation of a triad interacts with a soliton at $t = 0s$

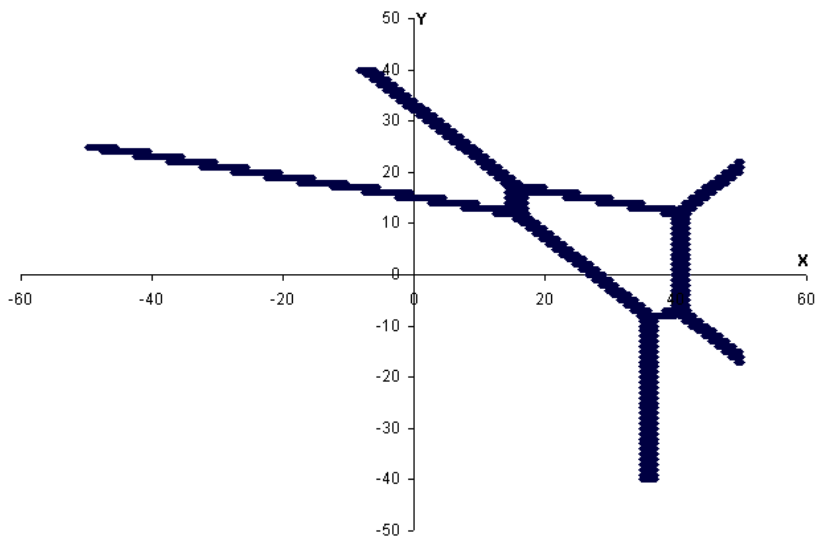

Figure 3.5: A triad interacts with a soliton at $t = -1s$

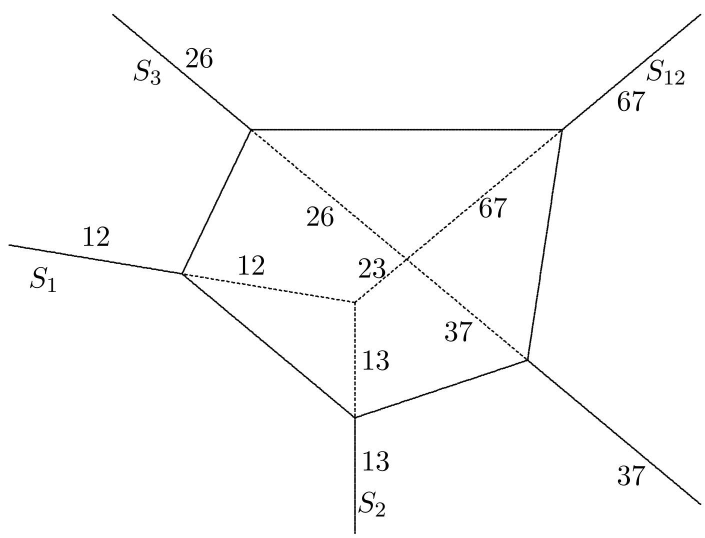

Figure 3.6: Geometrical representation of a triad interacts with a soliton at $t = -1s$

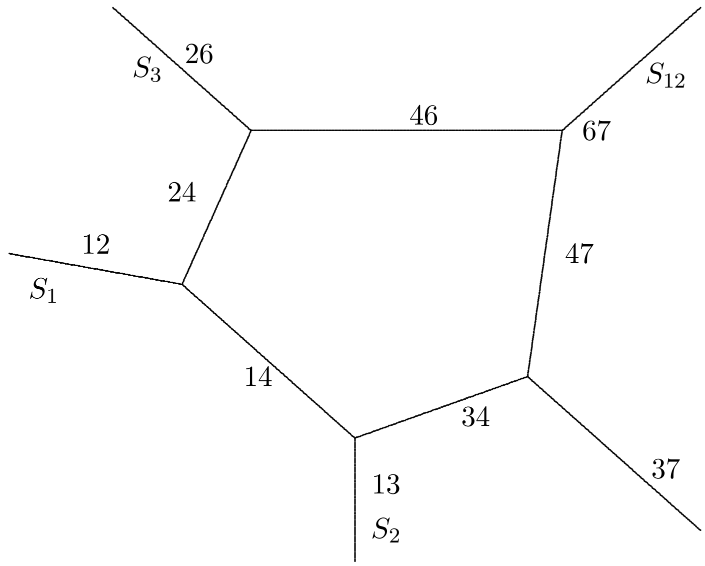

Figure 3.7: Geometrical representation of a triad interacts with a soliton at $t = -1s$

We can observe that the interaction between a triad and a soliton is from $t = 0.5$ to $t = -1$ and is not from negative value of t to positive value of t. This is due to the positive dispersion of the KP equation. If we considered the negative dispersion of the KP equation, then all interactions generated will be from negative t to positive t.

Before we discuss more about the interaction patterns, let us determine the position of S_1, S_2, S_3 and S_{12}. At $t = 0$, S_1, S_2 and S_3 are centered along the line $k_i x + m_i y = 0$. Thus we have

$$
\begin{aligned}
S_1 &: \quad x + 5y = 0, \qquad & y = -\frac{1}{5}x \\
S_2 &: \quad 6x + 0 = 0, \qquad & x = 0 \\
S_3 &: \quad (5 + 10^{-11})x + 5y = 0, \qquad & y \approx -x
\end{aligned}
\tag{3.88}
$$

However, each soliton will also centered along these lines in any value of t. For this case, we have chosen $n_1 = n_2$. Thus S_1 will interacts with S_2 to produce S_{12}, as shown in Figure 3.3 .

From the above interaction pattern, we can observe clearly that this interaction is between a triad and a soliton at $t = 0.5$ as shown in Figure 3.1. If the value t increases, the patterns will still remain the same but the distance between S_3 with S_1 and S_2 will be even further away.

When $t = 0$ which is from Figure 3.3, S_3 starts to interact with S_1 and S_2 at the beginning point of the resonant soliton and produce a square pattern. This square becomes bigger and bigger and finally becomes a pentagon as time increases. The pentagon will not change anymore to other form. However it is getting bigger as t changes. From Figure 3.5, we can observe that there were two quadruplets and a triad in that pattern.

These interaction patterns can be represented geometrically in Figure 3.2, Figure 3.4, Figure 3.6 and Figure 3.7. To explain the interaction patterns geometrically, the function $f(x, y, t)$ that we considered is the general solution which is Equation (3.7) so that we will have a complete list of combination of terms. Equation (3.7) can be rewritten as

$$\begin{aligned} f \;=\; & \underbrace{1}_{(1)} + \underbrace{\varepsilon_1 \exp(\eta_1)}_{(2)} + \underbrace{\varepsilon_2 \exp(\eta_2)}_{(3)} + \underbrace{\varepsilon_3 \exp(\eta_3)}_{(4)} \\ & + \underbrace{A_{12}\varepsilon_1\varepsilon_2 \exp(\eta_1 + \eta_2)}_{(5)} + \underbrace{A_{13}\varepsilon_1\varepsilon_3 \exp(\eta_1 + \eta_3)}_{(6)} \\ & + \underbrace{A_{23}\varepsilon_2\varepsilon_3 \exp(\eta_2 + \eta_3)}_{(7)} + \underbrace{A_{123}\varepsilon_1\varepsilon_2\varepsilon_3 \exp(\eta_1 + \eta_2 + \eta_3)}_{(8)}. \end{aligned} \tag{3.89}$$

Any two combination of the components of $f(x, y, t)$ will give a soliton solution. Therefore for each soliton we have,

$$\text{Soliton 1,} \quad S_1 \quad : \quad (12),\ (35),\ (46),\ (78); \tag{3.90}$$

$$\text{Soliton 2,} \quad S_2 \quad : \quad (13),\ (25),\ (47),\ (68); \tag{3.91}$$

$$\text{Soliton 3,} \quad S_3 \quad : \quad (14),\ (26),\ (37),\ (88); \tag{3.92}$$

$$\text{Resonant Soliton,} \quad S_{12} \quad : \quad (15),\ (23),\ (48),\ (67); \tag{3.93}$$

$$\text{Resonant Soliton,} \quad S_{13} \quad : \quad (16),\ (24),\ (38),\ (57); \tag{3.94}$$

$$\text{Resonant Soliton,} \quad S_{23} \quad : \quad (17),\ ,(28)\ (34),\ (56). \tag{3.95}$$

To explain this, let us take the combination of (46), then we have

$$f = \varepsilon_3 \exp(\eta_3) + A_{13}\varepsilon_1\varepsilon_3 \exp(\eta_1 + \eta_3), \tag{3.96}$$

$$= \varepsilon_3 \exp(\eta_3) \left[1 + A_{13}\varepsilon_1 \exp(\eta_1)\right]. \tag{3.97}$$

We know that the term $\exp(\eta_3)$ is linear in x, hence it will vanish when we substitute Equation (3.97) into Equation (3.8). Thus $f = 1 + A_{13}\varepsilon_1 \exp(\eta_1)$ which is the $f(x, y, t)$ of soliton 1. For the rest of soliton like S_1, S_2, S_3, S_{12}, S_{13} and S_{23}, the same procedure applies.

By referring to Figure 3.2, we see that S_1 (12) interacts with S_2 (13) to produce a resonant soliton S_{12} (23) and later interacts with S_3 (14). Since S_3 (14) has no term that is of common to S_{12} (23), so there is no interaction and they merely collide and cross over. But later as time passed, S_3 (14) will meet S_1 (12) and S_2 (13) and we get a square and later becomes a pentagon.

By referring to Figure 3.7, we can see that pentagon has all its sides representing a soliton formed by S_1 (46), S_{13} (24), S_3 (14), S_{23} (34) and S_2 (47). If we zoomed in, we realized that there are two quadruplets which is Q_{13} formed by S_1 and S_3 and Q_{23} formed by S_2 and S_3 and also a triad, T_{12} formed by S_1 and S_2.

Figure 3.4 and Figure 3.7 show that the interaction between a triad and a soliton moves as an entity although they have a complicated interaction. We still can see a soliton cross over the triad in these figures which is shown in dotted lines.

If we choose $n_1 = n_2 \neq n_3$ which is not $n_1 = n_2 \approx n_3$, then triad will only cross with soliton and this is shown in Figure 3.8, Figure 3.9 and Figure 3.10. In this case we choose

$$\begin{aligned} n_1 &= 3 \qquad n_2 = 3 \qquad n_3 = 5 \\ l_1 &= -2 \qquad l_2 = 3 \qquad l_3 = 2. \end{aligned} \tag{3.98}$$

From the parametric values, we still have a triad, T_{12} but it will only cross with S_3.

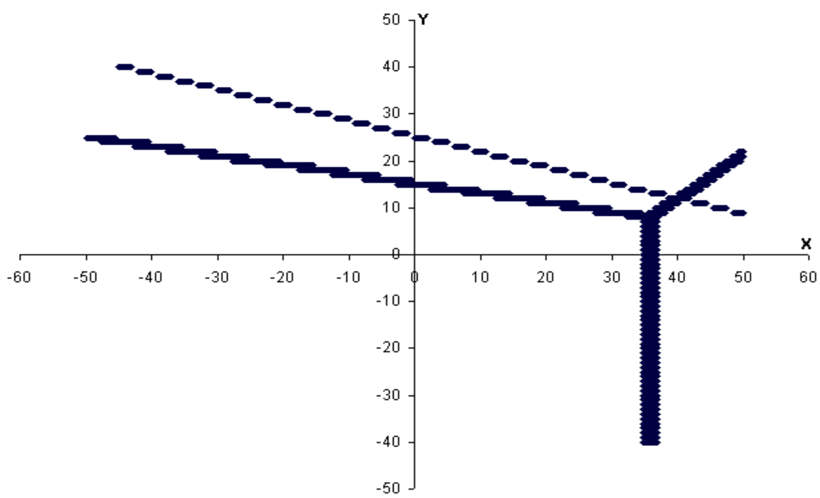

Figure 3.8: A triad cross with soliton at $t = -1s$

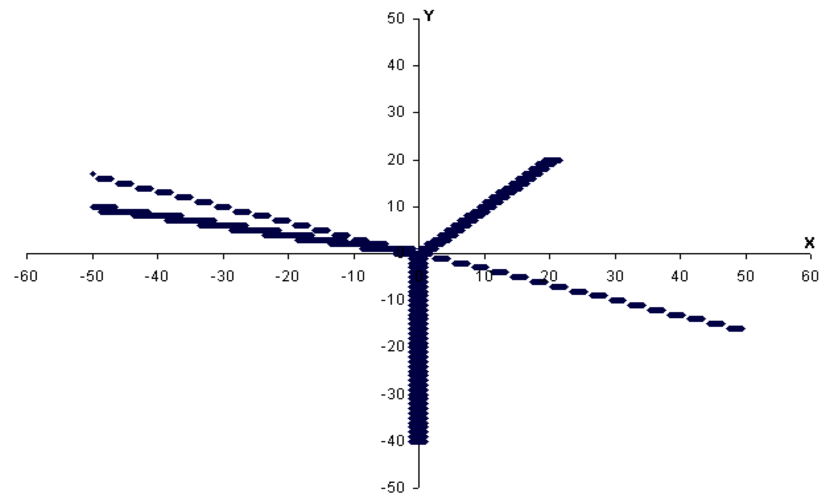

Figure 3.9: A triad cross with soliton at $t = 0s$

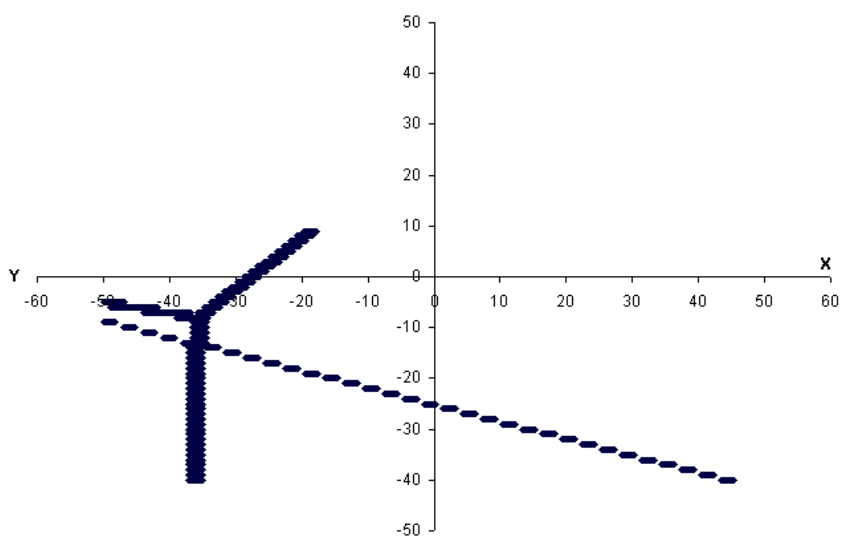

Figure 3.10: A triad cross with soliton at $t = 1s$

Another case is when we choose the value of n_3 is 1 which is less than 3, it will produce another type of interaction as shown in Figure 3.11, Figure 3.12 and Figure 3.13.

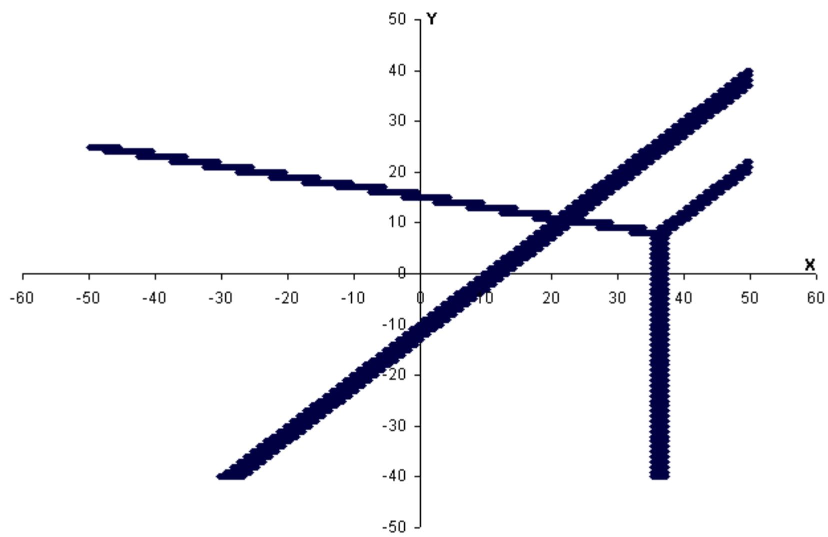

Figure 3.11: A triad cross with soliton at $t = -1s$

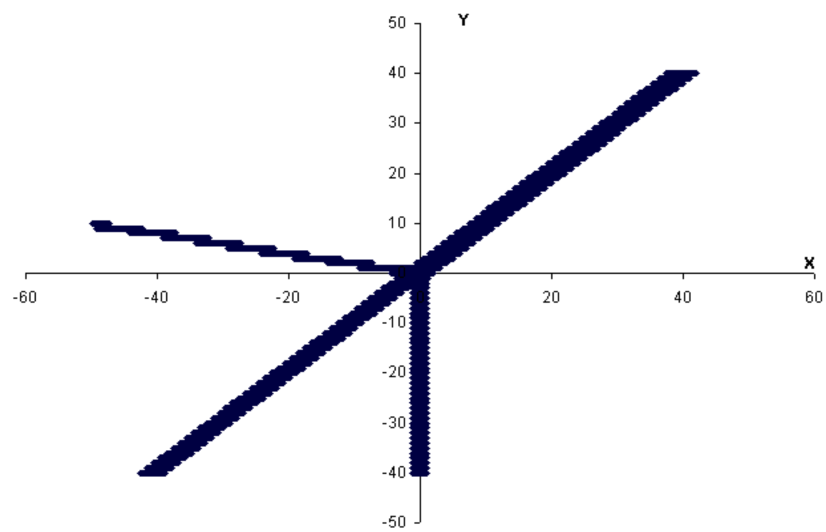

Figure 3.12: A triad cross with soliton at $t = 0s$

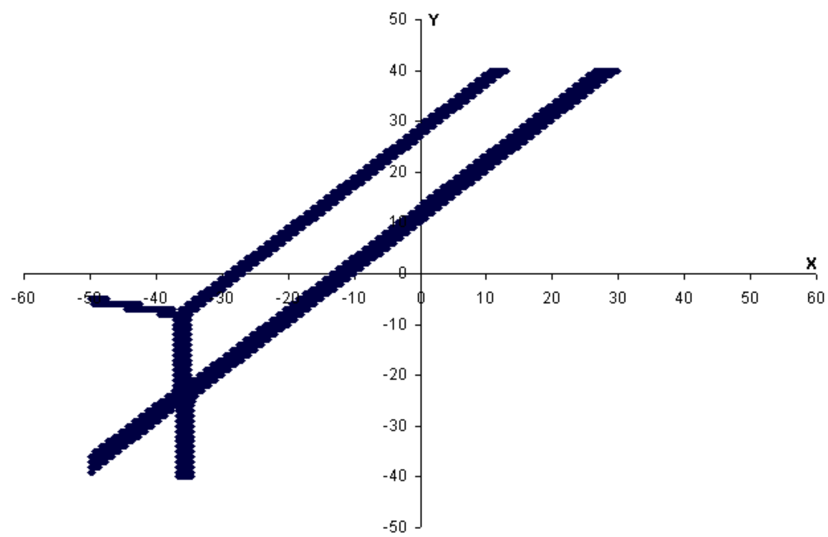

Figure 3.13: A triad cross with soliton at $t = 1s$

In Figure 3.12, we can observe that S_3 will only collide with S_{12} but they do not interacts among themselves.

3.6.1.2 Type 2: $n_2 = n_3 \approx n_1$

Now we will look at another type of interactions between a triad and a soliton. We will still use the parametric values from the previous version. The only difference is the sequence of the values of n_i. In the last subsection we have chosen $n_1 = n_2 \approx n_3$ but now we change the sequence to $n_1 \approx n_2 = n_3$. However we still use the same values of l_i.

$$\begin{aligned} n_1 &= 3 + 10^{-11} \qquad n_2 = 3 \qquad n_3 = 3 \\ l_1 &= -2 \qquad l_2 = 3 \qquad l_3 = 2 \end{aligned} \tag{3.99}$$

From the values we have chosen above we know that in this interaction, S_2 will interacts with S_3 to produce resonant soliton, S_{23} $(n_2 = n_3)$ and this triad, T_{23} will interacts with S_1 later. Before we look into the interaction pattern, let us determine the position of each soliton first. As we mentioned before,

$$\begin{aligned} S_1 &: (1+10^{-11})x + 5y = 0, \qquad y \approx -\frac{1}{5}x \\ S_2 &: 6x + 0 = 0, \qquad x = 0 \\ S_3 &: 5x + 5y = 0, \qquad y = -x \end{aligned} \tag{3.100}$$

So now let us look at Figure 3.14 to see clearly the position of each soliton.

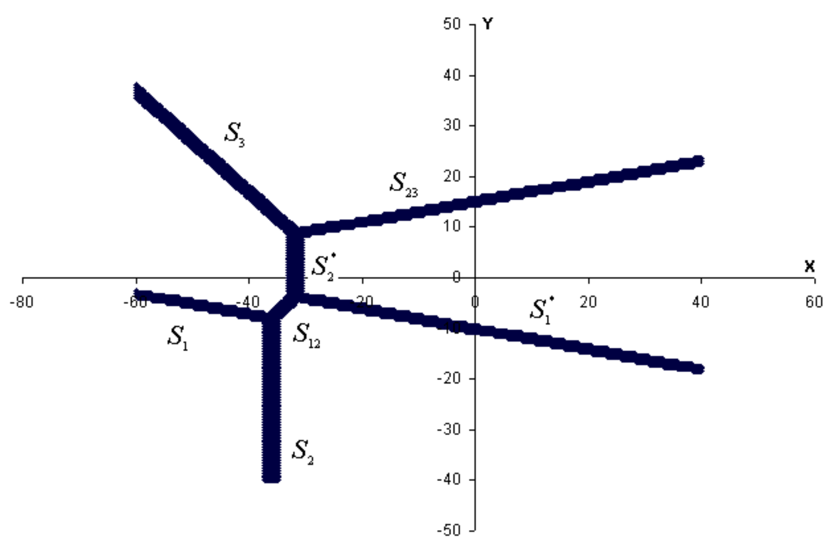

Figure 3.14: A triad interacts with soliton at $t = 1s$

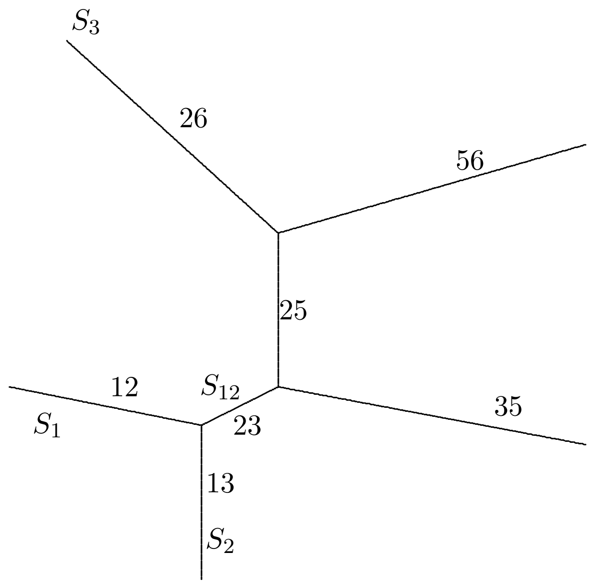

Figure 3.15: Geometrical representation of a triad interacts with a soliton at $t = 1s$

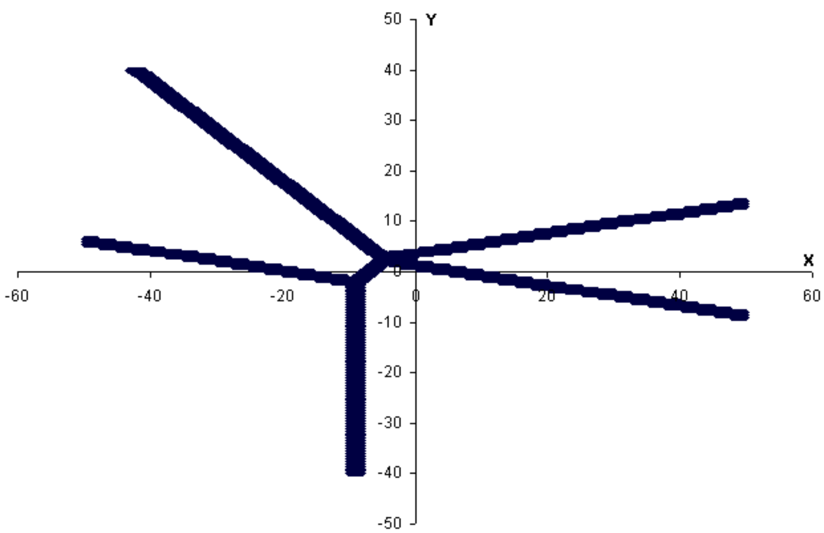

Figure 3.16: A triad interacts with soliton at $t = 0.25s$

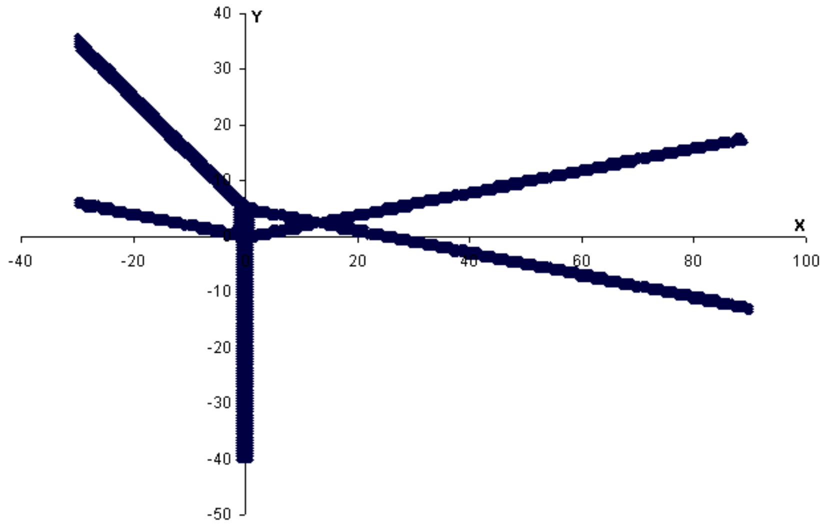

Figure 3.17: A triad interacts with soliton at $t = 0s$

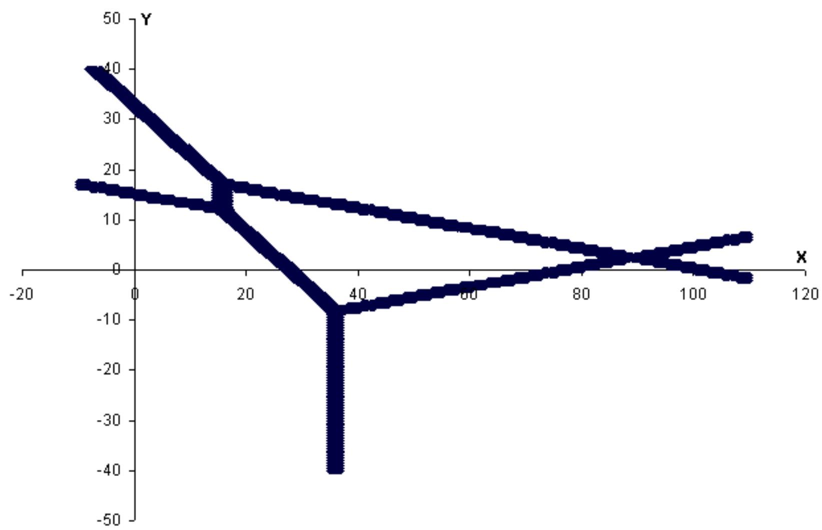

Figure 3.18: A triad interacts with soliton at $t = -1s$

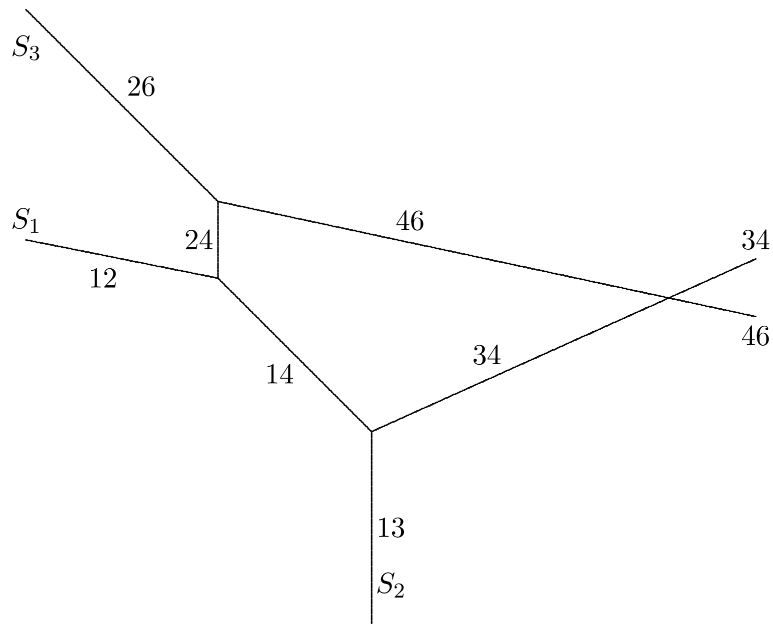

Figure 3.19: Geometrical representation of a triad interacts with a soliton at $t = -1s$

This type of interactions patterns also start from positive value of t to negative value of t. At $t = 1$, we can see clearly that there is a triad formed by the interaction of S_2 and S_3 to produce the resonant soliton, S_{23}. At first S_1 will interact with S_2. Since $n_1 \approx n_2$, therefore S_1 will interact with S_2 to produce a quadruplet, Q_{12}. Later Q_{12} will move upwards to interact with the triad or in other words is to interacts with S_3. Since $n_1 \approx n_3$ too, thus S_1 will interact with S_2 to produce another quadruplet. At the end of the interactions, the triad, T_{23} produced by S_2 and S_3 will still remain. This is important because this interaction is between T_{23} and S_1.

These interaction patterns can be represented in geometrical form. By referring to Figure 3.15, S_1 (12) and S_2 (13) is partially resonance at first and formed a S_{12} (23). After breaking up, S_2 (25) will interact with S_3 (26)and produced S_{23} (56). Therefore at the beginning of the interaction, we have a quadruplet, Q_{12} by S_1 and S_2 and a triad, T_{23} by S_2 and S_3.

Next we refer to Figure 3.19, we can see that the quadrilateral has all its sides representing a soliton formed by S_{13} (24), S_3 (14), S_{23} (34) and S_1 (46). If we zoomed in, we realized that there are a quadruplet and a triad which are Q_{13} formed by S_1 and S_3 and T_{23} formed by S_2 and S_3.

3.6.1.3 Type 3: $n_1 = n_3 \approx n_2$

Next we will look at another version of interactions between a triad and a soliton. As before, we will change again the sequence of the values of n_i. This time we have $n_1 = n_3 \approx n_2$. Following is the values that we have chosen:

$$\begin{aligned} n_1 &= 3 \qquad n_2 = 3 + 10^{-11} \qquad n_3 = 3 \\ l_1 &= -2 \qquad l_2 = 3 \qquad l_3 = 2 \end{aligned} \tag{3.101}$$

We know that there will be a triad ($n_1 = n_3$), T_{13} by S_1 and S_3 and later this T_{13} will interact with S_2. Now, let us determine the position of each soliton first. Using the same explanation before, each soliton will be centered along $k_i x + m_i y = 0$ and were shown respectively as following.

$$\begin{aligned} S_1 &: \ x + 5y = 0, \qquad y = -\frac{1}{5}x \\ S_2 &: \ (6 + 10^{11})x + 10^{-22}y = 0, \qquad x \approx 0 \\ S_3 &: \ 5x + 5y = 0, \qquad y = -x \end{aligned} \tag{3.102}$$

The position of each soliton is shown in Figure 3.23.

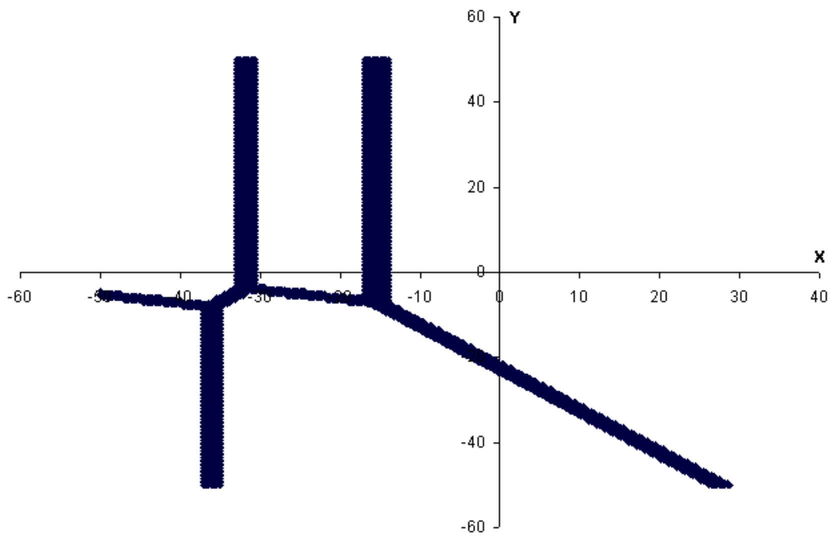

Figure 3.20: A triad interacts with a soliton at $t = 1s$

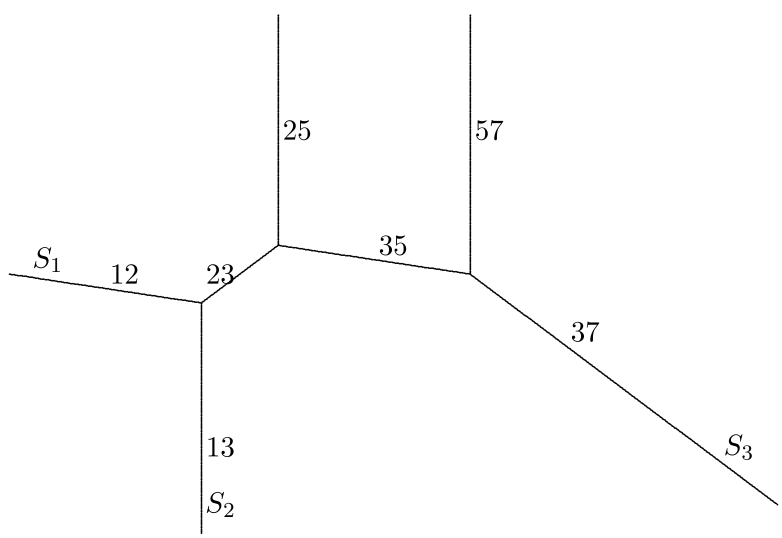

Figure 3.21: Geometrical representation of a triad interacts with a soliton at $t = 1s$

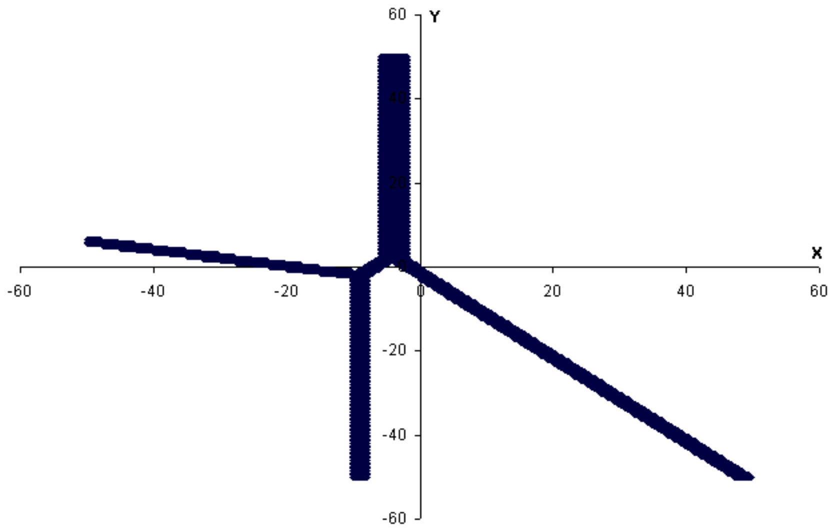

Figure 3.22: A triad interacts with a soliton at $t = 0.25s$

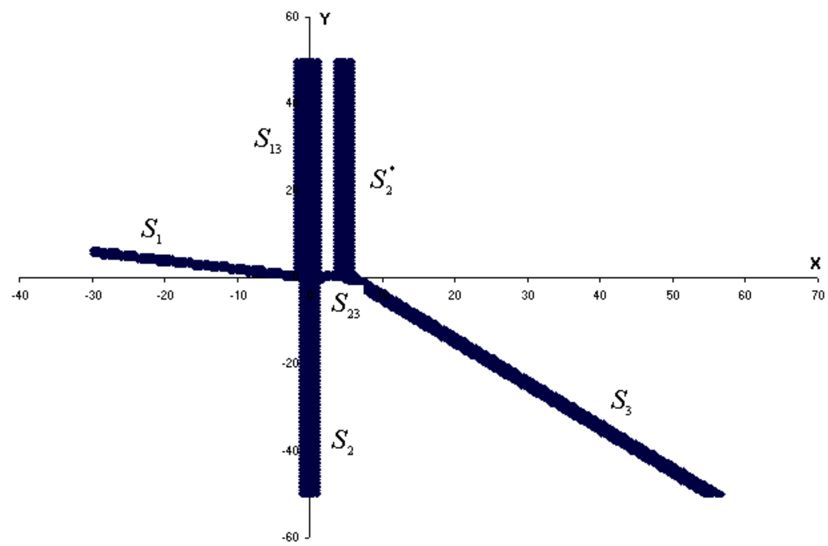

Figure 3.23: A triad interacts with a soliton at $t = 0s$

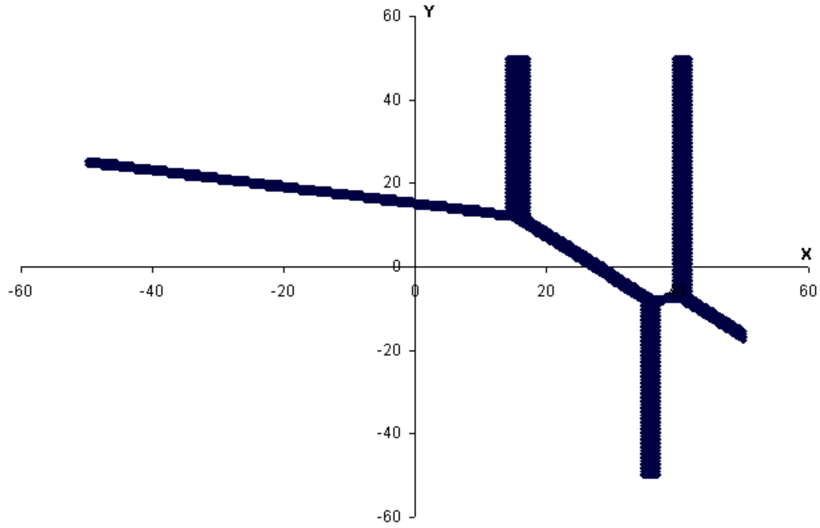

Figure 3.24: A triad interacts with a soliton at $t = -1s$

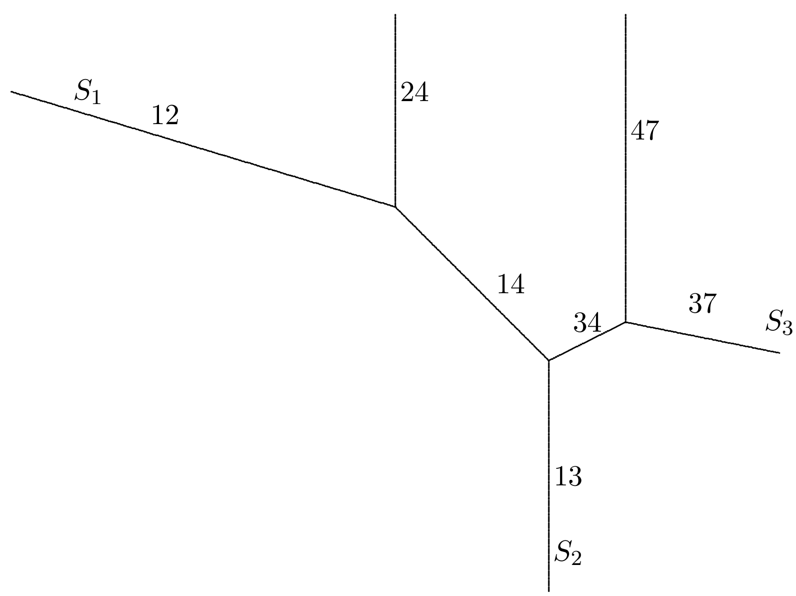

Figure 3.25: Geometrical representation of a triad interacts with a soliton at $t = -1s$

From Figure 3.23, it looks like that S_2 and S_{13} are straight line. However if we study it in detail, we can see that the thickness of line S_2 is different with line S_{13}. So now let us look at the interactions patterns in the following figures.

Since we have choose $n_1 = n_3$, so we must have the resonant soliton, S_{13} from the beginning of the interactions until the end of the interactions. In other word, we must have T_{13} in the interactions. From the above figures, we can observe that there are quadruplet, Q_{12} produced by the interaction between S_1 and S_2 and a triad, T_{13} produced by S_1 and S_3 at $t = 1$. This is true when we compared the interaction pattern with the parametric values.

The interactions start with S_2 interacting with S_1 to form Q_{12} because $n_1 \approx n_2$. As we know, the resonant soliton S_{12} in the quadruplet will then break up into soliton 1 and soliton 2 again with some phase-shifted and we denote these solitons as S_1^* and S_2^*. Later S_2^* will moved towards S_{13} and this makes the length of S_1^* becomes shorter and shorter.

Next S_2^* will merge with S_{13} and they just merely collide among themselves and is shown in Figure 3.22. S_{12} will also collides with S_{13}. After that S_{13} and S_2^* will break up and S_{13} and S_2 will become straight line while S_2^* will interacts with S_3 to produce another quadruplet, Q_{23}. This quadruplet will becomes more obvious as t changes. Finally, The interaction pattern will only left with T_{13} and Q_{23}.

By referring to Figure 3.21, we can see that S_1 (12) and S_2 (13) in partially resonance to form S_{12} and later they break up again which are S_1 (35) and S_2 (25). After that, S_1 (35) interacts with S_3 (37) to produce S_{13} (57). In Figure 3.25, we still have same triad, T_{13} by S_1 (12) and S_3 (14) but with different quadruplet which is Q_{23} by S_2 (13) and S_3 (14).

3.6.2 A Quadruplet Interacts With A Soliton

In the last section we had put a condition $n_i = n_j \approx n_k$ where i, j, $k = 1, 2, 3$. This condition will produce a triad since $n_i = n_j$ and this triad must interacts with a soliton because we have choose $n_k \approx n_i$ and n_j. We will use the same idea here. This time we will choose $n_i \approx n_j \approx n_k$. This implies that there must be a quadruplet interacting with a soliton.

In this case we choose the values as follow:

$$\begin{aligned} n_1 &= 3 + 10^{-11} \qquad n_2 = 3 + 10^{-15} \qquad n_3 = 3 \\ l_1 &= -2 \qquad l_2 = 3 \qquad l_3 = 2 \end{aligned} \tag{3.103}$$

We shall determine the position of each soliton first. Each soliton will be centered on line $k_i x + m_i y = 0$.

$$\begin{aligned} S_1 &: (1 + 10^{-11})x + 5y = 0, \qquad y \approx -\frac{1}{5}x \\ S_2 &: (6 + 10^{-15})x + 0 = 0, \qquad x \approx 0 \\ S_3 &: 5x + 5y = 0, \qquad y = -x \end{aligned} \tag{3.104}$$

This is shown in Figure 3.26. The following are figures for the interaction patterns between a quadruplet with a soliton.

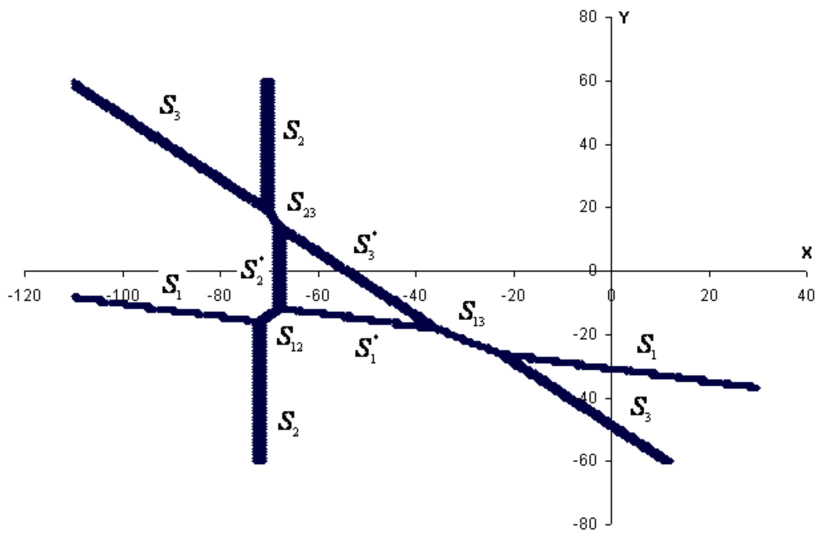

Figure 3.26: A quadruplet interacts with a soliton at $t = 2s$

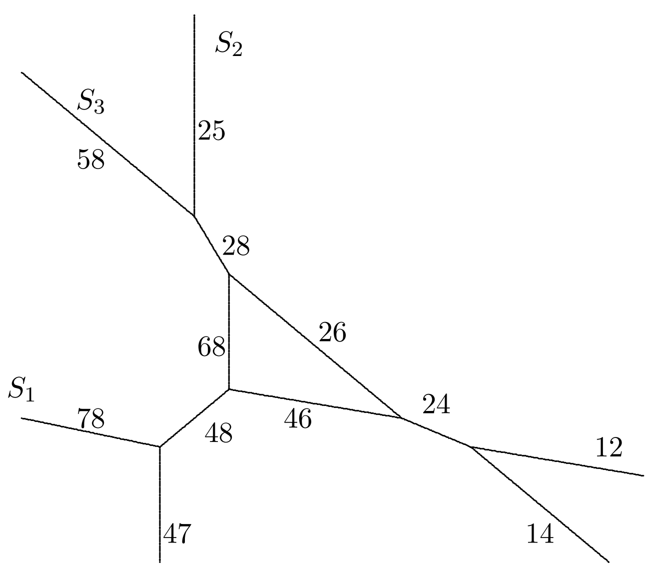

Figure 3.27: Geometrical representation of a quadruplet interacts with a soliton at $t = 2s$

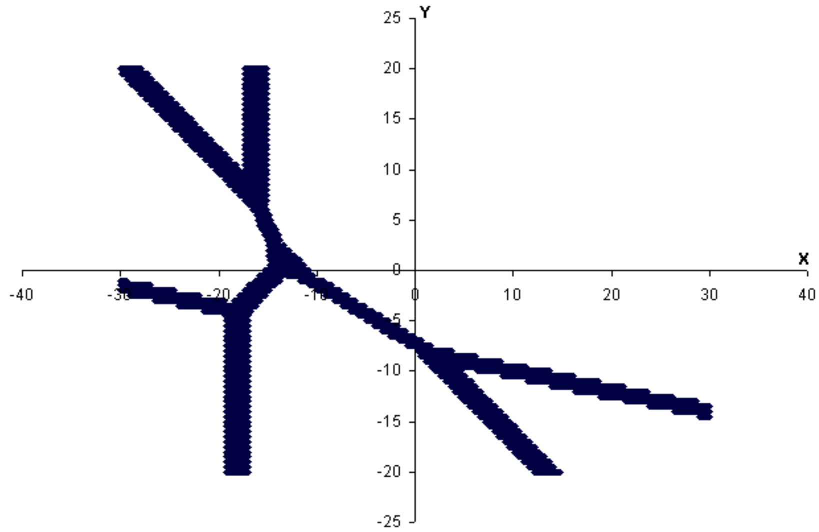

Figure 3.28: A quadruplet interacts with a soliton at $t = 0.5s$

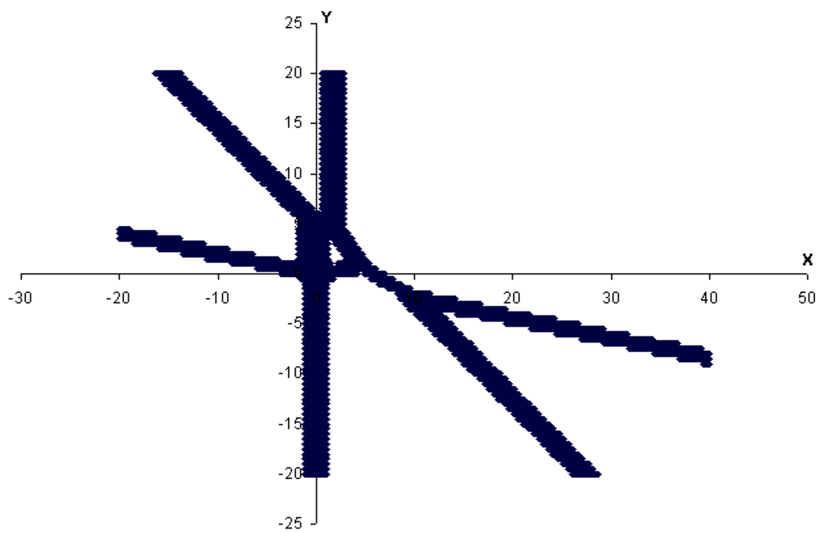

Figure 3.29: A quadruplet interacts with a soliton at $t = 0s$

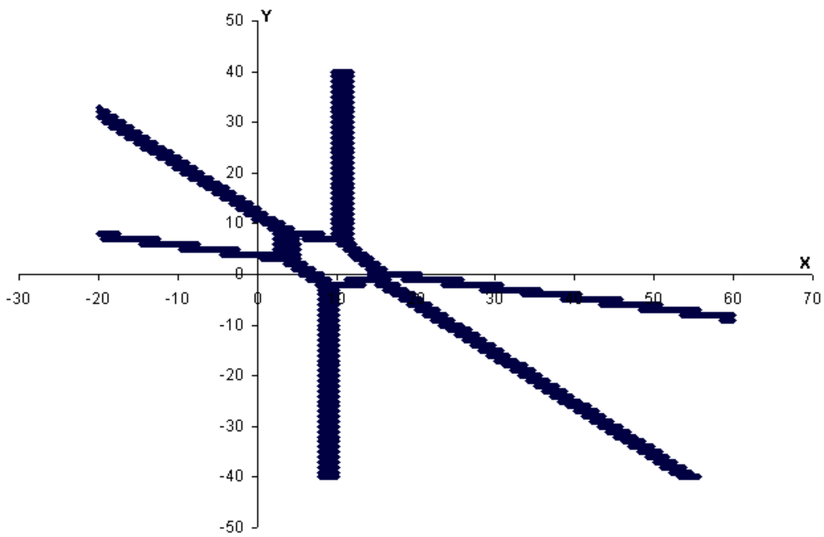

Figure 3.30: A quadruplet interacts with a soliton at $t = -0.25s$

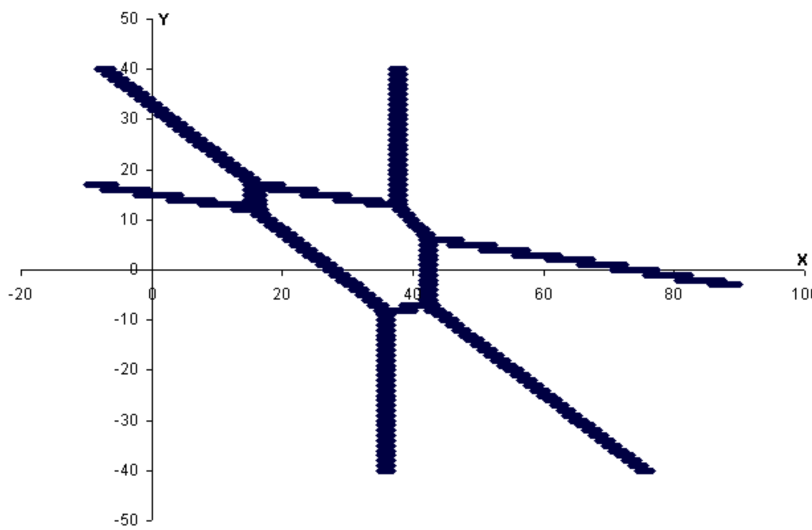

Figure 3.31: A quadruplet interacts with a soliton at $t = -1s$

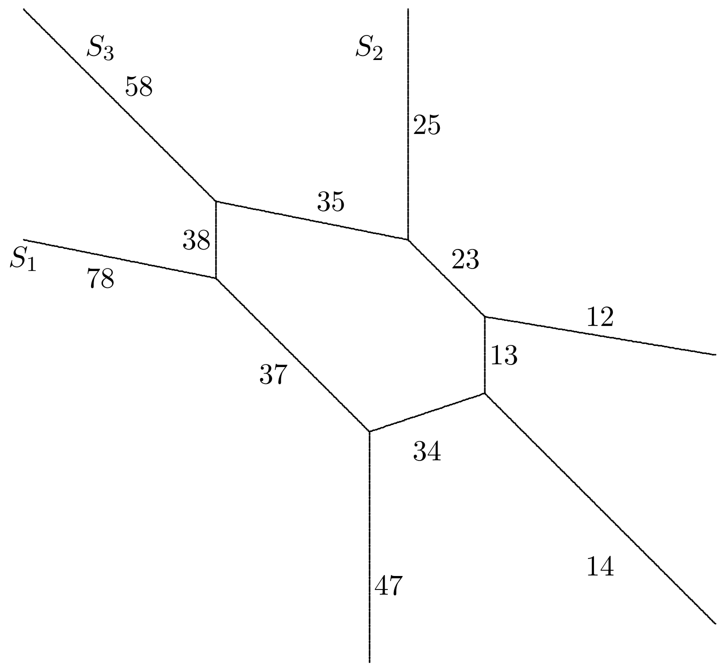

Figure 3.32: Geometrical representation of a quadruplet interacts with a soliton at $t = -1s$

From the above figures, we can observe obviously that there is a quadruplet formed by S_1 and S_2 and it will interact with a soliton at $t = 2$. Since $n_1 \approx n_2 \approx n_3$, therefore S_3 will interact with both S_1 and S_2 to produce another two quadruplets. So at the beginning of the interactions, we have three quadruplets. We named it as Q_{12}, Q_{13} and Q_{23}.

As the value of t changes, S_3 will move towards Q_{12} until it interacts with S_{12}. However there is much interaction happens. Later, the length of both S_{13} and S_{23} become shorter and shorter. After that, there is some interactions at the center or at the origin of the axis. It starts to break up into a pentagon at first and later change into a hexagon. If the value of t continues changing, the size of the hexagon will increase.

If we want to explain this interactions by using the combination of terms in the function $f(x, y, t)$, we still can use Equation (3.89) and the combination of the terms are listed from Equation (3.90) to Equation (3.95). This is shown in the Figure 3.27 and Figure 3.32.

By referring to Figure 3.27, we can see that there are three quadruplets which is Q_{12} by S_1 and S_2, Q_{13} by S_1 and S_3 and Q_{23} by S_2 and S_3. As t passed, these three quadruplets will interact among themselves and produce a pentagon and later becomes a hexagon. By referring to Figure 3.32, we see that hexagon has all its sides representing a soliton formed by S_1 (35), S_{13} (57), S_3 (37), S_{23} (34), S_2 (13) and S_{12} (23). If we zoomed in, we realized that there still have three quadruplets Q_{12}, Q_{13} and Q_{23}.

3.7 Conclusion

In this chapter we have discussed about three types of the interactions between a triad and a soliton and interactions of a quadruplet with a soliton. In these interactions, we have a few shapes after interactions which are quadrilateral, square, pentagon and hexagon.

CHAPTER IV

SUMMARY AND CONCLUSIONS

4.1 Introduction

Nonlinearity is a fascinating element of nature whose importance has been appreciated for many years when considering large-amplitude wave motions observed in various fields ranging from fluids and plasma to solid-state, chemical, biological, and geological systems. Localized large-amplitude waves called solitons, which propagate without spreading and have particle-like properties, represent one of the most striking aspects of nonlinear phenomena.

Today, many scientists see nonlinear science as the most deeply important frontier for fundamental understanding of nature. The soliton concept was firmly established after a gestation period of about 150 years. Since then, different kinds of solitons have been observed experimentally in various real systems, and today, they have captured the imagination of scientist in most physical discipline. They are widely accepted as a structural basis for viewing and understanding the dynamics behavior of complex nonlinear systems.

4.2 Summary

In this research we have studied the KP equation and the interactions patterns produced by solitons interacting. In first chapter, we did some literature review on history of soliton, properties of soliton, the KdV equation and finally the two dimensional equation of the KdV equation which is the KP equation.

Furthermore, we have discussed the condition of resonance of the KP equation in Chapter 2 and we found out that we must have a condition where $A_{12} \approx 0$ so that resonances occur. Therefore for two-soliton solution, we have three structures which are triad, quadruplet and cross. The position and direction for every soliton depends on the values of n_i and l_i. Although we choose and fix the value of n_i, we will have different structures for different value of l_i since every soliton is centered on the line $k_i x + m_i y = 0$ or specifically $(n_i + l_i)x + (n_i^2 - l_i^2)y = 0$.

From these structures, we can proceed to the interactions of the three KP solitons. In Chapter 3, we discussed about the interactions patterns between a triad with a soliton and between a quadruplet and a soliton. We have three versions of the interactions patterns between triad and a soliton. The difference between these versions is the sequence for the value of n_i with the same value of l_i. The choice of the sequence will determine the structure of the triad. Hence this will influence the interactions patterns. In the interactions of three KP solitons, we can see some patterns produced after the interactions. For example, we have square, pentagon and hexagon which have all its sides representing a soliton. On the other hand, we can noticed that the solution for the interaction between a triad and a soliton can be transformed into Wronskian determinant.

4.3 Suggestion

Since the KP equation has rich mathematical structures, we need to explore further. We only know little about this equation compared to the KdV equation. Thus there is a lot of things about the KP equation we can explore. Since the KP equation is the two dimensional equation for the KdV equation, it is more applicable to the physical problems. It can be applied into fluid dynamics and other problem in real world.

The KP equation has been an intensively studied model equation in fluid dynamics. Furthermore, Taniuti and Hasegawa (1991) recently have shown that the nonlinear development of the magneto sonic waves are governed by the KP equation with the positive dispersion in some parameter regions (Tajiri and Murakami, 1992). The above examples show that the KP equation describes many other physical phenomena which deserves details investigation on it.

Not only the KP equation with the positive dispersion, we also can do some research on the KP equation with the negative dispersion and forced KP equation.

4.4 Application Of The KP Equation

Before we discuss more about the application of the KP equation, let us look at the following figure.

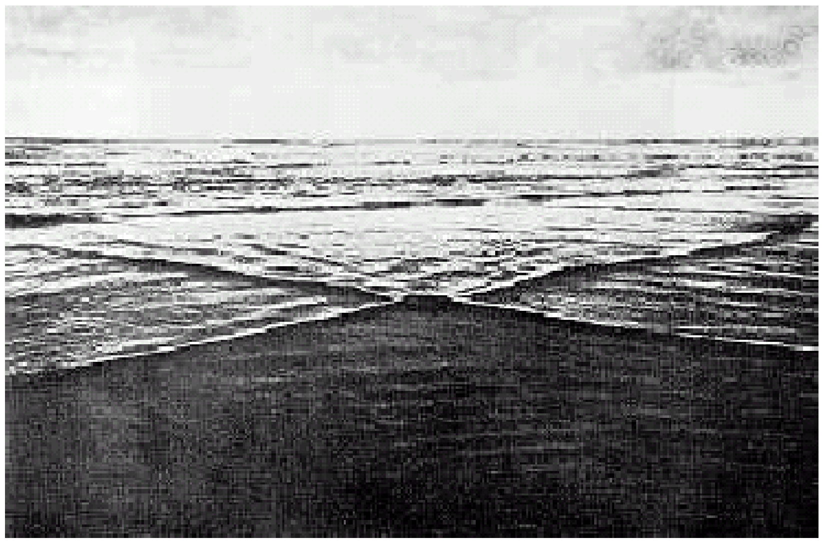

Figure 4.1: Oblique interaction of two nearly solitary waves in shallow water about one meter deep, off the coast of Oregon.

Figure 4.1 shows the oblique interaction of two solitary waves in shallow water and we can observed all the features of the soliton phenomena from this figure such as existence of stable solitary waves, wave crests in different sides of the interaction region are parallel with phase shift. Miles (1977a) had studied the oblique interaction of two solitary waves. However, he used the Boussinesq approximation to derive the two solitons solution directly from the boundary value problem for inviscid irrotational motion.

The KP equation also can be used to model weakly oblique interactions where it had been demonstrated in laboratory experiments to compare the oblique interaction of two waves train with the corresponding analytical solutions of the KP equation. It is shown that the KP equation is a remarkably good model for the phenomenon of oblique interaction of waves also for relatively large interaction angles (Peterson, 2001).

From Chapter 2, we know that one of the resonance structures in two KP solitons interactions is quadruplet. If we have many quadruplets and join them, then we have hexagonal patterns. This is study by Ong *et al* (1997). They observed hexagonal patterns of genus two-soliton interactions as shown in the following figure. This is another application of the KP equation.

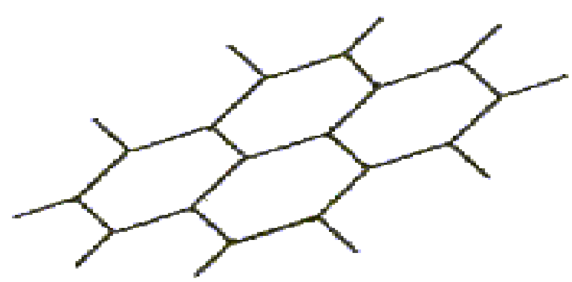

Figure 4.2: Hexagonal patterns of genus two solitons interactions.

Another application for the KP equation is the inverse problem by Peterson and Groesen (2000, 2001). They addressed a new 'inverse' problem for reconstructing the amplitudes of 2D surface waves from observation of the wave patterns (formed by wave crests). The aim of the inverse problem cf wave crests is to predict wave parameters (most important wave amplitudes) from the geometry of interaction patterns of waves. He showed that this problem can be solved when the waves are modeled by an equation that supports soliton solutions. Specifically, the explicit solution to the inverse problem is derived for two interacting solitons of the KP equation two wave interactions modeled by two solitons solution.

4.5 Future Research

For future research, we are looking into telecommunication sector, the nano–technology and also optical solitons. In 1973 Hasegawa and Tappert had proposed that soliton pulses could be used in optical communications through the balance of non-linearity and dispersion. They showed that these solitons would propagate according to the nonlinear Schrodinger equation (NLS), which had been solved by the inverse scattering method a year earlier by Zakharov and Shabat. At that time there was no capability to produce the fibers with the proper characteristics for doing this and the dispersive properties of optical fibers were not known. Also, the system required a laser which could produce very small wavelengths, which also was unavailable. It wasn't until seven years later, when Mollenauer, Stollen and Gordon at AT&T Bell Laboratories had experimentally demonstrated the propagation of solitons in optical fibers. The original communications systems employed pulse trains with widths of about one nanosecond. However, there was still some distortion due to fiber loss. This was corrected by placing repeaters every several of tens of kilometers. As the width of the available pulses was decreased, the spacing of the repeaters was increased. In the mid 1980's it was proposed that by sending in an additional pump wave along the fiber, the dispersion of a soliton could be halted through a process known as Raman scattering. In 1988 Mollenauer and his group had shown that this could be done by propagating a soliton over 6000 km without the need for repeaters.

Research into the use of solitons as information carriers in optical systems is still being heavily researched. Soliton pulses are not immune to fiber losses, leading to pulse broadening. As the pulses broaden, neighboring solitons will overlap and this overlap is not fully understood. Systems, which allow the propagation of envelope solitons, operate in the region of negative group velocity dispersion (GVD). However, for positive GVD it has been recently found that a different type of soliton, the so-called dark soliton, can propagate through optical fibers. The characteristics of these fibers is currently being investigated. These are just a few of the directions that theoretical and experimental investigations have taken over the last decade in this rapidly growing field. It is clear that solitons will play a major role in the next generation of optical communication systems.

Solitons have been studied in many fields of science; however, the most promising applications of soliton theory are in the field of optical communications. In such systems information is encoded into light pulses and transmitted through optical fibers over large distances. Optical communication systems are the pipe dreams of the future. Commercial systems have been in operation since 1977. Recently, a transatlantic undersea optical cable has been developed, which is expected to transmit around 40,000 telephone conversations simultaneously. The development of optical fibers, which are the basis of such systems, has lead to a revolution in communications technology.

Future commercial applications could include adding side branches similar to those used by the Japanese team to exhaust pipes and other machinery, which might tame the emitted noise by transforming the loud shock waves into milder solitary waves. And because the solitary waves can transport energy steadily over a long distance, the Japanese are exploring potential applications in acoustic compressors, heat engines, and even heat pumps. Other applications await conceptualization.

Maybe solitons will even find application in the exhaust system of motorcycles, which would decrease environmental noise pollution.

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