9

THE NONABELIAN TENSOR SQUARES OF ONE FAMILY OF BIEBERBACH GROUPS WITH POINT GROUP C2

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The nonabelian tensor square, $G \otimes G$, of a group G is generated by the symbols $g \otimes h$, where $g, h \in G$ subject to the relations $gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h)$ and $g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$ for all $g, g', h, h' \in G$, where ${}^{g}g' = gg'g^{-1}$ is the conjugation on the left. The nonabelian tensor square is a special case of the nonabelian tensor product which has its origins in homotopy theory. The Bieberbach groups are extensions of a point group and a free abelian group of finite rank. The rank of the free abelian group is the dimension of a Bieberbach group. In this study, we will compute the nonabelian tensor square of one family of Bieberbach groups with cyclic point group of order 2 and dimension 3 or, denoted by $B_2(3)$. This group is polycyclic since it is an extension of polycyclic groups. The nonabelian tensor squares are obtained using computational method developed by R. Blyth and R. F. Morse in 2008 for polycyclic groups.

9.1 OVERVIEW

An important algebraic invariant of flat manifold is its fundamental groups. These groups are known as the Bieberbach groups and turn out to have many interesting algebraic properties. The Bieberbach groups are torsion free crystallographic groups. These groups G are extension of a free abelian group L of finite rank by a group P. Thus, there is a short exact sequence shown as

$$1 \longrightarrow L \xrightarrow{\psi} G \longrightarrow P \longrightarrow 1.$$

such that $G/\psi(L) \cong P$. The group L is called the lattice group and P is a point group, also known as a holonomy group. The dimension of G is called the rank of L.

The nonabelian tensor square is a specialisation of the more general nonabelian tensor square product introduced by Brown, Johnson & Robertson [1]. This group's construction has its roots in algebraic K-theory and topology extending ideas of Whitehead [2]. The nonabelian tensor square appears independently in Dennis's work [3] on K-theory and is based on the ideas of Miller [4]. Brown, Johnson & Robertson [1] consider the nonabelian tensor square from a group theoretic point of view and they compute the nonabelian tensor squares of all nonabelian groups up to order 30. By computing the nonabelian tensor square we mean finding a standard or simplified presentation for the nonabelian tensor square for these groups. Many papers have appeared in computing the nonabelian tensor squares for various groups and classes of groups since the publication of Brown, Johnson and Robertson seminal's work. This include 2-generator nilpotent of class 2 groups in Kappe & Bacon [5], and Kappe, Sarmin & Visscher [6], metacyclic groups in Kappe & Beuerle [7] and free nilpotent groups in Blyth, Moravec & Morse [8]. The main goal of this paper is to calculate the nonabelian tensor squares of the Bieberbach group, given by

$$B_2(3) = \left\langle a, l_1, l_2, l_3 \mid a^2 = l_3, {}^{a}l_1 = l_2, {}^{a}l_2 = l_1, [a, l_3] = 1, [l_i, l_j] = 1 \text{ for } i, j = 1, 2, 3 \right\rangle.$$

Our method for computation will use the techniques developed by Blyth & Morse [9]. The following lemma identifies the structure of the group $B_2(3)$. The proof of the lemma is given in Section 9.3.

Lemma 9.1. *The groups* $B_2(3)$ *is a Bieberbach group of dimension 2 with point group* C_2*, where* C_n *the cyclic group of order n.*

By Lemma 1 we see that $B_2(3)$ is an extension of a finitely generated abelian group by a finite cyclic group. Any finitely generated abelian group is polycyclic. Since the polycyclic's property is closed under extensions, we have $B_2(3)$ is polycyclic. Our method for computing the nonabelian tensor squares of $B_2(3)$ will rely on the fact that this group is polycyclic. The following lemma [10] provides a method for creating a family of Bieberbach groups from a given Bieberbach group *B* that have the same point group.

Lemma 9.2. *Let G be any Bieberbach group of dimension n with point group P and lattice group L. Let* $B = G \times F_m{}^{ab}$ *where* $F_m{}^{ab}$ *is a free abelian group of rank m. Then B is a Bieberbach group of dimension* $n+m$ *with point group P.*

Lemmas 1 and 2 allow us to define the following family of Bieberbach groups.

Definition 9.1. *The group,*

$$B_2(n) = B_2(3) \times F_{n-3}{}^{ab}, \quad \textit{for } n \geq 3$$

is a Bieberbach group with point group C_2 *of dimension n, where* ${F_m}^{ab}$ *is a free abelian group of rank m.*

The first main result of the paper is the computation of the nonabelian tensor square of $B_2(3)$.

Theorem 9.1. *The nonabelian tensor square* $B_2(3)\otimes B_2(3)$ *is abelian and isomorphic to* $C_2\times C_0^{\,5}$*,*
where C_0 *is the infinite cyclic group.*

Computing the nonabelian tensor square of $B_2(n)$ follows from Theorem 9.1 and the following lemma found in Brown, Johnson & Robertson [1].

Lemma 9.3. *Let G be any group such that* $G = A\times B$*. Then*

$$G\otimes G = (A\times B)\otimes(A\times B) = (A\otimes A)\times(A^{ab}\otimes B^{ab})\times(B^{ab}\otimes A^{ab})\times(B\otimes B)$$

where $A^{ab} = A/A'$ *and* $B^{ab} = B/B'$ *are the abelianisations of A and B respectively.*

We note that the nonabelian tensor square of two abelian groups is equivalent to the ordinary tensor square for abelian groups. By our construction of $B_2(n)$, Lemma 9.3 reduces the computation of the nonabelian tensor square of these groups to computing the nonabelian tensor square of the group $B_2(3)$ and the ordinary tensor products of abelian groups.

Corollary 9.1. *For the Bieberbach group* $B_2(n)$*,*

$$B_2(n)\otimes B_2(n)\cong C_2\times C_0^{\,(n-1)^2+1},\ \textit{for}\ n\geq 3.$$

Section 9.2 outlines the techniques developed by Blyth & Morse [9] for computing the nonabelian tensor squares of polycyclic groups. In Section 9.3 we give structural results for $B_2(3)$, and the proofs of Theorem 9.1 and Corollary 9.1.

9.2 PRELIMINARIES

Our approach to computing the nonabelian tensor square involves the group $\nu(G)$ as defined below. This group was independently investigated by Rocco [11] and Ellis & Leonard [12].

Definition 9.2 [11]**.** *Let G be a group with presentation* $\langle \mathcal{G} | \mathcal{R} \rangle$ *and let* G^{φ} *be an isomorphic copy of G via the mapping* $\varphi : g \to g^{\varphi}$ *for all* $g \in G$. *The group* $\nu(G)$ *is defined as*

$$\nu(G) = \left\langle G, G^{\varphi} \;\middle|\; {}^{k}[g, h^{\varphi}] = [{}^{k}g, ({}^{k}h)^{\varphi}] = {}^{k\varphi}[g, h^{\varphi}],\ \forall g, h, k \in G \right\rangle .$$

The importance of $\nu(G)$ is that $G \otimes G$ is isomorphic to a subgroup of $\nu(G)$.

Theorem 9.2 ([11],[12]). *Let G be a group. Then the map* $\sigma : G \otimes G \to [G, G^{\varphi}] \lhd \nu(G)$ *defined by* $\sigma(g \otimes h) = [g, h^{\varphi}]$ *for all* $g, h \in G$ *is an isomorphism.*

Since there is an isomorphism from $G \otimes G$ to $[G, G^{\varphi}]$, then all tensor computations of a group G can be translated into commutator computations within the subgroup $[G, G^{\varphi}]$ of $\nu(G)$. A commutator calculus that explicitly translates commutator computations in $\nu(G)$ into tensor computations was initiated by Rocco [11] and have been extended further by Blyth & Morse [9] and Blyth, Moravec & Morse [8]. This commutator calculus allows us to make a computations of $G \otimes G$ within the subgroup $[G, G^{\varphi}]$.

Since all conjugations and commutators in this paper are done using conjugation from the left, a few basic commutator identities are given for the convenience of the reader. Let G be any group and x, y and z be elements of G. The conjugation of y by x is ${}^{x}y = xyx^{-1}$. The commutator of x and y is $[x, y] = xyx^{-1}y^{-1}$ and the commutators are left normed, i.e $[x, y, z] = [[x, y], z]$. There are list some of the commutator identities as below:

$$
\begin{aligned}
&{}^{x}y = [x, y]y;\\
&{}^{z}[x, y] = [{}^{z}x, {}^{z}y];\\
&[xy, z] = {}^{x}[y, z].[x, z];\\
&[x, yz] = [x, y].{}^{y}[x, z];\\
&[x^{-1}, y] = {}^{x^{-1}}[x, y]^{-1} = [x^{-1}, [x, y]^{-1}].[x, y]^{-1};\\
&[x, y^{-1}] = {}^{y^{-1}}[x, y]^{-1} = [y^{-1}, [x, y]^{-1}].[x, y]^{-1};\\
&[x^{-1}, y^{-1}] = [x^{-1}, [y^{-1}, [x, y]]].[y^{-1}, [x, y]].[x^{-1}, [x, y]].[x, y].
\end{aligned}
$$

The following lemmas record some basic identities used in this paper.

Lemma 9.4 ([8],[9]). *Let G be a group. The following relations hold in* $\nu(G)$:

(i) ${}^{[g_3, g_4^{\varphi}]}[g_1, g_2^{\varphi}] = {}^{[g_3, g_4]}[g_1, g_2^{\varphi}] = {}^{[g_3^{\varphi}, g_4]}[g_1, g_2^{\varphi}]$ *for all* g_1, g_2, g_3, g_4 *in G;*

(ii) $[g_1^{\varphi}, g_2, g_3] = [g_1, g_2, g_3^{\varphi}] = [g_1^{\varphi}, g_2, g_3^{\varphi}] = [g_1, g_2^{\varphi}, g_3] = [g_1^{\varphi}, g_2^{\varphi}, g_3] = [g_1, g_2^{\varphi}, g_3^{\varphi}]$ *for all* g_1, g_2, g_3 *in G;*

(iii) $[g_1, [g_2, g_3^{\varphi}]] = [g_2, g_3, g_1^{\varphi}]^{-1}$;

(iv) $[g, g^{\varphi}]$ *is central in* $\nu(G)$ *for all g in G;*

(v) $[g_1, g_2^{\varphi}][g_2, g_1^{\varphi}]$ *is central in* $\nu(G)$ *for all* g_1, g_2 *in G;*

(vi) $[g, g^{\varphi}] = 1$ *for all g in* G'.

The following identities can be found in [9] and [10].

Lemma 9.5. *Let g and h be elements of G such that* $[g,h]=1$. *Then in* $\nu(G)$,

(i) $[g^n, h^\varphi] = [g, h^\varphi]^n = [g, (h^\varphi)^n]$ *for all integers n;*

(ii) $[g^n, (h^m)^\varphi][h^m, (g^n)^\varphi] = ([g, h^\varphi][h, g^\varphi])^{nm}$;

(iii) $[g, h^\varphi]$ *is in the center of* $\nu(G)$.

The following three lemmas will be used in the sequel.

Lemma 9.6 [10]. *Let G and H be groups and let* $g \in G$. *Suppose* ϕ *is a homomorphism from G to H. If* $\phi(g)$ *has finite order then* $|\phi(g)|$ *divides* $|g|$. *Otherwise, the order of* $\phi(g)$ *equals the order of g.*

Lemma 9.7 [13]. *Let A, B and C be any abelian group. Consider the ordinary tensor product of two abelian groups. Then,*

(i) $C_0 \otimes A \cong A$,

(ii) $C_0 \otimes C_0 \cong C_0$,

(iii) $C_n \otimes C_m \cong C_{\gcd(n,m)}$ *, for n,m* $\in Z$ *, and*

(iv) $A \otimes (B \times C) = (A \otimes B) \times (A \otimes C)$.

Lemma 9.8 [10]. *Let A be a finitely generated abelian group and a is a basis element in A. If a has infinite order then* $a \otimes a$ *in* $A \otimes A$ *has infinite order.*

Blyth & Morse [9] have shown that if *G* is polycyclic, then $G \otimes G$ and $\nu(G)$ are polycyclics. Hence, for finite and infinite polycyclic groups both $G \otimes G$ and $\nu(G)$ are finitely presented. The following proposition gives us a finite generating set for $[G, G^\varphi]$ in terms of a polycyclic generating set of *G*, independent of the polycyclic presentation of $\nu(G)$.

Proposition 9.1 [9]. *Let G be a polycyclic group with a polycyclic generating sequence* $g_1, \ldots, g_k$ *. Then* $[G, G^{\varphi}]$ *a subgroup of* $\nu(G)$, is generated by

$[G,G^{\varphi}] = \left\langle\ [g_i, g_i^{\varphi}],\ [g_i^{\varepsilon}, (g_j^{\varphi})^{\delta}],\ [g_i, (g_j^{\varphi})][g_j, (g_i^{\varphi})] \right\rangle$ *for* $1 \le i, j \le k,\ \ i < j$,

where

$$\varepsilon = \begin{cases} 1 & \text{if } |g_i| < \infty; \\ \pm 1 & \text{if } |g_i| = \infty \end{cases} \quad \text{and} \quad \delta = \begin{cases} 1 & \text{if } |g_i| < \infty; \\ \pm 1 & \text{if } |g_i| = \infty. \end{cases}$$

We conclude this section with three statements about the structural results of the nonabelian tensor square.

Lemma 9.9 [10]. *Let G be any group such that* G' *is cyclic, then* $G \otimes G$ *is abelian.*

The following two theorems are from Brown, Johnson and Robertson [1].

Theorem 9.3. *Let G be a group. Then there exists a commutator mapping*

$$\kappa : G \otimes G \to G'$$

defined by $\kappa(g \otimes h) = [g, h]$. *The kernel of* κ *is in the center of* $G \otimes G$.

Theorem 9.4. *Let G and H be groups such that there is an epimorphism* $\varepsilon : G \to H$ *. Then there exists an epimorphism*

$$\alpha : G \otimes G \to H \otimes H$$

defined by $\alpha(g \otimes h) = \varepsilon(g) \otimes \varepsilon(h)$.

In the next section, all lemmas, proposition and theorems in this section will be applied in computing the nonabelian tensor squares of groups $B_2(3)$ and $B_2(n)$.

9.3 THE NONABELIAN TENSOR SQUARES OF $B_2(3)$ and $B_2(n)$

In this section we compute the nonabelian tensor squares of $B_2(3)$ and $B_2(n)$. We start the investigation of the structure of $B_2(3)$ by proving Lemma 1 and showing the generators in the definition of $B_2(3)$ form a polycyclic generating sequence. This information will allow us to apply Proposition 1 in computing the tensor square of this group.

Proof of Lemma 9.1. Consider the subgroup $L = \langle l_1, l_2, l_3 \rangle$ of $B_2(3)$. Since l_1, l_2 and l_3 commute to each other then L is an abelian group of rank 3. We can see that for any element l_i in L and x in $B_2(3)$, $xl_ix^{-1} \in L$. Hence L is a normal subgroup in $B_2(3)$. All elements of $B_2(3)$ have infinite orders, so that all elements in L are elements of infinite order. It follows that L is a free abelian of rank 3. Since $B_2(3)/L = \langle a, l_1, l_2, l_3 \rangle \big/ \langle l_1, l_2, l_3 \rangle \cong \langle a \rangle \cong C_2$, then $B_2(3)$ is an extension of group L by C_2. We conclude that $B_2(3)$ is a Bieberbach group with point group C_2 and dimension 3.

The following lemma is an immediate consequence of the proof of Lemma 9.1.

Lemma 9.10. *The generators given in the definition of* $B_2(3)$ *form a polycyclic generating sequence.*

Proof. The normal subgroup $L = \langle l_1, l_2, l_3 \rangle$ from the proof of Lemma 9.1 is polycyclic and the factor $B_2(3)\big/L$ is cyclic of order 2. Hence a, l_1, l_2 and l_3 form a polycyclic generating sequence.

The following lemma will be used to prove the nonabelian tensor square of $B_2(3)$ is abelian.

Lemma 9.11. *The groups* $B_2(3)$ *has cyclic derived subgroup and its abelianisation* $B_2(3)/B_2(3)' \cong C_0 \times C_0$ *is generated by the cosets* $l_2B_2(3)'$ *and* $l_3B_2(3)'$ *of order infinity, respectively.*

Proof. From the relations of group $B_2(3)$, since ${}^a l_1 = l_2$ and ${}^a l_2 = l_1$, we have $al_1a^{-1}l_1^{-1} = {}^a l_1 l_1^{-1} = l_2 l_1^{-1}$
$\neq 1$ and $al_2a^{-1}l_2^{-1} = {}^a l_2 l_2^{-1} = l_1 l_2^{-1} \neq 1$. It follows that $[a, l_1]$ and $[a, l_2]$ are the only nontrivial elements in $B_2(3)'$ and therefore $\langle l_2 l_1^{-1} \rangle = B_2(3)'$.

The factor group $B_2(3)/B_2(3)'$ is generated by the cosets $aB_2(3)'$, $l_1B_2(3)'$, $l_2B_2(3)'$ and $l_3B_2(3)'$. However $aB_2(3)' \cap l_3B_2(3)'$ and $l_1B_2(3)' \cap l_2B_2(3)'$ are not trivial. Hence $aB_2(3)' = l_3B_2(3)'$ and $l_1B_2(3)' = l_2B_2(3)'$. It follows that $l_2B_2(3)'$ and $l_3B_2(3)'$ of infinite order respectively. Therefore, $B_2(3)/B_2(3)' = \langle l_2B_2(3)', l_3B_2(3)' \rangle \cong C_0 \times C_0$ as needed.

To complete the proof of Theorem 9.1, we prove a series of lemmas specific to $\nu(B_2(3))$.

Lemma 9.12. *The following identities hold in* $\nu(B_2(3))$:

(i) $[l_1, l_2^{\varphi}] = [l_2, l_1^{\varphi}]$; *and*

(ii) $[l_1, l_1^{\varphi}] = [l_2, l_2^{\varphi}]$.

Proof. From the relation of group $B_2(3)$, ${}^a l_1 = l_2$ and ${}^a l_2 = l_1$. Then, from equation (9.2) and Lemma 5(iii) we obtain

$$[l_1, l_2^{\varphi}] = [{}^a l_2, {}^a l_1^{\varphi}] = {}^a[l_2, l_1^{\varphi}] = [l_2, l_1^{\varphi}].$$

Also, $[l_1, l_1^{\varphi}] = [{}^a l_2, {}^a l_2^{\varphi}] = {}^a[l_2, l_2^{\varphi}] = [l_2, l_2^{\varphi}]$.

Lemma 9.13. *The elements* $[a^{-1},l_1^{\varphi}]$, $[a^{-1},l_1^{-\varphi}]$, $[a,l_1^{-\varphi}]$, $[a^{-1},l_2^{\varphi}]$, $[a^{-1},l_2^{-\varphi}]$, *and* $[a,l_2^{-\varphi}]$ *of* $\nu(B_2(3))$ *can be written as* $[a^{-1},l_1^{\varphi}]=[a,l_2^{\varphi}]^{-1}$,

$[a,l_1^{-\varphi}]=[l_1,l_2^{\varphi}][l_1,l_1^{\varphi}]^{-1}[a,l_1^{\varphi}]^{-1}$, $[a^{-1},l_2^{\varphi}]=[a,l_1^{\varphi}]^{-1}$,

$[a,l_2^{-\varphi}]=[l_1,l_2^{\varphi}][l_2,l_2^{\varphi}]^{-1}[a,l_2^{\varphi}]^{-1}$, $[a^{-1},l_1^{-\varphi}]=[l_1,l_2^{\varphi}]^{-1}[l_2,l_2^{\varphi}][a,l_2^{\varphi}]$,

and

$[a^{-1},l_2^{-\varphi}]=[l_1,l_2^{\varphi}]^{-1}[l_1,l_1^{\varphi}][a,l_1^{\varphi}]$.

Proof. Using the relations of $B_2(3)$, we have,

$${}^{a^{-1}}l_1=a^{-1}l_1a=a^{-1}al_2a^{-1}a=l_2, \text{ and}$$
$${}^{a^{-1}}l_2=a^{-1}l_2a=a^{-1}al_1a^{-1}a=l_1.$$

Using these two equations we obtain,

$$\begin{aligned}
[a^{-1},l_1^{\varphi}] &= {}^{a^{-1}}[a,l_1^{\varphi}]^{-1} && \text{(from 9.5)}\\
&= [a,({}^{a^{-1}}l_1)^{\varphi}]^{-1} && \text{(from 9.2)}\\
&= [a,l_2^{\varphi}]^{-1} && \text{(from 9.8);}\\
[a,l_1^{-\varphi}] &= {}^{l_1^{-1}}[a,l_1^{\varphi}]^{-1} && \text{(from 9.6)}\\
&= [l_1^{-1},[a,l_1^{\varphi}]^{-1}][a,l_1^{\varphi}]^{-1} && \text{since } {}^{a}b=[a,b]b\\
&= [l_1^{-1},[a,l_1]^{-\varphi}][a,l_1^{\varphi}]^{-1}\\
&= [l_1^{-1},(l_2l_1^{-1})^{\varphi}][a,l_1^{\varphi}]^{-1}\\
&= [l_1^{-1},l_2^{-\varphi}]\,{}^{l_2^{-1}}[l_1^{-1},l_1^{\varphi}]\,[a,l_1^{\varphi}]^{-1} && \text{(from 9.4)}\\
&= [l_1,l_2^{\varphi}]\,[l_1,l_1^{\varphi}]^{-1}\,[a,l_1^{\varphi}]^{-1};\\
[a^{-1},l_1^{-\varphi}] &= {}^{a^{-1}}[a,l_1^{-\varphi}]^{-1} && \text{(from 9.5)}\\
&= [a,({}^{a^{-1}}l_1)^{-\varphi}]^{-1} && \text{(from 9.2)}\\
&= [a,l_2^{-\varphi}]^{-1} && \text{(from 9.8);}\\
[a^{-1},l_2^{\varphi}] &= {}^{a^{-1}}[a,l_2^{\varphi}]^{-1} && \text{(from 9.5)}\\
&= [a,({}^{a^{-1}}l_2)^{\varphi}]^{-1} && \text{(from 9.2)}\\
&= [a,l_1^{\varphi}]^{-1} && \text{(from 9.9);}\\
[a,l_2^{-\varphi}] &= {}^{l_2^{-1}}[a,l_2^{\varphi}]^{-1} && \text{(from 9.6)}\\
&= [l_2^{-1},[a,l_2^{\varphi}]^{-1}][a,l_2^{\varphi}]^{-1} && \text{since } {}^{a}b=[a,b]b
\end{aligned}$$

$$= [l_2^{-1},(l_2 l_1^{-1})^{\varphi}][a,l_2^{\varphi}]^{-1}$$
$$= [l_2^{-1},l_2^{\varphi}]\ {}^{l_2}[l_2,l_1^{\varphi}]\ [a,l_2^{\varphi}]^{-1} \qquad \text{(from 9.4)}$$
$$= [l_1,l_2^{\varphi}]\ [l_2,l_2^{\varphi}]^{-1}\ [a,l_2^{\varphi}]^{-1};$$
$$[a^{-1},l_2^{-\varphi}] = {}^{a^{-1}}[a,l_2^{-\varphi}]^{-1} \qquad \text{(from 9.5)}$$
$$= [a,({}^{a^{-1}}l_2)^{-\varphi}]^{-1} \qquad \text{(from 9.2)}$$
$$= [a,l_1^{-\varphi}]^{-1} \qquad \text{(from 9.9)}$$
$$= [l_1,l_2^{\varphi}]^{-1}\ [l_1,l_1^{\varphi}]\ [a,l_1^{\varphi}]; \quad \text{and finally,}$$
$$[a^{-1},l_1^{-\varphi}] = [a,l_2^{-\varphi}]^{-1} = [l_1,l_2^{\varphi}]^{-1}\ [l_2,l_2^{\varphi}]\ [a,l_2^{\varphi}].$$

Lemma 9.14. *The nonabelian tensor square of* $B_2(3)$ *is generated by the elements* $[a,a^{\varphi}]$, $[l_1,l_1^{\varphi}]$, $[l_1,l_2^{\varphi}]$, $[a,l_1^{\varphi}]$, $[a,l_2^{\varphi}]$, $[a,l_1^{\varphi}][l_1,a^{\varphi}]$ *and* $[a,l_2^{\varphi}][l_2,a^{\varphi}]$.

Proof. From Lemma 9.10, the group $B_2(3)$ is a polycyclic group with polycyclic generating sequence $\langle a,l_1,l_2,l_3\rangle$. Therefore we can apply Proposition 9.1 and obtain that $B_2(3)\otimes B_2(3)$ is generated by

$$\begin{aligned}
&[a,a^{\varphi}],\ [l_1,l_1^{\varphi}],\ [l_2,l_2^{\varphi}],\\
&[l_3,l_3^{\varphi}],\ [a,l_1^{\varphi}],\ [a^{-1},l_1^{\varphi}],\ [a,l_1^{-\varphi}],\ [a^{-1},l_1^{-\varphi}],\ [a,l_2^{\varphi}],\\
&[a^{-1},l_2^{\varphi}],\ [a,l_2^{-\varphi}],\ [a^{-1},l_2^{-\varphi}],\ [a,l_3^{\varphi}],\ [a^{-1},l_3^{\varphi}],\\
&[a,l_3^{-\varphi}],\ [a^{-1},l_3^{-\varphi}],\ [l_1,l_2^{\varphi}],\ [l_1^{-1},l_2^{\varphi}],\ [l_1,l_2^{-\varphi}],\\
&[l_1^{-1},l_2^{-\varphi}],\ [l_1,l_3^{\varphi}],\ [l_1^{-1},l_3^{\varphi}],\ [l_1,l_3^{-\varphi}],\ [l_1^{-1},l_3^{-\varphi}],\\
&[l_2,l_3^{\varphi}],\ [l_2^{-1},l_3^{\varphi}],\ [l_2,l_3^{-\varphi}],\ [l_2^{-1},l_3^{-\varphi}],\ [a,l_1^{\varphi}][l_1,a^{\varphi}],\\
&[a,l_2^{\varphi}][l_2,a^{\varphi}],\ [a,l_3^{\varphi}][l_3,a^{\varphi}],\ [l_1,l_2^{\varphi}][l_2,l_1^{\varphi}],\\
&[l_1,l_3^{\varphi}][l_3,l_1^{\varphi}] \text{ and } [l_2,l_3^{\varphi}][l_3,l_2^{\varphi}].
\end{aligned}$$

These generators need not be independent. We show that each of these generators can be expressed as products of powers of $[a,a^{\varphi}]$, $[l_1,l_1^{\varphi}]$, $[l_1,l_2^{\varphi}]$, $[a,l_1^{\varphi}]$, $[a,l_1^{\varphi}][l_1,a^{\varphi}]$, $[a,l_2^{\varphi}]$, and $[a,l_2^{\varphi}][l_2,a^{\varphi}]$. By Lemma 9.12 (i) and (ii),

$$[l_2,l_2^{\varphi}] = [l_1,l_1^{\varphi}], \text{ and } [l_1,l_2^{\varphi}][l_2,l_1^{\varphi}] = [l_1,l_2^{\varphi}]^2.$$

Since $a^2 = l_3$, we can substitute and using Lemma 9.5(i) to have as follows:

$[l_3, l_3^{\varphi}] = [a^2, a^{2\varphi}] = [a, a^{\varphi}]^4$;

$[a, l_3^{\varphi}] = [a, a^{2\varphi}] = [a, a^{\varphi}]^2$;

$[a^{-1}, l_3^{\varphi}] = [a^{-1}, a^{2\varphi}] = [a, a^{\varphi}]^{-2}$;

$[a, l_3^{-\varphi}] = [a, (a^{-2})^{\varphi}] = [a, a^{\varphi}]^{-2}$;

$[a^{-1}, l_3^{-\varphi}] = [a^{-1}, a^{-2\varphi}] = [a, a^{\varphi}]^2$; and

$[a, l_3^{\varphi}][l_3, a^{\varphi}] = [a, (a^2)^{\varphi}][a^2, a^{\varphi}] = [a, a^{\varphi}]^4$.

By Lemma 9.13, we have

$[a^{-1}, l_1^{\varphi}] = [a, l_2^{\varphi}]^{-1}$;

$[a, l_1^{-\varphi}] = [l_1, l_2^{\varphi}]\,[l_1, l_1^{\varphi}]^{-1}\,[a, l_1^{\varphi}]^{-1}$;

$[a^{-1}, l_1^{-\varphi}] = [l_1, l_2^{\varphi}]^{-1}\,[l_2, l_2^{\varphi}]\,[a, l_2^{\varphi}]$;

$[a^{-1}, l_2^{\varphi}] = [a, l_1^{\varphi}]^{-1}$;

$[a, l_2^{-\varphi}] = [l_1, l_2^{\varphi}]\,[l_2, l_2^{\varphi}]^{-1}\,[a, l_2^{\varphi}]^{-1}$;

$[a^{-1}, l_2^{-\varphi}] = [l_1, l_2^{\varphi}]^{-1}\,[l_1, l_1^{\varphi}]\,[a, l_1^{\varphi}]$.

Since l_1, l_2 and l_3 commute in $B_2(3)$, by Lemma 9.5 (i)

$[l_1^{-1}, l_2^{\varphi}] = [l_1, l_2^{\varphi}]^{-1}$, $[l_1, l_2^{-\varphi}] = [l_1, l_2^{\varphi}]^{-1}$,

$[l_1^{-1}, l_2^{-\varphi}] = [l_1, l_2^{\varphi}]$, $[l_1^{-1}, l_3^{\varphi}] = [l_1, l_3^{\varphi}]^{-1}$,

$[l_1, l_3^{-\varphi}] = [l_1, l_3^{\varphi}]^{-1}$, $[l_1^{-1}, l_3^{-\varphi}] = [l_1, l_3^{\varphi}]$,

$[l_2^{-1}, l_3^{\varphi}] = [l_2, l_3^{\varphi}]^{-1}$, $[l_2, l_3^{-\varphi}] = [l_2, l_3^{\varphi}]^{-1}$,

$[l_2^{-1}, l_3^{-\varphi}] = [l_2, l_3^{\varphi}]$.

Next,

$[l_1, l_3^{\varphi}] = [l_1, a^{2\varphi}] = [l_1, a^{\varphi}]^{a}[l_1, a^{\varphi}] = [l_1, a^{\varphi}][l_2, a^{\varphi}]$ and

$[l_2, l_3^{\varphi}] = [l_2, a^{2\varphi}] = [l_2, a^{\varphi}]^{a}[l_2, a^{\varphi}] = [l_2, a^{\varphi}][l_1, a^{\varphi}]$.

Note that $[l_1, a^{\varphi}]$ and $[l_2, a^{\varphi}]$ can be written as $[a, l_1^{\varphi}]^{-1}[a, l_1^{\varphi}][l_1, a^{\varphi}]$ and $[a, l_2^{\varphi}]^{-1}[a, l_2^{\varphi}][l_2, a^{\varphi}]$. Using Lemma 9.5 (ii) and substituting $a^2 = l_3$ we obtain

$$
\begin{aligned}
([a,l_1^{\varphi}][l_1,a^{\varphi}])^2 &= [a^2,l_1^{\varphi}][l_1,a^{2\varphi}] \\
&= [l_3,l_1^{\varphi}][l_1,l_3^{\varphi}] \\
&= [l_1,l_3^{\varphi}]^{-1}[l_1,l_3^{\varphi}][l_3,l_1^{\varphi}][l_1,l_3^{\varphi}] \\
&= [l_1,l_3^{\varphi}][l_3,l_1^{\varphi}], \quad \text{and} \\
([a,l_2^{\varphi}][l_2,a^{\varphi}])^2 &= [a^2,l_2^{\varphi}][l_2,a^{2\varphi}] \\
&= [l_3,l_2^{\varphi}][l_2,l_3^{\varphi}] \\
&= [l_2,l_3^{\varphi}]^{-1}[l_2,l_3^{\varphi}][l_3,l_2^{\varphi}][l_2,l_3^{\varphi}] \\
&= [l_2,l_3^{\varphi}][l_3,l_2^{\varphi}].
\end{aligned}
$$

Therefore, $[l_1,l_3^{\varphi}]$, $[l_1,l_3^{\varphi}][l_3,l_1^{\varphi}]$ and $[l_2,l_3^{\varphi}][l_3,l_2^{\varphi}]$ can be expressed in terms of the generating elements proposed. Then, all the generators of $B_2(3)\otimes B_2(3)$ given by Proposition 9.1 can be expressed as products and powers of the generating set given in the Lemma and the proof is complete.

Proof of Theorem 9.1. By Lemma 9.11 the derived subgroup of $B_2(3)$ is cyclic. Therefore the nonabelian tensor square of $B_2(3)$ is abelian by Lemma 9.9. We first compute the orders of each of the seven generators of $B_2(3)\otimes B_2(3)$ identified in Lemma 9.14. The derived subgroup of $B_2(3)$ is generated by $l_2l_1^{-1}$. Then,

$$
\begin{aligned}
&([l_1,l_2^{\varphi}]^{-1}[l_1,l_1^{\varphi}])^2 \\
&=[l_1,l_2^{\varphi}]^{-2}[l_1,l_1^{\varphi}]^2 \\
&=[l_1,l_2^{\varphi}]^{-1}\,[l_1,l_2^{\varphi}]^{-1}\,[l_1,l_1^{\varphi}]\,[l_1,l_1^{\varphi}] \\
&=[l_1,l_1^{\varphi}]\,[l_1,l_1^{\varphi}]\,[l_1,l_2^{\varphi}]^{-1}\,[l_1,l_2^{\varphi}]^{-1} && \text{(by Lemma 9.4(iv))} \\
&=[l_2,l_2^{\varphi}]\,[l_1,l_1^{\varphi}]\,[l_1^{-1},l_2^{\varphi}]\,[l_2,l_1^{-\varphi}] && \text{(by Lemma 9.5 \& 9.12)} \\
&=[l_2l_1^{-1},(l_2l_1^{-1})^{\varphi}] && \text{(by (9.3) \& (9.4))} \\
&= 1
\end{aligned}
$$

Hence, $[l_1,l_2^{\varphi}]^{-1}[l_1,l_1^{\varphi}]$ has order dividing 2.

The mapping κ from $B_2(3)\otimes B_2(3)$ to $B_2(3)'$ defined in Theorem 9.3 gives $k([a,l_1^{\varphi}])=[a,l_1]$ and $k([a,l_2^{\varphi}])=[a,l_2]$. Since

$[a,l_1]=l_2l_1^{-1}\neq 1$ and $[a,l_2]=l_1l_2^{-1}\neq 1$ in $B_2(3)$ has infinite order it follows from Lemma 9.6 that $[a,l_1^{\varphi}]$ and $[a,l_2^{\varphi}]$ have infinite orders. We define the abelianisation of $B_2(3)$ by $B_2(3)^{ab}$ with natural homomorphism

$$\varepsilon: B_2(3)\to B_2(3)^{ab}.$$

By Theorem 9.4, there is an epimorphism

$$\alpha: B_2(3)\otimes B_2(3)\to B_2(3)^{ab}\otimes B_2(3)^{ab}$$
$$\cong (C_0\times C_0)\otimes(C_0\times C_0).$$

By Lemma 9.7, we have

$$(C_0\times C_0)\otimes(C_0\times C_0)\cong C_0\times C_0\times C_0\times C_0.$$

By Lemma 9.11 the group $B_1(2)^{ab}$ is generated by $\varepsilon(l_1)$ and $\varepsilon(l_2)$ of order 2 and infinity respectively. Lemma 9.7 gives

$$\langle\varepsilon(l_2)\otimes\varepsilon(l_2)\rangle\cong C_0$$
$$\langle\varepsilon(l_2)\otimes\varepsilon(l_3)\rangle\cong C_0$$
$$\langle\varepsilon(l_3)\otimes\varepsilon(l_2)\rangle\cong C_0$$
$$\langle\varepsilon(l_3)\otimes\varepsilon(l_3)\rangle\cong C_0.$$

Therefore the image $\alpha(l_3\otimes l_3)=\varepsilon(l_3)\otimes\varepsilon(l_3)$ has infinite order. Hence by Lemma 9.6, $l_3\otimes l_3$ has infinite order. However,

$$[l_2,l_2^{\varphi}]=[a^2,(a^2)^{\varphi}]=[a,a^{\varphi}]^4.$$

We conclude that $[a,a^{\varphi}]$ has infinite order as needed. Also, if $B_2(3)^{ab}$ is generated by $\varepsilon(a)$ and $\varepsilon(l_1)$,

$$\langle\varepsilon(a)\otimes\varepsilon(a)\rangle\cong C_0$$

$$\langle \varepsilon(a)\otimes\varepsilon(l_1)\rangle \cong C_0$$
$$\langle \varepsilon(l_1)\otimes\varepsilon(a)\rangle \cong C_0$$
$$\langle \varepsilon(l_1)\otimes\varepsilon(l_1)\rangle \cong C_0 .$$

Since the image of $\alpha(l_1\otimes a)=\varepsilon(l_1)\otimes\varepsilon(a)$ has infinite order, hence by Lemma 9.6, $l_1\otimes a$ has infinite order. Then $[l_1,a^{\varphi}]$ has infinite order. With similar argument, if $B_2(3)^{ab}$ is generated by $\varepsilon(a)$ and $\varepsilon(l_2)$, this will give $[l_2,a^{\varphi}]$ has infinite order. However, $[a,l_1^{\varphi}][l_1,a^{\varphi}]$ and $[a,l_2^{\varphi}][l_2,a^{\varphi}]$ generate the same elements as $[l_1,a^{\varphi}]$ and $[l_2,a^{\varphi}]$. Therefore, $[a,l_1^{\varphi}][l_1,a^{\varphi}]$ and $[a,l_2^{\varphi}][l_2,a^{\varphi}]$ have infinite orders.

We have shown that $[l_1,l_2^{\varphi}]^{-1}[l_1,l_1^{\varphi}]$ has order 2 and the orders of $[a,a^{\varphi}]$, $[a,l_1^{\varphi}]$, $[a,l_2^{\varphi}]$, $[a,l_1^{\varphi}][l_1,a^{\varphi}]$ and $[a,l_2^{\varphi}][l_2,a^{\varphi}]$ are all infinite. Therefore we have

$$B_2(3)\otimes B_2(3)\cong C_2\times C_0^{\;5},$$

as needed.

The proof of Corollary 9.1 is an application of Theorem 9.1, Lemma 9.3 and Lemma 9.7.

Proof of Corollary 9.1. Since the group $B_2(3)=B_2(3)\times F_{n-3}^{ab}$ for $n\geq 3$, by Lemma 9.3

$$B_2(n)\otimes B_2(n)=(B_2(3)\otimes B_2(3))\times(B_2(3)\times F_{n-3}^{ab})=(F_{n-3}^{ab}\otimes B_2(3))\times(F_{n-3}^{ab}\otimes F_{n-3}^{ab})$$

By Theorem 9.1, $B_2(3)\otimes B_2(3)\cong C_2\times C_0^{\;5}$. By Lemma 9.11 we have $B_2(3)^{ab}\cong C_0\times C_0$ and $F_{n-3}^{ab}\cong C_0^{n-3}$. Using Lemma 9.7, this leads to

$$\begin{aligned} B_2(3)\otimes F_{n-3}^{ab} &\cong (C_0\times C_0)\otimes C_0^{n-3} \\ &= (C_0\otimes C_0^{n-3})\times(C_0\otimes C_0^{n-3}) \\ &= C_0^{n-3}\otimes C_0^{n-3}. \end{aligned}$$

By symmetry, $F_{n-3}^{ab} \otimes B_2(3)^{ab} = C_0^{n-3} \times C_0^{n-3}$. Finally, $F_{n-3}^{ab} \otimes F_{n-3}^{ab} = C_0^{n-3} \times C_0^{n-3} = C_0^{(n-3)^2}$ by repeated use of Lemma 9.7. Therefore,

$$B_2(n) \otimes B_2(n) = C_2 \times C_0^5 \times C_0^{n-3} \times C_0^{n-3} \times C_0^{n-3} \times C_0^{n-3} \times C_0^{(n-3)^2} \quad .$$

Collecting terms, we get

$$\begin{aligned} B_2(n) \otimes B_2(n) &= C_2 \times C_0^{5+(n-3)+(n-3)+(n-3)+(n-3)+(n-3)^2} \\ &= C_2 \times C_0^{n^2-2n+2} \\ &= C_2 \times C_0^{(n-1)^2+1} \end{aligned}$$

and this completes the proof.

9.4 CONCLUSION

Any Bieberbach group with point group C_2 is a polycyclic group. In this paper, using computational methods developed by Blyth and Morse for polycyclic groups, we calculated the nonabelian tensor squares for one family of Bieberbach groups of order 2 and dimension 3, labeled by $B_2(3)$ with point group C_2 whose nonabelian tensor square is abelian. Here we generalized the nonabelian tensor square of this group to *n* dimension, i.e $B_2(n)$.

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