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The nonabelian tensor squares and homological functors for certain classes of groups

Nor Haniza Sarmin
Department of Mathematics, Faculty of Science,
Universiti Teknologi Malaysia, Skudai, Malaysia.
nhs@utm.my

Abstract

The nonabelian tensor squares and homological functors for many groups have been determined by many authors. The nonabelian tensor squares have been determined for all 2-Engel groups in 1997 by Bacon, Kappe and Morse. In 2003, Bacon and Kappe determined some homological functors for the non-abelian 2-generator p-groups of class 2. In this talk, the nonabelian tensor squares and homological functors for certain other classes of groups will be presented. Some of the results have been verified by Groups, Algorithms and Programming (GAP) software.

Keyword and phrases: Nonabelian tensor square, Homological functors, Nilpotent groups

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1 Introduction

The nonabelian tensor square $G \otimes G$ of a group G is a special case of the nonabelian tensor product $G \otimes H$ of two arbitrary groups G and H that was introduced by Brown and Loday in [3,4] and arises from applications of a generalized Van Kampen theorem in homotopy theory.

For all $g, h \in G$ let ${}^{g}h = ghg^{-1}$ and $[g, h] = ghg^{-1}h^{-1}$. Then $G \otimes G$ is defined as the group generated by the symbols $g \otimes h$, for $g, h \in G$, subject to the relations;

$$gh \otimes k = ({}^{g}h \otimes {}^{g}k)(g \otimes k) \text{ and } g \otimes hk = (g \otimes h)({}^{h}g \otimes {}^{h}k),$$

for all $g, h, k \in G$.

The definition guarantees the existance of an epimorphism $\kappa : G \otimes G \to G'$, defined on the generators by $\kappa\,(g \otimes h) = [g, h]$, for all $g, h \in G$. Let $J\,(G)$ be the kernel of the map κ, i.e $J\,(G) = \ker\,(\kappa)$ and $J(G)$ is a G-trivial subgroup of $G \otimes G$ contained in its center. Consistent with the notation and terminology in [5], let $\nabla\,(G)$ denote the subgroup of $J(G)$ generated by the elements $x \otimes x$ for $x \in G$ for $x, y \in G$. The group $(G \otimes G)/\nabla\,(G)$ is called the nonabelian exterior square of G, and is denoted as $G \wedge G$. The map κ factorizes modulo $\nabla(G)$, thus inducing an epimorphism $\kappa' : G \wedge G \to G'$. By results in [6,7] the kernel of the map κ' is isomorphic to the Schur multiplicator $M(G)$ of G.
The classification of infinite nonabelian 2-generator groups of nilpotency class 2 has been given by [8]. The classification are given in the following Proposition 1.1 and Proposition 1.2.

Proposition 1.1[8] *Let $G = \langle a, b\rangle$ be a $2-$generator group of nilpotency class less than or equal to 2 of the form $G = P \rtimes \langle b\rangle$, where $\langle b\rangle$ is an infinite cyclic group and $P = \langle [a, b]\rangle\langle a\rangle$ is a p-group. Then G is isomorphic to exactly one group of the following types*:
(1.1.1) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, *where* $[a,b]=c$, $[a,c]=[b,c]=1$, $|a|=p^{\alpha}$, $|c|=p^{\gamma}$, $\alpha \geq \gamma \geq 1$;
(1.1.2) $G \cong \langle a\rangle \rtimes \langle b\rangle$, *where* $[a,b]=a^{p^{\alpha-\gamma}}$, $|a|=p^{\alpha}$, $\alpha \geq 2\gamma \geq 2$;
(1.1.3) $G \cong (\langle a\rangle \times \langle c\rangle) \rtimes \langle b\rangle$, *where* $[a,b]=a^{p^{\alpha-\gamma}}c$, $[c,b]=a^{-p^{2(\alpha-\gamma)}}$, $|a|=p^{\alpha}$, $|c|=p^{\sigma}$, $\gamma > \sigma \geq 1$, $\alpha + \sigma \geq 2\gamma$;

(1.1.4) $G \cong \langle a\rangle \rtimes \langle b\rangle$, *where* $[a,b]=1$, $|a|=p^{\alpha}$.

The groups in the above list have nilpotency class two precisely for (1.1.1), (1.1.2), and (1.1.3) and are abelian for (1.1.4).

Proposition 1.2[8] *Let G be an infinite* $2-$*generator group of nilpotency class two. Then G is isomorphic to exactly one group of the following types*:
(1.2.1) $G \cong (\langle a\rangle \times \langle c\rangle) \rtimes \langle b\rangle$, *where* $[a,b]=c$, $[a,c]=[b,c]=1$, $|a|=\infty$, $|b|=\infty$, $|c| \leq \infty$;
(1.2.2) $G \cong (P_1 \times P_2 \times ... \times P_i \times ... \times P_n) \rtimes \langle b\rangle$, $n \geq 1$, *where, for* $i = 1, ..., n$, *the component* P_i *is a* p_i*-group*, $p_i \neq p_j$ *for* $i \neq j$, $|b|=\infty$ *and* $P_1 \rtimes \langle b\rangle$ *is of type* (1.1.1), (1.1.2), (1.1.3) and (1.1.4) *respectively*.

2 Main Results

The nonabelian tensor product has its origin in homotopy theory. In this research, we focus on the tensor square $G \otimes G$ of a group G. Using the explicit knowledge of the nonabelian tensor squares of groups in the classification (Proposition 1.1 and Proposition 1.2), the nonabelian tensor squares and the homological functors for infinite nonabelian 2-generator groups of nilpotency class two are computed and listed below.

Theorem 2.1 *Let G be a group of type* (1.1.1) *with* $p \neq 2$. *Then*

$$G \otimes G = \langle a \otimes a\rangle \times \langle a \otimes b\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle a \otimes c\rangle \times \langle b \otimes c\rangle \times \langle b \otimes b\rangle \cong \mathbb{Z}^3_{p^{\alpha}} \times \mathbb{Z}^2_{p^{\gamma}} \times \mathbb{Z},$$

$$J(G) = \langle a \otimes a\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle a \otimes c\rangle \times \langle b \otimes c\rangle \times \langle (a \otimes b)^{p^{\gamma}}\rangle \times \langle b \otimes b\rangle \cong \mathbb{Z}^2_{p^{\alpha}} \times \mathbb{Z}^2_{p^{\gamma}} \times \mathbb{Z}_{p^{\alpha-\gamma}} \times \mathbb{Z},$$

$$\nabla(G) = \langle a \otimes a\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle b \otimes b\rangle \cong \mathbb{Z}^2_{p^{\alpha}} \times \mathbb{Z},$$

$$G \wedge G = \langle a \wedge c\rangle \times \langle b \wedge c\rangle \times \langle a \wedge b\rangle \cong \mathbb{Z}^2_{p^{\gamma}} \times \mathbb{Z}_{p^{\alpha}},$$

$$M(G) = \langle a \wedge c\rangle \times \langle b \wedge c\rangle \times \langle (a \wedge b)^{p^{\gamma}}\rangle \cong \mathbb{Z}^2_{p^{\gamma}} \times \mathbb{Z}_{p^{\alpha-\gamma}},$$

$$\triangle(G) = \langle (a \otimes a)^2\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle (b \otimes b)^2\rangle \cong \mathbb{Z}^2_{p^{\alpha}} \times \mathbb{Z},$$

$$G \widetilde{\otimes} G = \langle a \widetilde{\otimes} c\rangle \times \langle b \widetilde{\otimes} c\rangle \times \langle a \widetilde{\otimes} b\rangle \times \langle b \widetilde{\otimes} b\rangle \cong \mathbb{Z}^2_{p^{\gamma}} \times \mathbb{Z}_{p^{\alpha}} \times \mathbb{Z}_2,$$

$$\widetilde{J}(G) = \langle a \widetilde{\otimes} c\rangle \times \langle b \widetilde{\otimes} c\rangle \times \langle (a \widetilde{\otimes} b)^{p^{\gamma}}\rangle \times \langle b \widetilde{\otimes} b\rangle \cong \mathbb{Z}^2_{p^{\gamma}} \times \mathbb{Z}_{p^{\alpha-\gamma}} \times \mathbb{Z}_2,$$

Theorem 2.2 *Let G be a group of type* (1.1.1) *with* $p = 2$, *then*

$$G \otimes G = \langle a \otimes a\rangle \times \langle a \otimes b\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle a \otimes c\rangle \times \langle b \otimes c\rangle \times \langle b \otimes b\rangle \cong \mathbb{Z}^3_{2^{\alpha}} \times \mathbb{Z}^2_{2^{\gamma}} \times \mathbb{Z},$$

$$J(G) = \langle a \otimes a\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle (a \otimes b)^{2^{\gamma}}\rangle \times \langle a \otimes c\rangle \times \langle b \otimes c\rangle \times \langle b \otimes b\rangle \cong \mathbb{Z}^2_{2^{\alpha}} \times \mathbb{Z}_{2^{\alpha-\gamma}} \times \mathbb{Z}^2_{2^{\gamma}} \times \mathbb{Z},$$

$$\nabla(G) = \langle a \otimes a\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle b \otimes b\rangle \cong \mathbb{Z}^2_{2^{\alpha}} \times \mathbb{Z},$$

$$G \wedge G = \langle a \wedge c\rangle \times \langle b \wedge c\rangle \times \langle a \wedge b\rangle \cong \mathbb{Z}^2_{2^{\gamma}} \times \mathbb{Z}_{2^{\alpha}},$$

$$M(G) = \langle a \wedge c\rangle \times \langle b \wedge c\rangle \times \langle (a \wedge b)^{2^{\gamma}}\rangle \cong \mathbb{Z}^2_{2^{\gamma}} \times \mathbb{Z}_{2^{\alpha-\gamma}},$$

$$\triangle(G) = \langle (a \otimes a)^2\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle (b \otimes b)^2\rangle \cong \mathbb{Z}_{2^{\alpha-1}} \times \mathbb{Z}_{2^{\alpha}} \times \mathbb{Z},$$

$G\widetilde{\otimes}G=\langle a\widetilde{\otimes}c\rangle\times\langle b\widetilde{\otimes}c\rangle\times\langle a\widetilde{\otimes}a\rangle\times\langle b\widetilde{\otimes}b\rangle\times\langle a\widetilde{\otimes}b\rangle \cong \mathbb{Z}_{2^\gamma}^2 \times \mathbb{Z}_2^2 \times \mathbb{Z}_{2^\alpha}$,

$\widetilde{J}(G)=\langle a\widetilde{\otimes}c\rangle\times\langle b\widetilde{\otimes}c\rangle\times\langle a\widetilde{\otimes}a\rangle\times\langle b\widetilde{\otimes}b\rangle\times\langle (a\widetilde{\otimes}b)^{2^\gamma}\rangle \cong \mathbb{Z}_{2^\gamma}^2 \times \mathbb{Z}_2^2 \times \mathbb{Z}_{2^{\alpha-\gamma}}$,

Theorem 2.3 *Let* $G\cong(\langle c\rangle \times \langle a\rangle)\rtimes\langle b\rangle$, *where* $[a,b]=c$, $[a,c]=[b,c]=1$, $|a|=2^\gamma$, $|b|=\infty$, $|c|=2^\gamma$, $\gamma\geq 1$. *Then*

$G\otimes G=\langle a\otimes a\rangle\times\langle (a\otimes b)(b\otimes a)\rangle\times\langle (a\otimes b)^2(b\otimes c)\rangle\times\langle a\otimes b\rangle\times\langle (a\otimes b)^2(a\otimes c)\rangle\times\langle b\otimes b\rangle \cong \mathbb{Z}_{2^\gamma}^3 \times \mathbb{Z}_{2^{\gamma+1}} \times \mathbb{Z}_{2^{\gamma-1}} \times \mathbb{Z}$,

$J(G)=\langle a\otimes a\rangle\times\langle (a\otimes b)(b\otimes a)\rangle\times\langle ((a\otimes b)^2(b\otimes c))^{2^{\gamma-1}}\rangle\times\langle (a\otimes b)^{2^\gamma}\rangle\times\langle b\otimes b\rangle \cong\mathbb{Z}_{2^\gamma}^2 \times \mathbb{Z}_2^2 \times \mathbb{Z}$,

$\nabla(G)=\langle a\otimes a\rangle\times\langle (a\otimes b)(b\otimes a)\rangle\times\langle b\otimes b\rangle \cong \mathbb{Z}_{2^\gamma}^2 \times \mathbb{Z}$,

$G\wedge G=\langle (a\wedge b)^2(b\wedge c)\rangle\times\langle a\wedge b\rangle\times\langle (a\wedge b)^2(a\wedge c)\rangle \cong \mathbb{Z}_{2^\gamma} \times \mathbb{Z}_{2^{\gamma+1}} \times \mathbb{Z}_{2^{\gamma-1}}$,

$M(G)=\langle ((a\wedge b)^2(b\wedge c))^{2^{\gamma-1}}\rangle\times\langle (a\wedge b)^{2^\gamma}\rangle\cong \mathbb{Z}_2^2$,

$\triangle(G)=\langle (a\otimes a)^2\rangle\times\langle (a\otimes b)(b\otimes a)\rangle\times\langle (b\otimes b)^2\rangle \cong \mathbb{Z}_{2^{\gamma-1}} \times \mathbb{Z}_{2^\gamma} \times \mathbb{Z}$,

$G\widetilde{\otimes}G=\langle a\widetilde{\otimes}a\rangle\times\langle b\widetilde{\otimes}b\rangle\times\langle (a\widetilde{\otimes}b)^2(b\widetilde{\otimes}c)\rangle\times\langle a\widetilde{\otimes}b\rangle\times\langle (a\widetilde{\otimes}b)^2(a\widetilde{\otimes}c)\rangle \cong \mathbb{Z}_2^2 \times \mathbb{Z}_{2^\gamma} \times \mathbb{Z}_{2^{\gamma+1}} \times \mathbb{Z}_{2^{\gamma-1}}$,

$\widetilde{J}(G)=\langle ((a\widetilde{\otimes}b)^2(b\widetilde{\otimes}c))^{2^{\gamma-1}}\rangle\times\langle (a\widetilde{\otimes}b)^{2^\gamma}\rangle\times\langle a\widetilde{\otimes}a\rangle\times\langle b\widetilde{\otimes}b\rangle\cong \mathbb{Z}_2^4$.

Theorem 2.4 *Let* $G\cong\langle a\rangle\rtimes\langle b\rangle$, *where* $[a,b]=a^{p^{\alpha-\gamma}}$, $|a|=p^\alpha$, $|b|=\infty$, $\alpha\geq 2\gamma\geq 2$, $z=[a,b]$ *and* p *odd prime. Then*

$G\otimes G=\langle a\otimes a\rangle\times\langle (a\otimes b)(b\otimes a)\rangle\times\langle a\otimes b\rangle\times\langle b\otimes b\rangle \cong \mathbb{Z}_{p^{\alpha-\gamma}}^2 \times \mathbb{Z}_{p^\alpha} \times \mathbb{Z}$,

$J(G)=\langle a\otimes a\rangle\times\langle (a\otimes b)(b\otimes a)\rangle\times\langle (a\otimes b)^{p^\gamma}\rangle\times\langle b\otimes b\rangle \cong\mathbb{Z}_{p^{\alpha-\gamma}}^3 \times \mathbb{Z}$,

$\nabla(G)=\langle a\otimes a\rangle\times\langle (a\otimes b)(b\otimes a)\rangle\times\langle b\otimes b\rangle \cong \mathbb{Z}_{p^{\alpha-\gamma}}^2 \times \mathbb{Z}$,

$G\wedge G=\langle a\wedge b\rangle\cong\mathbb{Z}_{p^\alpha}$,

$M(G)=\langle (a\wedge b)^{p^\gamma}\rangle\cong \mathbb{Z}_{p^{\alpha-\gamma}}$,

$\triangle(G)=\langle (a\otimes a)^2\rangle\times\langle (a\otimes b)(b\otimes a)\rangle\times\langle (b\otimes b)^2\rangle \cong \mathbb{Z}_{p^{\alpha-\gamma}}^2 \times \mathbb{Z}$,

$G\widetilde{\otimes}G=\langle a\widetilde{\otimes}b\rangle\times\langle b\widetilde{\otimes}b\rangle\cong \mathbb{Z}_{p^\alpha} \times \mathbb{Z}_2$,

$\widetilde{J}(G)=\langle (a\widetilde{\otimes}b)^{p^\gamma}\rangle\times\langle b\widetilde{\otimes}b\rangle\cong \mathbb{Z}_{p^{\alpha-\gamma}} \times \mathbb{Z}_2$.

Theorem 2.5 *Let* $G\cong\langle a\rangle\rtimes\langle b\rangle$, *where* $[a,b]=a^{2^{\alpha-\gamma}}$, $|a|=2^\alpha$, $|b|=\infty$, $\alpha\geq 2\gamma\geq 2$, $z=[a,b]$. *Then*

$G\otimes G=\langle a\otimes a\rangle\times\langle (a\otimes b)(b\otimes a)\rangle\times\langle a\otimes b\rangle\times\langle b\otimes b\rangle \cong \mathbb{Z}_{2^{\alpha-\gamma+1}}\mathbb{Z}_{2^{\alpha-\gamma}} \times \mathbb{Z}_{2^\alpha} \times \mathbb{Z}$,

$J(G){=}\langle a\otimes a\rangle{\times}\langle(a\otimes b)(b\otimes a)\rangle{\times}\langle(a\otimes b)^{2^\gamma}\rangle{\times}\langle b\otimes b\rangle\cong\mathbb{Z}_{2^{\alpha-\gamma+1}}\times\mathbb{Z}^2_{2^{\alpha-\gamma}}\times\mathbb{Z}$,

$\nabla(G){=}\langle a\otimes a\rangle{\times}\langle(a\otimes b)(b\otimes a)\rangle{\times}\langle b\otimes b\rangle\cong\mathbb{Z}_{2^{\alpha-\gamma+1}}\times\mathbb{Z}_{2^{\alpha-\gamma}}\times\mathbb{Z}$,

$G{\wedge}G{=}\langle a\wedge b\rangle{\cong}\mathbb{Z}_{2^\alpha}$,

$M(G){=}\langle(a\wedge b)^{p^\gamma}\rangle{\cong}\mathbb{Z}_{2^{\alpha-\gamma}}$,

$\triangle(G){=}\langle(a\otimes a)^2\rangle{\times}\langle(a\otimes b)(b\otimes a)\rangle{\times}\langle(b\otimes b)^2\rangle\cong\mathbb{Z}^2_{2^{\alpha-\gamma}}\times\mathbb{Z}$,

$G\widetilde{\otimes}G{=}\langle a\widetilde{\otimes}b\rangle{\times}\langle a\widetilde{\otimes}a\rangle{\times}\langle b\widetilde{\otimes}b\rangle{\cong}\mathbb{Z}_{2^\alpha}\times\mathbb{Z}^2_2$,

$\widetilde{J}(G){=}\langle(a\widetilde{\otimes}b)^{2^\gamma}\rangle{\times}\langle a\widetilde{\otimes}a\rangle{\times}\langle b\widetilde{\otimes}b\rangle{\cong}\mathbb{Z}_{2^{\alpha-\gamma}}\times\mathbb{Z}^2_2$.

Theorem 2.6 *Let* $G{\cong}(\langle a\rangle{\times}\langle c\rangle){\rtimes}\langle b\rangle$*, where* $z=[a,b]=a^{p^{\alpha-\gamma}}c$*,* $[c,b]=a^{-p^{2(\alpha-\gamma)}}c^{-p^{\alpha-\gamma}}$*,* $|a|=p^\alpha$*,* $|c|=p^\sigma$*,* $\gamma>\sigma\geq1$*,* $\alpha+\sigma\geq2\gamma$ *and* p *odd prime. Then*

$G{\otimes}G{=}\langle a\otimes a\rangle{\times}\langle(a\otimes b)(b\otimes a)\rangle{\times}\langle(a\otimes a)^{p^{\alpha-\gamma}}(a\otimes z)\rangle{\times}\langle(a\otimes b)^{p^{\alpha-\gamma}}(b\otimes z)\rangle{\times}\langle a\otimes b\rangle{\times}\langle b\otimes b\rangle$ $\cong\mathbb{Z}^2_{p^{\alpha-\gamma+\sigma}}\times\mathbb{Z}^2_{p^\sigma}\times\mathbb{Z}_{p^\alpha}\times\mathbb{Z}$,

$J(G){=}\langle a{\otimes}a\rangle{\times}\langle(a{\otimes}b)(b{\otimes}a)\rangle{\times}\langle(a{\otimes}a)^{p^{\alpha-\gamma}}(a{\otimes}z)\rangle{\times}\langle((a{\otimes}b)^{p^{\alpha-\gamma}}(b{\otimes}z))^{p^{2\gamma-\alpha}}\rangle{\times}\langle(a{\otimes}b)^{p^\gamma}\rangle{\times}\langle b{\otimes}b\rangle$ ${\cong}\mathbb{Z}^2_{p^{\alpha-\gamma+\sigma}}\times\mathbb{Z}_{p^\sigma}\times\mathbb{Z}_{min(p^{\sigma-2\gamma+\alpha},\sigma)}\times\mathbb{Z}_{max(p^{\alpha-\gamma},\gamma)}\times\mathbb{Z}$,

$\nabla(G){=}\langle a\otimes a\rangle{\times}\langle(a\otimes b)(b\otimes a)\rangle{\times}\langle b\otimes b\rangle\cong\mathbb{Z}^2_{p^{\alpha-\gamma+\sigma}}\times\mathbb{Z}$,

$G{\wedge}G{=}\langle(a\wedge a)^{p^{\alpha-\gamma}}(a\wedge z)\rangle{\times}\langle(a\wedge b)^{p^{\alpha-\gamma}}(b\wedge z)\rangle{\times}\langle a\wedge b\rangle{\cong}\mathbb{Z}^2_{p^\sigma}\times\mathbb{Z}_{p^\alpha}$,

$M(G){=}\langle(a\wedge a)^{p^{\alpha-\gamma}}(a\wedge z)\rangle{\times}\langle((a\wedge b)^{p^{\alpha-\gamma}}(b\wedge z))^{p^{2\gamma-\alpha}}\rangle{\times}\langle(a\wedge b)^{p^\gamma}\rangle{\cong}\mathbb{Z}_{p^\sigma}\times\mathbb{Z}_{min(p^{\sigma-2\gamma+\alpha},\sigma)}\times$ $\mathbb{Z}_{max(p^{\alpha-\gamma},\gamma)}$,

$\triangle(G){=}\langle(a\otimes a)^2\rangle{\times}\langle(a\otimes b)(b\otimes a)\rangle{\times}\langle(b\otimes b)^2\rangle\cong\mathbb{Z}^2_{p^{\alpha-\gamma+\sigma}}\times\mathbb{Z}$,

$G\widetilde{\otimes}G{=}\langle(a\widetilde{\otimes}a)^{p^{\alpha-\gamma}}(a\widetilde{\otimes}z)\rangle{\times}\langle(a\widetilde{\otimes}b)^{p^{\alpha-\gamma}}(b\widetilde{\otimes}z)\rangle{\times}\langle a\widetilde{\otimes}b\rangle{\times}\langle b\widetilde{\otimes}b\rangle{\cong}\mathbb{Z}^2_{p^\sigma}\times\mathbb{Z}_{p^\alpha}\times\mathbb{Z}_2$,

$\widetilde{J}(G){=}\langle(a\widetilde{\otimes}a)^{p^{\alpha-\gamma}}(a\widetilde{\otimes}z)\rangle{\times}\langle((a\widetilde{\otimes}b)^{p^{\alpha-\gamma}}(b\widetilde{\otimes}z))^{p^{2\gamma-\alpha}}\rangle{\times}\langle(a\widetilde{\otimes}b)^{p^\gamma}\rangle{\times}\langle b\widetilde{\otimes}b\rangle{\cong}\mathbb{Z}_{p^\sigma}\times\mathbb{Z}_{min(p^{\sigma-2\gamma+\alpha},\sigma)}\times$ $\mathbb{Z}_{max(p^{\alpha-\gamma},\gamma)}\times\mathbb{Z}_2$.

Theorem 2.7 *Let* $G{\cong}(\langle a\rangle{\times}\langle c\rangle){\rtimes}\langle b\rangle$*, where* $[a,b]=a^{2^{\alpha-\gamma}}c$*,* $[c,b]=a^{-p^{2(2\alpha-\gamma)}}c^{-2^{\alpha-\gamma}}$*,* $[a,b]=a^{2^{\alpha-\gamma}}c$*,* $[c,b]=a^{-2^{2(\alpha-\gamma)}}c^{-2^{\alpha-\gamma}}$*,* $|a|=2^\alpha$*,* $|c|=2^\sigma$*,* $\gamma>\sigma\geq1$*. Then*

$G{\otimes}G{=}\langle a\otimes a\rangle{\times}\langle(a\otimes b)(b\otimes a)\rangle{\times}\langle a\otimes b\rangle{\times}\langle(a\otimes a)^{2^{\alpha-\gamma}}(a\otimes z)\rangle{\times}\langle(a\otimes b)^{2^{\alpha-\gamma}}(a\otimes z)^{-2^{\alpha-\gamma-1}}(b\otimes$ $z)\rangle{\times}\langle b\otimes b\rangle\cong\mathbb{Z}_{2^{\alpha-\gamma+\sigma+1}}\times\mathbb{Z}_{2^{\alpha-\gamma+\sigma}}\times\mathbb{Z}_{2^\alpha}\times\mathbb{Z}^2_{2^\alpha}\times\mathbb{Z}$,

$J(G){=}\langle a{\otimes}a\rangle{\times}\langle(a{\otimes}b)(b{\otimes}a)\rangle{\times}\langle(a{\otimes}a)^{2^{\alpha-\gamma}}(a{\otimes}z)\rangle{\times}\langle((a{\otimes}b)^{2^{\alpha-\gamma}}(a{\otimes}z)^{-2^{\alpha-\gamma-1}}(b{\otimes}z))^{2^{2\gamma-\alpha}}\rangle{\times}\langle(a{\otimes}$ $b)^{2^\gamma}\rangle{\times}\langle b\otimes b\rangle\cong\mathbb{Z}_{2^{\alpha-\gamma+\sigma+1}}\times\mathbb{Z}_{2^{\alpha-\gamma+\sigma}}\times\mathbb{Z}_{2^\sigma}\times\mathbb{Z}_{min(p^{\sigma-2\gamma+\alpha},\sigma)}\times\mathbb{Z}_{max(p^{\alpha-\gamma},\gamma)}\times\mathbb{Z}$,

$\nabla(G){=}\langle a\otimes a\rangle{\times}\langle(a\otimes b)(b\otimes a)\rangle{\times}\langle b\otimes b\rangle\cong\mathbb{Z}_{2^{\alpha-\gamma+\sigma+1}}\times\mathbb{Z}_{2^{\alpha-\gamma+\sigma}}\times\mathbb{Z}$,

$G{\wedge}G{=}\langle(a\wedge a)^{2^{\alpha-\gamma}}(a\wedge z)\rangle{\times}\langle(a\wedge b)^{2^{\alpha-\gamma}}(a\wedge z)^{-2^{\alpha-\gamma-1}}(b\wedge z)\rangle{\times}\langle a\wedge b\rangle{\cong}\mathbb{Z}^2_{2^\sigma}\times\mathbb{Z}_{2^\alpha}$,

$M(G)=\langle (a \wedge a)^{2^{\alpha-\gamma}}(a \wedge z)\rangle \times \langle ((a \wedge b)^{2^{\alpha-\gamma}}(a \wedge z)^{-2^{\alpha-\gamma-1}}(b \wedge z))^{2^{2\gamma-\alpha}}\rangle \times \langle (a \wedge b)^{2^{\gamma}}\rangle \cong \mathbb{Z}_{2^{\sigma}} \times \mathbb{Z}_{min(p^{\sigma-2\gamma+\alpha},\sigma)} \times \mathbb{Z}_{max(p^{\alpha-\gamma},\gamma)}$,

$\triangle(G)=\langle (a \otimes a)^2\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle (b \otimes b)^2\rangle \cong \mathbb{Z}_{2^{\alpha-\gamma+\sigma-1}} \times \mathbb{Z}_{2^{\alpha-\gamma+\sigma}} \times \mathbb{Z}$,

$G\widetilde{\otimes}G=\langle (a\widetilde{\otimes}a)^{2^{\alpha-\gamma}}(a\widetilde{\otimes}z)\rangle \times \langle (a\widetilde{\otimes}b)^{2^{\alpha-\gamma}}(a\widetilde{\otimes}z)^{-2^{\alpha-\gamma-1}}(b\widetilde{\otimes}z)\rangle \times \langle a\widetilde{\otimes}b\rangle \times \langle a\widetilde{\otimes}a\rangle \times \langle b\widetilde{\otimes}b\rangle \cong \mathbb{Z}_{2^{\sigma}}^2 \times \mathbb{Z}_{2^{\alpha}} \times \mathbb{Z}_2^2$,

$\widetilde{J}(G)=\langle (a\widetilde{\otimes}a)^{2^{\alpha-\gamma}}(a\widetilde{\otimes}z)\rangle \times \langle ((a\widetilde{\otimes}b)^{2^{\alpha-\gamma}}(a\widetilde{z})^{-2^{\alpha-\gamma-1}}(b\widetilde{\otimes}z))^{2^{2\gamma-\alpha}}\rangle \times \langle (a\widetilde{\otimes}b)^{2^{\gamma}}\rangle \times \langle a\widetilde{\otimes}a\rangle \times \langle b\widetilde{\otimes}b\rangle \cong \mathbb{Z}_{2^{\sigma}} \times \mathbb{Z}_{min(p^{\sigma-2\gamma+\alpha},\sigma)} \times \mathbb{Z}_{max(p^{\alpha-\gamma},\gamma)} \times \mathbb{Z}_2^2$.

Theorem 2.8 *Let $G \cong \langle a\rangle \times \langle b\rangle$, where $[a,b]=1$, $|a|=p^{\alpha}$, and p odd prime. Then*

$G\otimes G=\langle a \otimes a\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle a \otimes b\rangle \times \langle b \otimes b\rangle \cong \mathbb{Z}_{p^{\alpha}}^3 \times \mathbb{Z}$,

$J(G)=\langle a \otimes a\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle a \otimes b\rangle \times \langle b \otimes b\rangle \cong \mathbb{Z}_{p^{\alpha}}^3 \times \mathbb{Z}$,

$\nabla(G)=\langle a \otimes a\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle b \otimes b\rangle \cong \mathbb{Z}_{p^{\alpha}}^2 \times \mathbb{Z}$,

$G\wedge G=\langle a \wedge b\rangle \cong \mathbb{Z}_{p^{\alpha}}$,

$M(G)=\langle a \wedge b\rangle \cong \mathbb{Z}_{p^{\alpha}}$,

$\triangle(G)=\langle (a \otimes a)^2\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle (b \otimes b)^2\rangle \cong \mathbb{Z}_{p^{\alpha}}^2 \times \mathbb{Z}$,

$G\widetilde{\otimes}G=\langle a\widetilde{\otimes}b\rangle \times \langle b\widetilde{\otimes}b\rangle \cong \mathbb{Z}_{p^{\alpha}} \times \mathbb{Z}_2$,

$\widetilde{J}(G)=\langle a\widetilde{\otimes}b\rangle \times \langle b\widetilde{\otimes}b\rangle \cong \mathbb{Z}_{p^{\alpha}} \times \mathbb{Z}_2$.

Theorem 2.9 *Let $G \cong \langle a\rangle \times \langle b\rangle$, where $[a,b]=1$, $|a|=p^{\alpha}$, and $p=2$. Then*

$G\otimes G=\langle a \otimes a\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle a \otimes b\rangle \times \langle b \otimes b\rangle \cong \mathbb{Z}_{2^{\alpha}}^3 \times \mathbb{Z}$,

$J(G)=\langle a \otimes a\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle a \otimes b\rangle \times \langle b \otimes b\rangle \cong \mathbb{Z}_{2^{\alpha}}^3 \times \mathbb{Z}$,

$\nabla(G)=\langle a \otimes a\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle b \otimes b\rangle \cong \mathbb{Z}_{2^{\alpha}}^2 \times \mathbb{Z}$,

$G\wedge G=\langle a \wedge b\rangle \cong \mathbb{Z}_{2^{\alpha}}$,

$M(G)=\langle a \wedge b\rangle \cong \mathbb{Z}_{2^{\alpha}}$,

$\triangle(G)=\langle (a \otimes a)^2\rangle \times \langle (a \otimes b)(b \otimes a)\rangle \times \langle (b \otimes b)^2\rangle \cong \mathbb{Z}_{2^{\alpha-1}} \times \mathbb{Z}_{2^{\alpha}} \times \mathbb{Z}$,

$G\widetilde{\otimes}G=\langle a\widetilde{\otimes}b\rangle \times \langle a\widetilde{\otimes}a\rangle \times \langle b\widetilde{\otimes}b\rangle \cong \mathbb{Z}_{2^{\alpha}} \times \mathbb{Z}_2^2$,

$\widetilde{J}(G)=\langle a\widetilde{\otimes}b\rangle \times \langle a\widetilde{\otimes}a\rangle \times \langle b\widetilde{\otimes}b\rangle \cong \mathbb{Z}_{2^{\alpha}} \times \mathbb{Z}_2^2$.

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