Paper 258

Proceedings of the Ninth International Conference
on Computational Structures Technology,
B.H.V. Topping and M. Papadrakakis, (Editors),
Civil-Comp Press, Stirlingshire, Scotland

On the Correlation of Theory and Experiment for Transversely Isotropic Nonlinear Incompressible Solids

M.H.B.M. Shariff[1], B.A. Mahad[2] and A.A. Zainal[2]
[1] Kalifa University of Science, Technology and Research
United Arab Emirates
[2] University of Technology
Malaysia

Abstract

A novel strain energy function for finite strain deformations of transversely isotropic elastic solids which is a function five invariants that have immediate physical interpretation has recently been developed. Three of the five invariants are the principal stretch ratios and the other two are squares of the dot product between the preferred direction and two principal directions of the right stretch tensor. A strain energy function, expressed in terms of these invariants, has a symmetrical property almost similar to that of an isotropic elastic solid written in terms of principal stretches. This constitutive equation is attractive if principal axes techniques are used in solving boundary value problems and experimental advantage is demonstrated by showing a simple triaxial test can vary a single invariant while keeping the remaining invariants fixed. Explicit expressions for the weighted Cauchy response functions are easily obtained since the response function basis is almost mutually orthogonal. In this paper a specific form of the strain energy function for incompressible materials which is linear with respect to its physical parameters is developed. When a curve fitting method is (sensibly) applied on an experimental data, the values of the parameters are obtained uniquely via a linear positive definite system of equations. The theory compares well with experimental data and the performance of the proposed specific form is discussed. A constitutive inequality, which may reasonably be imposed upon the material parameters, is discussed.

Keywords: transversely-isotropic, nonlinear, constitutive-equation, principal-axes.

1 Introduction

The invariants [1]

$$I_1 = tr\boldsymbol{C}\,, \quad I_2 = \frac{1}{2}\left((tr\boldsymbol{C})^2 - tr\boldsymbol{C}^2\right)\,,$$

$$I_3 = det\boldsymbol{C}\,, \quad I_4 = \boldsymbol{a} \bullet \boldsymbol{C}\boldsymbol{a}\,, \quad I_5 = \boldsymbol{a} \bullet \boldsymbol{C}^2\boldsymbol{a}\,, \tag{1}$$

are commonly used in the literature, where $\boldsymbol{C}$ is the right Cauchy-Green deformation tensor, tr denotes the trace of a second order tensor and $\boldsymbol{a}$ is the preferred direction in the reference configuration. The variables $\sqrt{I_3}$ and $\sqrt{I_4}$ represent the volume change and the stretch in the preferred direction, respectively, of a deformed material. However, the other three invariants do not have immediate physical interpretation. A strain energy function written in terms of the invariants given in Equation (1) is not experimentally friendly. For example, a simple isochoric deformation such as uniaxial stretch in the preferred direction, perturbs the invariants I_1, I_2, I_4 and I_5 and a pure dilatation deformation perturbs all of the invariants. These deformations are not ideal in obtaining the functional form of a strain energy function if the functional form is determined by doing tests that hold four out five invariants constant so that the dependence in the remaining invariant can be identified. There are several sets of invariants proposed for transversely isotropic that appeared in the literature, see e.g., the works of [2], [3] and [4]. These sets are equivalent (in the sense of one to one correspondence) to the set of invariants given in Equation (1) and are formulated to serve some purposes; there is no set which is generally suitable for all purposes.

In isotropic elasticity, strain energy functions that depend explicitly on the physically interpreted principal extension ratios λ_1, λ_2 and λ_3 have been widely and successfully used in predicting elastic deformations [5] and in terms of such variables the stress-deformation relations take on a concise and transparent mathematical form. In this paper, we extend this principal-ratio dependent to model strain energy functions of transversely isotropic elastic solids. Hence, we introduce a strain energy function which depends on five variables that have immediate physical interpretation. Three of the variables are the principal extension ratios $\lambda_i > 0$ $(i = 1, 2, 3)$ and the other two are $\zeta_1 = (a \bullet \boldsymbol{e}_1)^2 > 0$ and $\zeta_2 = (a \bullet \boldsymbol{e}_2)^2 > 0$, where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are any two of the principal directions of the right stretch tensor $\boldsymbol{U}$. The physical meaning of λ_i is obvious and it is clear that $a \bullet \boldsymbol{e}_i$ $(i = 1, 2)$ is the cosine of the angle between the principal direction e_i and the preferred direction $\boldsymbol{a}$. In addition to the simple and direct physical interpretation of our invariants, our model has an experimental advantage where a triaxial test can vary a single invariant while keeping the remaining invariants fixed. In view that 9 out of 10 inner products of our response terms vanish, and the response terms are nearly orthogonal, the Cauchy stress response functions (defined in Section 4) can be explicitly expressed in terms of stress and deformation. This offers an advantage over many previous constitutive equations in the sense that a specific strain energy function for a particular material can be obtained with mathematical rigour; previous specific forms are generally obtained, heuristically. The form of strain energy function written in terms of the proposed variables enjoys a symmetrical property almost similar to the symmetry possessed by a strain energy function of an isotropic elastic solid written in terms of principal stretches. *We note that strain energy functions that appeared in the literature are not symmetrical with respect to their invariants*.

In Section 6, we proposed a simple specific form, in the sense that the functions in the strain energy function depend only on a single variable which is easy to analyse, and when a principal axes technique is employed, the stress-deformation relations take on a concise and transparent mathematical form. The theory compares well with experimental data.

In Section 8, Hill's constitutive inequality [6] is used in a discussion to obtain sufficient conditions for the inequality.

2 Strain Energy Function

We first recall some essential kinematics of finite deformation of a transversely isotropic elastic material. Consider a body occupying the region B_0 in some reference configuration. Let $\boldsymbol{F}$ be the deformation tensor and $\boldsymbol{X}$ a position vector of a point in B_0. Under this deformation the point moves to a new position $\boldsymbol{x}(\boldsymbol{X}) \in B$, where B is the current configuration of the deformed body.

The principal stretch λ_i $(i = 1, 2, 3)$ is given by

$$\lambda_i = \sqrt{\boldsymbol{e}_i \bullet \boldsymbol{U}^2 \boldsymbol{e}_i}\,, \tag{2}$$

where $\boldsymbol{U}^2 = \boldsymbol{F}^T\boldsymbol{F}$ and, as mentioned before, $\boldsymbol{e}_i$ is a principal direction of $\boldsymbol{U}$. In this communication all subscripts i and j take the values 1, 2 and 3, unless stated otherwise.

The material response of a transversely isotropic solid is indifferent to arbitrary rotations about the direction $\boldsymbol{a}$ and by replacement of $\boldsymbol{a}$ by -$\boldsymbol{a}$. Such material can be characterised with a strain energy function W_e which depends on $\boldsymbol{U}$ and the tensor $\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{a}$ ($\otimes$ denotes the dyadic product), i.e. ,

$$W_e = \hat{W}(\boldsymbol{U}, \boldsymbol{A})\,. \tag{3}$$

Since

$$\boldsymbol{U} = \lambda_1 \boldsymbol{E}_1 + \lambda_2 \boldsymbol{E}_2 + \lambda_3 \boldsymbol{E}_3\,, \tag{4}$$

where $\boldsymbol{E}_i = \boldsymbol{e}_i \otimes \boldsymbol{e}_i$. We can express

$$\hat{W}(\boldsymbol{U}, \boldsymbol{A}) = \bar{W}(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3, \boldsymbol{A})\,. \tag{5}$$

Shariff [7] has shown that the strain energy function can be written in the form

$$W_e = W_f(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3)\,, \tag{6}$$

The function W_f enjoys the symmetrical property [7]

$$W_f(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3) = W_f(\lambda_2, \lambda_1, \lambda_3, \zeta_2, \zeta_1, \zeta_3) = W_f(\lambda_3, \lambda_2, \lambda_1, \zeta_3, \zeta_2, \zeta_1)\,. \tag{7}$$

However, ζ_3 depends on ζ_1 and ζ_2, i.e.,

$$\zeta_3 = 1 - \zeta_1 - \zeta_2\,. \tag{8}$$

Hence, we can omit ζ_3 from the list in Equation (6) and we then have

$$W_e = \tilde{W}(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2) = W_f(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, 1 - \zeta_1 - \zeta_2)\,. \tag{9}$$

The commonly used invariants can be written explicitly in terms of the physical variables, i.e.,

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2\,, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2\,, \quad I_3 = (\lambda_1\lambda_2\lambda_3)^2$$

$$I_4 = \lambda_1^2\zeta_1 + \lambda_2^2\zeta_2 + \lambda_3^2\zeta_3\,, \quad I_5 = \lambda_1^4\zeta_1 + \lambda_2^4\zeta_2 + \lambda_3^4\zeta_3\,. \tag{10}$$

For an incompressible material $\lambda_1\lambda_2\lambda_3 = 1$, the number of variables is reduce to 4 and we can express

$$W_e = W(\lambda_1, \lambda_2, \zeta_1, \zeta_2) = \tilde{W}(\lambda_1, \lambda_2, \frac{1}{\lambda_1\lambda_2}, \zeta_1, \zeta_2)\,. \tag{11}$$

In the reference state $\boldsymbol{U} = \boldsymbol{I}$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, any orthonormal set of vectors can represent the principal directions of $\boldsymbol{U}$. For simplicity, we let $\boldsymbol{a} = \boldsymbol{e}_3$ and it is clear that $\zeta_3 = 1$, $\zeta_1 = \zeta_2 = 0$ in this state. To be consistent with the classical linear theory of incompressible transversely isotropic elasticity, appropriate for infinitesimal deformations, we must have the relations

$$\frac{\partial^2 W}{\partial \lambda_1^2}(1,1,0,0) = \frac{\partial^2 W}{\partial \lambda_2^2}(1,1,0,0) = 4\mu_L + \beta\,,$$
$$\frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2}(1,1,0,0) = 4\mu_L - 2\mu_T + \beta\,,$$

$$\frac{\partial^2 W}{\partial \lambda_i \partial \zeta_j}(1,1,0,0) = \frac{\partial^2 W}{\partial \zeta_i \partial \zeta_j}(1,1,0,0) = 0\,, \quad i,j = 1,2\,, \tag{12}$$

where μ_T and μ_L, represent the elastic shear moduli in the ground state and β can be related to other elastic constant which has more direct physical interpretation, such as the extension modulus.

3 Stress

The incompressible Cauchy stress is given by the relation

$$\boldsymbol{\sigma} = 2\boldsymbol{F}\frac{\partial W_e}{\partial \boldsymbol{C}}\boldsymbol{F}^T - p\boldsymbol{I}\,, \tag{13}$$

where p is the Lagrange multiplier associated with the incompressibility constraint. The proposed alternative formulation requires the symmetric components $\left(\frac{\partial W_e}{\partial \boldsymbol{C}}\right)_{ij}$ of $\frac{\partial W_e}{\partial \boldsymbol{C}}$ relative to the basis $\{\boldsymbol{e}_i\}$. They are [7];

$$\left(\frac{\partial W_e}{\partial \boldsymbol{C}}\right)_{ii} = \frac{1}{2\lambda_i}\frac{\partial \tilde{W}}{\partial \lambda_i} \quad (i \text{ not summed}) \tag{14}$$

and the shear components

$$\left(\frac{\partial W_e}{\partial \boldsymbol{C}}\right)_{ij} = \frac{\dfrac{\partial \tilde{W}}{\partial \zeta_i} - \dfrac{\partial \tilde{W}}{\partial \zeta_j}}{(\lambda_i^2 - \lambda_j^2)}\boldsymbol{e}_i \bullet \boldsymbol{A}\boldsymbol{e}_j \quad i \neq j\,, \quad i,j = 1,2\,,$$

$$\left(\frac{\partial W_e}{\partial \boldsymbol{C}}\right)_{\alpha 3} = \frac{\dfrac{\partial \tilde{W}}{\partial \zeta_\alpha}}{(\lambda_\alpha^2 - \lambda_3^2)}\boldsymbol{e}_\alpha \bullet \boldsymbol{A}\boldsymbol{e}_3\,, \qquad \alpha = 1,2\,. \tag{15}$$

It is assumed that $\tilde{W}$ has sufficient regularity to ensure that, as λ_i and λ_α approach λ_j and λ_3, respectively, the relations in (15) have limits. It is explicit in Equations (13), (14) and (15) that the Cauchy stress stress is coaxial with $\boldsymbol{V}$ (the left stretch tensor) when the preferred direction $\boldsymbol{a}$ is parallel to one of the principal directions. This explicitness may not be obtained if the strain energy function is expressed in terms of other invariants found in the literature.

4 Orthogonality

The Cauchy stress-strain relation can be written as

$$\boldsymbol{\sigma} + \bar{p}\boldsymbol{I} = \sum_{k=1}^{4}\frac{\partial W}{\partial \eta_k}\hat{\boldsymbol{A}}_k\,, \tag{16}$$

where

$$\bar{p} = p + \lambda_3\frac{\partial \tilde{W}}{\partial \lambda_3}\,,$$

$$\hat{\boldsymbol{A}}_i = \lambda_i\hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_i\,, \quad i = 1,2\,,$$

$$\hat{\boldsymbol{A}}_3 = \beta_1(\hat{\boldsymbol{e}}_1 \otimes \hat{\boldsymbol{e}}_2 + \hat{\boldsymbol{e}}_2 \otimes \hat{\boldsymbol{e}}_1) + \beta_2(\hat{\boldsymbol{e}}_1 \otimes \hat{\boldsymbol{e}}_3 + \hat{\boldsymbol{e}}_3 \otimes \hat{\boldsymbol{e}}_1)\,,$$

$$\hat{\boldsymbol{A}}_4 = -\beta_1(\hat{\boldsymbol{e}}_1 \otimes \hat{\boldsymbol{e}}_2 + \hat{\boldsymbol{e}}_2 \otimes \hat{\boldsymbol{e}}_1) + \beta_3(\hat{\boldsymbol{e}}_2 \otimes \hat{\boldsymbol{e}}_3 + \hat{\boldsymbol{e}}_3 \otimes \hat{\boldsymbol{e}}_2)\,, \tag{17}$$

$$\beta_1 = \frac{2\lambda_1\lambda_2}{\lambda_1^2 - \lambda_2^2}\boldsymbol{e}_1\boldsymbol{A}\boldsymbol{e}_2\,, \quad \beta_2 = \frac{2\lambda_1\lambda_3}{\lambda_1^2 - \lambda_3^2}\boldsymbol{e}_1\boldsymbol{A}\boldsymbol{e}_3\,, \quad \text{and} \quad \beta_3 = \frac{2\lambda_2\lambda_3}{\lambda_2^2 - \lambda_3^2}\boldsymbol{e}_2\boldsymbol{A}\boldsymbol{e}_3\,. \tag{18}$$

and $\eta_1 = \lambda_1$, $\eta_2 = \lambda_2$, $\eta_3 = \zeta_1$ and $\eta_4 = \zeta_2$. We call the term $\dfrac{\partial W}{\partial \eta_k}$ a response function of the extra Cauchy stress and the set $\{\hat{\boldsymbol{A}}_k\}$ the response basis function.

The inner products have the values

$$tr(\hat{\boldsymbol{A}}_1\hat{\boldsymbol{A}}_k) = 0\,, \quad k = 2, 3, 4\,,$$

$$tr(\hat{\boldsymbol{A}}_2\hat{\boldsymbol{A}}_l) = 0\,, \quad l = 3, 4\,,$$

$$tr(\hat{\boldsymbol{A}}_3\hat{\boldsymbol{A}}_4) = -\beta_1^2 \tag{19}$$

With this almost mutually orthogonal basis, the response functions are explicitly obtained, i.e.,

$$\frac{\partial W_e}{\partial \lambda_1} = \frac{tr((\boldsymbol{\sigma} + \bar{p}\boldsymbol{I})\hat{\boldsymbol{A}}_1)}{\lambda_1^2}\,, \quad \frac{\partial W_e}{\partial \lambda_2} = \frac{tr((\boldsymbol{\sigma} + \bar{p}\boldsymbol{I})\hat{\boldsymbol{A}}_2)}{\lambda_2^2}\,,$$

$$\frac{\partial W_e}{\partial \zeta_1} = btr((\boldsymbol{\sigma} + \bar{p}\boldsymbol{I})\hat{\boldsymbol{A}}_3) - ctr((\boldsymbol{\sigma} + \bar{p}\boldsymbol{I})\hat{\boldsymbol{A}}_4)\,,$$

$$\frac{\partial W_e}{\partial \zeta_2} = -ctr((\boldsymbol{\sigma} + \bar{p}\boldsymbol{I})\hat{\boldsymbol{A}}_3) + atr((\boldsymbol{\sigma} + \bar{p}\boldsymbol{I})\hat{\boldsymbol{A}}_4)\,, \tag{20}$$

where $a = \dfrac{tr(\hat{\boldsymbol{A}}_3\hat{\boldsymbol{A}}_3)}{det}$, $b = \dfrac{tr(\hat{\boldsymbol{A}}_4\hat{\boldsymbol{A}}_4)}{det}$, $c = \dfrac{tr(\hat{\boldsymbol{A}}_3\hat{\boldsymbol{A}}_4)}{det}$ and
$det = tr(\hat{\boldsymbol{A}}_3\hat{\boldsymbol{A}}_3)tr(\hat{\boldsymbol{A}}_4\hat{\boldsymbol{A}}_4) - tr(\hat{\boldsymbol{A}}_3\hat{\boldsymbol{A}}_4)^2$.

5 Experimental Advantage

In a triaxial test of an incompressible solid, where $W_e = W(\lambda_1, \lambda_2, \zeta_1, \zeta_2)$, the principal stretches λ_1 and λ_2 can be varied independently. The invariants ζ_1 and ζ_2 can be varied independently by taking different samples, of the same material, with different preferred directions (relative to a principal direction (say)). Hence, it allows us to determine the functional form of W by doing tests that holds three out four invariants constant so that the dependence of W on the remaining invariant can be identified. We note in passing that the invariants I_1, I_2, I_4 and I_5 cannot be varied independently during a triaxial test. In addition to this, in view of the orthogonal properties given in Section 4, a response function can be explicitly expressed in terms of stress and deformation. Hence, the functional form of W can be easily obtained by integrating expressions in Equation (20). We note that for a strain energy function expressed in terms of the invariants I_1, I_2, I_4 and I_5, explicit expressions for the extra Cauchy stress response functions require a non-numeric inversion of a 4×4 matrix; this may be difficult (or impossible) to obtain.

6 Specific Form

Using series expansion techniques Shariff [7] has shown that the strain energy function can be written as

$$W_e = \sum_{i=1}^{3} \hat{f}(\lambda_i, \zeta_i) + \hat{g}(\lambda_1, \lambda_2, \zeta_1, \zeta_2) + \hat{g}(\lambda_1, \lambda_3, \zeta_1, \zeta_3) + \hat{g}(\lambda_2, \lambda_3, \zeta_2, \zeta_3)\,, \quad (21)$$

where $\lambda_3 = \dfrac{1}{\lambda_1 \lambda_2}$, the function $\hat{g}$ has the symmetry $\hat{g}(x, y, \phi, \psi) = \hat{g}(y, x, \psi, \phi)$. A special case of (21) is the augmented form

$$W_e = W_{iso}(\lambda_1, \lambda_2, \lambda_3) + W_{trn}(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3)\,, \quad (22)$$

where

$$W_{iso}(\lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{3} r(\lambda_i) + \bar{g}(\lambda_1, \lambda_2) + \bar{g}(\lambda_1, \lambda_3) + \bar{g}(\lambda_2, \lambda_3)\,, \quad (23)$$

$$W_{trn}(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3) = \sum_{i=1}^{3} f(\lambda_i, \zeta_i) + g(\lambda_1, \lambda_2, \zeta_1, \zeta_2) +$$

$$g(\lambda_1, \lambda_3, \zeta_1, \zeta_3) + g(\lambda_2, \lambda_3, \zeta_2, \zeta_3)\,, \quad (24)$$

g has the same symmetric property as $\hat{g}$ and $\bar{g}(x, y) = \bar{g}(y, x)$. W_{iso} is a strain energy function for an isotropic material. A special form of the augmented strain energy (22) with its isotropic base taking the Valanis & Landel [8] form, is the semi-linear form

$$W_e = \sum_{i=1}^{3} \mu_T (\lambda_i - 1)^2 + 2(\mu_L - \mu_T) \sum_{i=1}^{3} \zeta_i (\lambda_i - 1)^2 + \frac{\beta}{2} \sum_{i,j=1}^{3} \zeta_i \zeta_j (\lambda_i - 1)(\lambda_j - 1)\,. \quad (25)$$

Based on (22) and (25), and for mathematical simplicity, we proposed the special form of W_e which is linear in its parameters, i.e.,

$$W_e = \sum_{i=1}^{3} \mu_T r(\lambda_i) + 2(\mu_L - \mu_T) \sum_{i=1}^{3} \zeta_i s(\lambda_i) + \frac{\beta}{2} \sum_{i,j=1}^{3} \zeta_i \zeta_j t(\lambda_i) t(\lambda_j)\,, \quad (26)$$

with the properties $r(1) = s(1) = t(1) = 0$, $r'(1) = s'(1) = 0$, $t'(1) = 1$, $r''(1) = s''(1) = 2$,

$$xr'(x) = f(x) = \phi_0(x) + \sum_{i=1}^{n} a_i\phi_i(x)$$

$$xs'(x) = g(x) = \bar{\phi}_0(x) + \sum_{i=1}^{m} b_i\bar{\phi}_i(x)$$

$$xt'(x) = h(x) = \tilde{\phi}_0(x) + \sum_{i=1}^{p} c_i\tilde{\phi}_i(x)\,, \tag{27}$$

where $\phi_0(1) = \bar{\phi}_0(1) = 0$, $\tilde{\phi}_0(1) = 1$,$\phi_0'(1) = \bar{\phi}_0'(1) = 2$, $\phi_k(1) = \bar{\phi}_k(1) = \phi_k'(1) = \bar{\phi}_k'(1) = 0$ and $\tilde{\phi}_k(1) = 0$, $k = 1, 2, \ldots$. We note that the sets of functions $\{\phi_k\}$, $\{\bar{\phi}_k\}$ and $\{\tilde{\phi}_k\}$ are linearly independent. With this specific form, we show in the next section, the stress-strain components in the axes of the Eulerian strain ellipsoid have simple forms .

7 Correlation with Experiment

There are several ways to fit a theoretical curve to an experimental data. In this communication we only consider the standard least squares fit with weighting. The weighting [5]

$$g_j = \frac{\dfrac{L}{c + t_j^2}}{\sum_j \dfrac{1}{c + t_j^2}} \tag{28}$$

is used to give a higher weight at low stress, where L is the number of data points, t_j are the values of the experimental stress and $c \geq 0$ is a constant. The theoretical curves are compared with the Humphrey's et al. [9] soft tissue (passive myocardium) data. We note that the soft tissue in [9] in not purely elastic and that the loading and unloading properties are not the same. We only use the loading portion of the data from equibiaxial stretching of a thin sheet of material. First, we consider the biaxial deformation of a thin sheet defined by

$$x_1 = \lambda_1 X_1\,, \quad x_2 = \lambda_2 X_2\,, \quad x_3 = \lambda_3 X_3\,, \tag{29}$$

where x_i and X_i are the Cartesian components of $\boldsymbol{x}$ and $\boldsymbol{X}$, respectively, and $\lambda_3 = \dfrac{1}{\lambda_1\lambda_2}$. For this deformation the deformation tensor $\boldsymbol{F} = \boldsymbol{U}$ and the principal axes of the deformation coincide with the Cartesian coordinate directions and are fixed as the values of the stretches change. Thus, $\boldsymbol{F} \equiv diag(\lambda_1, \lambda_2, \lambda_3)$. We only consider a

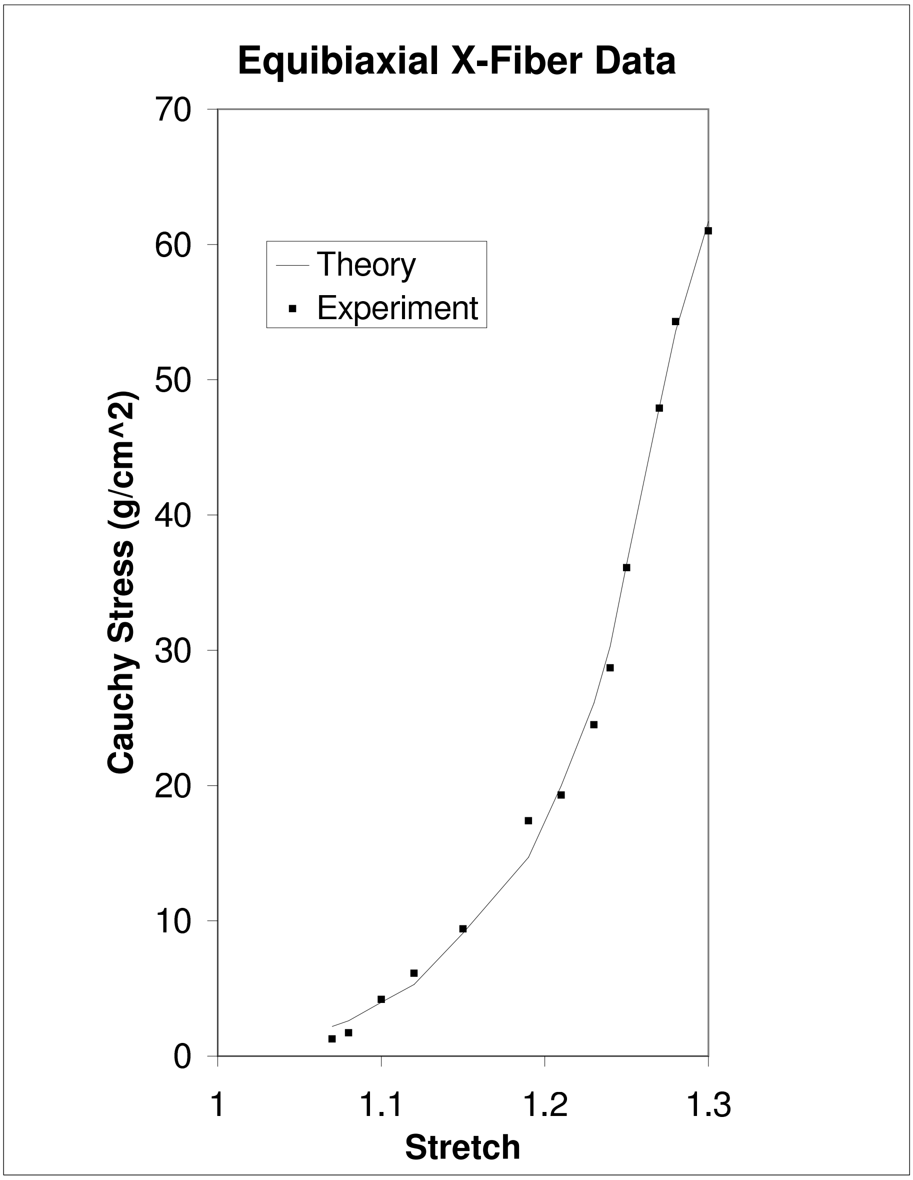

Figure 1: σ_{22} : $\mu_T = 7.06\, gm/cm^2$, $a_1 = 26.7$, $a_2 = 18.03$, $c = 10^6$

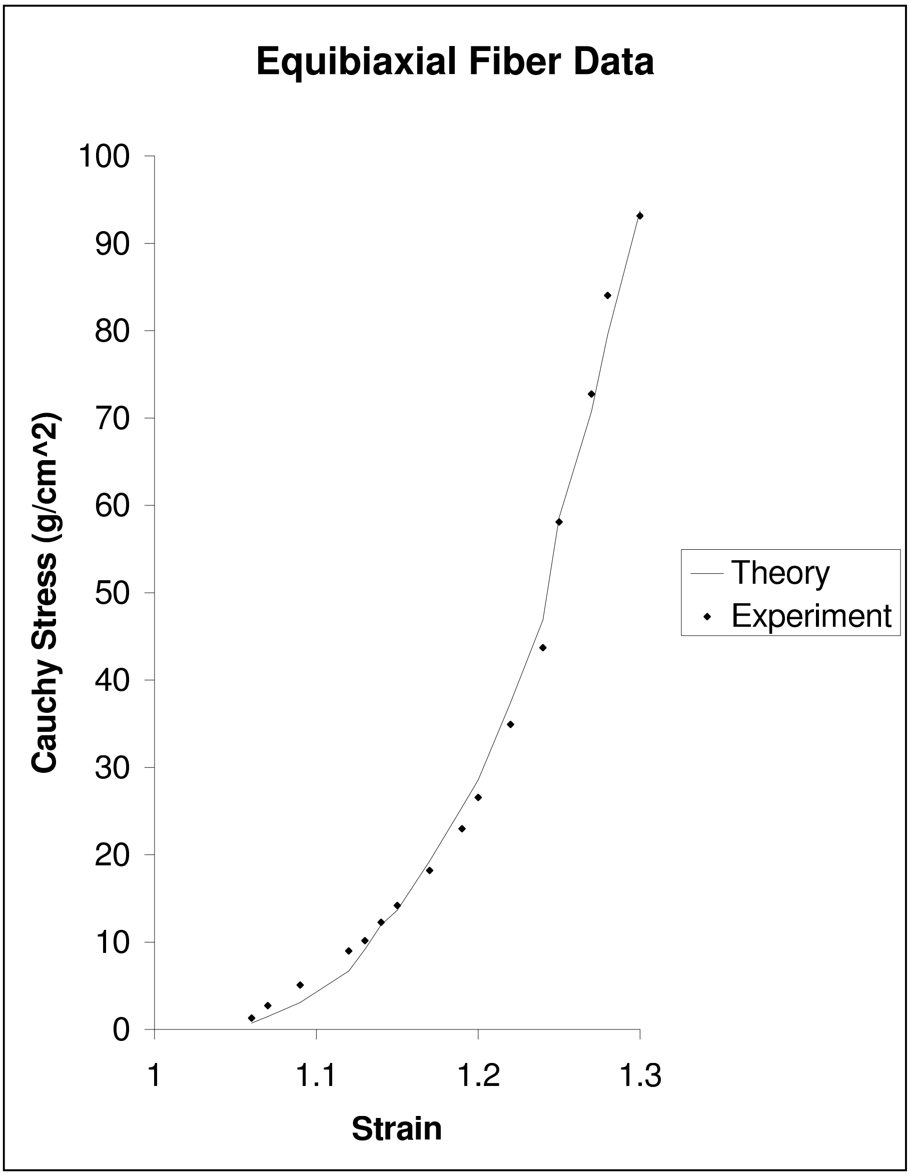

Figure 2: σ_{11} : $\mu_L = 261.93\, gm/cm^2$, $\beta = -583$, $c = 10^6$

biaxial deformation applied on a thin sheet that lies on the (X_1, X_2)-plane with the Cauchy stress component $\sigma_{33} = 0$. The preferred direction $\boldsymbol{a}$ is parallel to the X_1 axis, hence in this case $\zeta_1 = 1$ and $\zeta_2 = \zeta_3 = 0$. For equibiaxial deformation, we have, $\lambda_1 = \lambda_2 = \lambda$. The Cartesian components of the Cauchy stress then have the expression

$$\sigma_{11} = \mu_T \left(\lambda r'(\lambda) - \frac{1}{\lambda^2} r'(\frac{1}{\lambda^2}) \right) + 2(\mu_L - \mu_T)\lambda s'(\lambda) + \beta t(\lambda)\lambda t'(\lambda) \tag{30}$$

$$\sigma_{22} = \mu_T \left(\lambda r'(\lambda) - \frac{1}{\lambda^2} r'(\frac{1}{\lambda^2}) \right) \tag{31}$$

For simplicity we use

$$f(x) = \phi_0 + a_1\phi_1 + a_2\phi_2 \,,$$

$$s(x) = \phi_0 \quad \text{and} \quad h(x) = \tilde{\phi}_0(x) = \frac{1}{x} \,, \tag{32}$$

where $\phi_0(x) = 2ln(x)$, $\phi_1(x) = e^{(x-1)} - x$ and $\phi_2(x) = \dfrac{(x-1)^3}{x^{3.6}}$. Hence, we only use five parameters, μ_T, μ_L, β, a_1 and a_2, to predict the experiment. We could use other sets of basis to improve the performance of the specific form but this will be done in the near future. The values of the parameters μ_T, a_1 and a_2 are obtained using the σ_{22} data. These values are then substituted in σ_{11} to obtain the values of μ_L and β using the σ_{11} data. The parameter values are uniquely obtained via a positive definite linear system of equations. This has an advantage over previous specific forms that are not linear in their parameters. The least-square errors for the σ_{22} and σ_{11} data are 5.166 and 8.04, respectively. It is clear from figures (1) and (2) that the theory compares well with the experimental data. In the near future, the theory will be compared with various types of experimental data and with various types of materials.

8 Constitutive Inequality

A problem intimately related to that of determining forms of the strain energy function is determining the restrictions which are to be placed on the strain energy function to ensure physically reasonable response. Material inequalities proposed in the literature are often used to restrict the forms of strain energy functions. However, none of the material inequalities proposed in the literature are adequate for all elastic materials. In this paper, we shall only discuss Hill's inequality [6] which seems adequate for incompressible elastic materials. Hill's inequality exerts that

$$tr(\hat{\boldsymbol{\sigma}}\boldsymbol{E}) > 0 \,, \tag{33}$$

where $\hat{\boldsymbol{\sigma}}$ is the rigid-body derivative (the rate of change on axes rotating rigidly with the local body spin) of the Cauchy stress and $\boldsymbol{E}$ is the Eulerian strain rate. Expressed in terms of components on the axes of the Eulerian strain ellipsoid, Equation (33) takes the form

$$\hat{\sigma}_{ij}E_{ij} > 0\,, \tag{34}$$

where $\hat{\sigma}_{ij}$ and E_{ij} are the components of $\hat{\boldsymbol{\sigma}}$ and $\boldsymbol{E}$, respectively. After some algebra we can show that

$$\hat{\sigma}_{rr} = \dot{\sigma}_{rr} + \sum_{j\neq r} \Omega_{rj}\sigma_{rj}\,, \quad r \;\; \text{not summed}\,, \tag{35}$$

where the superposed dot represents material time derivative, σ_{ij} are the components of $\boldsymbol{\sigma}$ on the Eulerian strain ellipsoid axes and Ω_{rj} are the components of the spin of the Eulerian strain ellipsoid axes on the same axes. For the shear components, we have

$$\hat{\sigma}_{ij} = \dot{\sigma}_{ij} + (\sigma_{jj} - \sigma_{ii})\Omega_{ij} + \sigma_{kj}\Omega_{ik} + \sigma_{ik}\Omega_{jk}\,, \quad i \neq j \neq k\,. \tag{36}$$

It can be shown that [6]

$$-\Omega_{ij} = \frac{\lambda_i^2 + \lambda_j^2}{\lambda_i^2 - \lambda_j^2}E_{ij}\,, \quad i \neq j\,, \quad \lambda_i \neq \lambda_j \tag{37}$$

and

$$\frac{\dot{\lambda}_i}{\lambda_i} = E_{ii}\,. \tag{38}$$

From (13), (14) and (15) we have

$$\sigma_{ii} = \lambda_i \frac{\partial \tilde{W}}{\partial \lambda_i} - p \tag{39}$$

and

$$\sigma_{ij} = 2\lambda_i\lambda_j \frac{\dfrac{\partial \tilde{W}}{\partial \zeta_i} - \dfrac{\partial \tilde{W}}{\partial \zeta_j}}{(\lambda_i^2 - \lambda_j^2)} \boldsymbol{e}_i \bullet \boldsymbol{A}\boldsymbol{e}_j \;\; i \neq j\,, \quad i,j = 1,2\,, \tag{40}$$

$$\sigma_{\alpha 3} = 2\lambda_\alpha\lambda_3 \frac{\dfrac{\partial \tilde{W}}{\partial \zeta_\alpha}}{(\lambda_\alpha^2 - \lambda_3^2)} \boldsymbol{e}_\alpha \bullet \boldsymbol{A}\boldsymbol{e}_3\,, \qquad \alpha = 1,2\,. \tag{41}$$

Sufficient conditions to satisfy (33) may be obtained when (35)-(41) are substituted in in (33). However, in this paper, we shall not derive these conditions. This will be done in the near future. Necessary conditions are, however, not so straightforward (or impossible) to obtained. We note that the pressure p term does not appear in the expression when (35)-(41) are substituted in (33) because $tr(\boldsymbol{E}) = 0$.

References

[1] A.J.M. Spencer, Constitutive Theory of the Mechanics of Fiber Reinforced Composites, CISM Courses and Lectures No. 282. Wien: Springer, 1984.

[2] G. deBotton, L. Hariton and E.A. Socolsky, *Neo-hookean fiber-reinforced composites in finite elasticity*, J. Mech. Phys. Solids 54, 533-559, 2006.

[3] Z.Y. Guo, X.Q. Peng, and B. Moran, *A composites-based hyperelastic constitutive model for soft tissue with application to the human annulus fibroses*, J. Mech. Phys. Solids 54, 1952-1971, 2006.

[4] J. Lu and L. Zhang, *Physically motivated invariant formulation for transversely isotropic hyperelasticity*, Int. J. Solids Struct. 42, 6015-6031, 2005.

[5] M.H.B.M. Shariff, *Strain energy function for filled and unfilled rubberlike material*, Rubber Chem. Technol. 73, 1-21, 2000.

[6] R. Hill, *Constitutive inequalities for isotropic elastic solids under finite strain*, Proc. Roy. soc. Lond. A 314, 457-472, 1970.

[7] M.H.B.M. Shariff, *Nonlinear transversely isotropic solid: An alternative representation*, Q. J. Mech. Appl. Math., DOI:10.1093/qjmam/hbm028, 2008.

[8] K.C. Valanis and R.F. Landel, *The strain-energy function of hyperelastic material in terms of the extension ratios*, J. Appl. Phys. 38, 2997-3002, 1967.

[9] J.D. Humphrey, R.K. Strumph and F.C.P. Yin, *Determination of a constitutive relation for passive myocardium: II - Parameter Estimation*, Trans. ASME, 112, 340-346, 1990.