Approximate Solution of
Forced Korteweg-de Vries Equation

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Abstract Several findings on forced solitons generated by the forced Korteweg-de Vries equation (fKdV) are discussed in this paper. This equation has lost group symmetries due to the forcing term. The traditional group-theoretical approach can no longer generate analytic solution of solitons, because there are no infinitely many conservation laws. Approximate solution and numerical simulation seem to be the only way to solve fKdV equations. In this paper we show how approximate scheme can be used to solve the fKdV equation and generate uniform forced solitons. A detail derivation of the approximate solution was provided and various profiles of fKdV such as the depth of depression zone; h_d, amplitude; a_s, speed; s and the period; T_s of generation of forced uniform solitons was given.

Keywords Forced soliton, uniform soliton, soliton collision and forced Korteweg de-Vries equation.

Abstrak *Beberapa keputusan tentang penjanaan soliton paksaan oleh persamaan paksaan Korteweg-de Vries (fKdV) telah dibincangkan dalam kertas kerja ini. Sistem persamaan seperti ini telah hilang sifat simetri kumpulannya akibat dari gangguan atau paksaan ke atasnya. Kaedah teori kumpulan tidak lagi mampu memberikan penyelesaian secara analitik kerana tidak wujud lagi ketakterhinggaan banyaknya hukum keabadian. Dengan itu kaedah penyelesaian secara penghampiran dan berangka sahaja yang mampu menyelesaikannya. Dalam kertas kerja ini kita akan tunjukkan bagaimana penyelesaian secara penghampiran mampu menyelesaikan persamaan fKdV dan seterusnya menjana soliton paksaan seragam. Penyelesaian hampir telah diterbitkan secara terperinci dan beberapa profil bagi fKdV seperti kedalaman zon tertekan; h_d, amplitud; a_s, laju; s dan tempoh; T_s penjanaan soliton paksaan seragam telah diberikan.*

Katakunci *Soliton paksaan, soliton seragam, perlanggaran soliton dan persamaan paksaan Korteweg-de Vries.*

1   Introduction

In the last ten years, several researchers have conducted extensive studies on Korteweg-de Vries (KdV) equation and they were able to get free solitons generated, Shen (1993) [8]. With forcing terms added to the original KdV equation, it has lost group symmetries and the traditional group-theoretical approach can no longer generate analytical solution of solitons, because there is no infinitely many conservation laws. Approximate solution and numerical simulation seem to be the only way to solve fKdV equations, Shen(2002)[6].

When a fluid flow interacts with a topographic feature, and the fluid can support wave propagation, then there is the potential for waves to be generated upstream or downstream. In many cases when the topographic feature has a small amplitude, the situation can be successfully described by using a linearized theory and any nonlinear effects are determined as a small perturbation on the linear theory. However, when the flow is critical, that is, the system supports a long wave with zero group velocity in the reference frame of the topographic feature, then the linear theory failed and hence an intrinsically nonlinear theory need to be developed. It is now known that in many cases such a transcritical, weakly dispersive theory leads to a fKdV equation.

The first evidence of the existence of such solitons was provided by the celebrated discovery of the upstream radiated waves by a Caltech fluid mechanics group led by Wu T.Y. in 1982 [3]. This phenomenon is given in Figure (1). They claimed that these solitary waves are solitons.
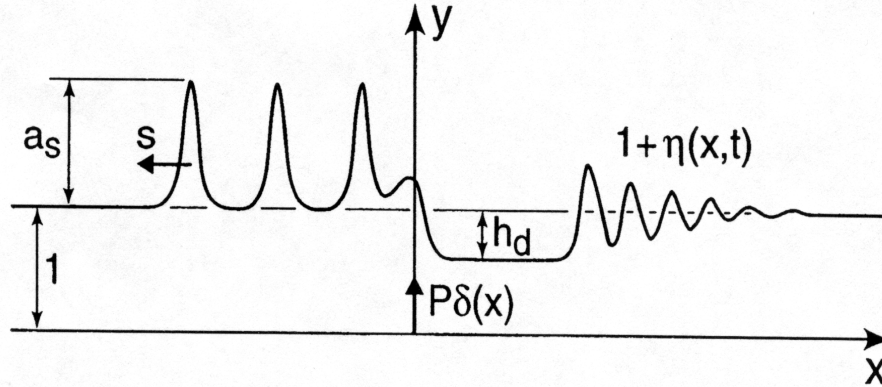

Figure 1: An illustration of the schematic solution $\eta(x,t)$ of fKdV for a fixed time t.

These waves have been successfully modelled by Shen in 1996 [9] and he was able to radiate these forced solitons. In free KdV equation, there exist infinitely many conservation laws but this is not true when we are dealing with fKdV; whereby certain symmetries such as the translation invariant property is broken or certain conservation laws, such as conservation of momentum is not satisfied. These equations have lost group symmetries due to forcing. The traditional group-theoretical approach can no longer generate analytical

solution of solitons because there is no infinitely many conservation laws. Approximate solution and numerical simulation seem to be the only way to solve the forced nonlinear evolution equations in asymmetric systems.

In Section 2 we will derive the approximate solitary wave solution of fKdV and provide the profiles of fKdV solitons. Conclusion and discussion are given in Section 3.

2   Approximate Solution of fKdV

Since there is no analytical solution for fKdV equation and we would like to understand the behavior of forced solitons in fKdV, therefore an approximate scheme will be developed to solve the fKdV model given by equation (1).

$$\eta_t + \lambda\eta_x + 2\alpha\eta\eta_x + \beta\eta_{xxx} = \frac{\gamma}{2}f'(x), \quad -\infty < x < \infty. \tag{1}$$

In this case, $\eta(x,t)$ describes the free surface profile of the water flows over a bump, λ measures the deviation of the bump speed from the shallow water velocity, γ is computed from the cross section area of the bump, whereas

$$f'(x) = \delta_x(x)$$

is an isolated forcing function of Dirac-delta function, x is the spatial coordinate along the channel, t is time, $\alpha < 0$ and $\beta < 0$ are constants. The control parameters in this model are the bump size parameter γ and the bump speed parameter λ. The initial condition for equation (1) is $\eta(x,0) = 0$ which is the water surface profile at rest. The solution consists of a forced soliton region generated upstream with amplitude; a_s and speed s, a depression region with depth h_d immediately on the lee side of the bump and a lee diminishing cnoidal wave further downstream. The schematic solution of equation (1) is shown in Figure (1).

2.1   Derivation of Approximate Solution of fKdV

The kth upstream soliton of the fKdV, Shen (1993),[8] may be expressed in the following form

$$\eta^{(k)}(x,t) = a_s\text{sech}^2\left\{\frac{\sqrt{3a_s}}{2}(x + st - \delta_k)\right\}, \tag{2}$$

where δ_k is the specific phase shift for the kth soliton, s is the upstream advancing speed of the solitons and $a_s = 2(\lambda + s)$ is the amplitude of the soliton.

For each soliton $\eta^{(k)}$, with the first three conservation laws one has

$$m_s^{(k)} = \int_{-\infty}^{\infty}\eta^{(k)}dx \quad = \quad 4\left[\frac{a_s}{3}\right]^{\frac{1}{3}} \quad \text{(mass)}$$

$$\int_{-\infty}^{\infty}(\eta^{(k)})^2dx = 8[\frac{2}{3}(\lambda + s)]^{\frac{3}{2}} \quad = \quad 8\left[\frac{a_s}{3}\right]^{\frac{3}{2}} \quad \text{(momentum)}$$

$$\int_{-\infty}^{\infty}[(\eta^{(k)})^3 + \frac{1}{3}(\eta_x^{(k)})^2]dx = \frac{32}{3}(\lambda + s)^{\frac{5}{2}} \quad = \quad \frac{4\sqrt{2}}{3}[a_s]^{\frac{5}{2}} \quad \text{(energy)}$$

Based upon the mass balance postulate that the upstream soliton mass comes solely from the downstream depression when time is sufficiently large, one can derive approximate expressions of the depression depth h_d, soliton amplitude a_s, soliton propagation speed s, and soliton generation period T_s in terms of the control parameters γ and λ.

In the stationary state with $\alpha = -\frac{3}{4}$ and $\beta = -\frac{1}{6}$, equation (1) can be reduced to;

$$\lambda\eta_x - \frac{3}{2}\eta\,\eta_x - \frac{1}{6}\eta_{xxx} = \frac{\gamma}{2}\delta_x(x). \tag{3}$$

In this case, we know that $\eta_t = 0$ (stationary state), $\eta(-\infty) = h_s$ and $\eta(\infty) = -h_d$. By letting $\eta(x) = \xi(x) + h_s$, equation(3) becomes

$$\lambda\xi_x - \frac{3}{2}(\xi + h_s)\xi_x - \frac{1}{6}\xi_{xxx} = \frac{\gamma}{2}\delta_x(x), \tag{4}$$

and thus gives us

$$\left[\lambda - \frac{3}{2}h_s\right]\xi_x - \frac{3}{2}\xi\xi_x - \frac{1}{6}\xi_{xxx} = \frac{\gamma}{2}\delta_x(x), \tag{5}$$

with $\xi(-\infty) = 0$ and $\xi(\infty) = -(h_s + h_d)$.

Equation (5) is only solvable when $\xi(x)$ is a smooth fall from the upstream zero solution to a downstream solitary wave tail. So $\lambda - \frac{3}{2}h_s < 0$ and $\xi(x) = 0$ for all x in the domain $(-\infty, 0)$. Integrating equation (5) in the domain $(0, \infty)$ gives

$$\left[\lambda - \frac{3}{2}h_s\right]\xi - \frac{3}{4}\xi^2 - \frac{1}{6}\xi_{xx} = \frac{\gamma}{2}\delta(x). \tag{6}$$

In the case $\delta(x) = 0$ if $x > 0$ and that reduce equation (6) is reduced to

$$\left[\lambda - \frac{3}{2}h_s\right]\xi - \frac{3}{4}\xi^2 - \frac{1}{6}\xi_{xx} = 0, \tag{7}$$

when $x > 0$, $\xi(0^+) = 0$; $\xi_x(0^+) = -3\gamma$ and $\xi(\infty) = -(h_s + h_d)$.

By integrating again equation (7) after multiplying by ξ_x the following expression is obtained

$$2[\lambda - \frac{3}{2}h_s]\xi^2 - \xi^3 - \frac{1}{3}\xi_x^2 + \frac{1}{3}(9\gamma^2) = 0,$$

which can be further simplified into

$$\frac{1}{3}\xi_x^2 = 2[\lambda - \frac{3}{2}h_s]\xi^2 - \xi^3 + 3\gamma^2. \tag{8}$$

Equation (8) is solvable only when the third order polynomial on the right hand side has a double roots. So,

$$2[\lambda - \frac{3}{2}h_s]\xi^2 - \xi^3 + 3\gamma^2 = 0. \tag{9}$$

On differentiating equation (9) once, we obtained

$$4[\lambda - \frac{3}{2}h_s]\xi - 3\xi^2 \;=\; 0.$$

So, a root of this equation is

$$\xi \;=\; \frac{4}{3}(\lambda - \frac{3}{2}h_s). \tag{10}$$

Since $\xi(\infty) = -(h_s + h_d)$,

$$h_d \;=\; h_s - \frac{4}{3}\lambda. \tag{11}$$

We know one of the roots of equation (9) is given by $r_0 = \frac{4}{3}(\lambda - \frac{3}{2}h_s)$ and if we substitute it into equation (9) we obtain

$$h_s \;=\; \frac{2}{3}\lambda + (\frac{3\gamma^2}{4})^{\frac{1}{3}}, \tag{12}$$

and

$$h_d \;=\; (\frac{3\gamma^2}{4})^{\frac{1}{3}} - \frac{2}{3}\lambda. \tag{13}$$

Equation (8) can now be written as

$$\frac{1}{3}\xi_x^2 = \xi(s_1 - \xi)(\xi - s_1 + s_2), \tag{14}$$

where the roots are $s_1 = r_1 - r_2 > 0$; $s_2 = r_1 - r_3 > 0$ and r_1, r_2, r_3 are roots of the equation. So thus equation (8) has a double roots and this give us $r_1 = r$; $r_2 = r_3 = r_0$ with $r_0 = \frac{4}{3}(\lambda - \frac{3}{2}h_s)$. Therefore $s_1 = r - r_0$ and $s_2 = r - r_0 = s_1$. So therefore equation (14) becomes

$$\frac{1}{3}\xi_x^2 \;=\; \xi^2(s_1 - \xi) \tag{15}$$
$$\xi_x^2 \;=\; 3\xi^2(s_1 - \xi)$$
$$\xi_x \;=\; \sqrt{3}\,\xi\,\sqrt{(s_1 - \xi)}$$
$$\frac{d\xi}{\sqrt{3}\xi\sqrt{(s_1 - \xi)}} \;=\; dx$$
$$\xi \;=\; s_1 \mathrm{sech}^2\frac{\sqrt{3s_1}}{2}(x). \tag{16}$$

Since $s_1 = h_s = \frac{2}{3}\lambda + (\frac{3\gamma^2}{4})^{\frac{1}{3}}$ so we can replace it in equation (16) to obtain

$$\xi = \left[\frac{2}{3}\lambda + (\frac{3\gamma^2}{4})^{\frac{1}{3}}\right]\mathrm{sech}^2\sqrt{\frac{3[\frac{2}{3}\lambda + (\frac{3\gamma^2}{4})^{\frac{1}{3}}]}{4}}\;\;(x). \tag{17}$$

In order to know more profiles related to fKdV, we will integrate equation (1) from $-\infty$ to 0^- with respect to x to obtain

$$\int_{-\infty}^{0^-} \eta_t\, dx + \lambda\left[\eta\right]_{-\infty}^{0^-} - \frac{3}{4}\left[\eta^2\right]_{-\infty}^{0^-} - \frac{1}{6}\left[\eta_{xx}\right]_{-\infty}^{0^-} = \left[\frac{\gamma}{2}\delta(x)\right]_{-\infty}^{0^-}. \tag{18}$$

But the term $\left[\dfrac{\gamma}{2}\delta(x)\right]_{-\infty}^{0^-}$ is always zero in the region of $(-\infty, 0^-)$ and $-\dfrac{1}{6}\Big[\eta_{xx}\Big]_{-\infty}^{0^-} = 0$ due to the "jump". With this in mind equation (18) will become

$$\left(\int_{-\infty}^{0^-} \eta\, dx\right)_t + \lambda\eta(0,t) - \frac{3}{4}\eta^2(0,t) = 0. \tag{19}$$

By letting

N =number of upstream solitons,

m_s =mass of one soliton and

T_s =period of generating one soliton.

We can now define the rate of change of mass as

$$\frac{d}{dt}\left(\int_{-\infty}^{0^-} \eta_1\, dx\right) = \frac{N\, m_s}{N\, T_s} = \frac{m_s}{T_s},$$

so equation (19) will become

$$\frac{m_s}{T_s} = -\lambda\eta(0,t) + \frac{3}{4}\eta^2(0,t). \tag{20}$$

By integrating [equation (1) multiply by $\eta(x,t)$] from $-\infty$ to 0^- with respect to x will yield

$$\int_{-\infty}^{0^-} \eta\eta_t\, dx \quad + \quad \lambda\int_{-\infty}^{0^-} \eta\eta_x\, dx - \frac{3}{2}\int_{-\infty}^{0^-} \eta^2\eta_x\, dx$$

$$- \quad \frac{1}{6}\int_{-\infty}^{0^-} \eta\eta_{xxx}\, dx = \frac{\gamma}{2}\int_{-\infty}^{0^-} \eta\delta_x(x)\, dx.$$

This can be further simplified into

$$\frac{d}{dt}\left(\int_{-\infty}^{0^-} \eta^2\, dx\right) \quad = \quad -\lambda\eta^2(0^-,t) + \eta^3(0^-,t)$$

$$+ \quad \frac{1}{3}\Big[\eta(0^-,t)\eta_{xx}(0^-,t)\Big] - \frac{1}{6}\Big[\eta_x^2(0^-,t)\Big]. \tag{21}$$

By denoting $\dfrac{d}{dt}\left(\displaystyle\int_{-\infty}^{0^-} \eta^2\, dx\right)$ as the rate of change of momentum which is equal to $\dfrac{N\, M_{hs}}{N\, T_s} = \dfrac{M_{hs}}{T_s}$; equation (21) becomes

$$\frac{M_{hs}}{T_s} \quad = \quad -\lambda\eta^2(0^-,t) + \eta^3(0^-,t)$$

$$+ \quad \frac{1}{3}\Big[\eta(0^-,t)\eta_{xx}(0^-,t)\Big] - \frac{1}{6}\Big[\eta_x^2(0^-,t)\Big]. \tag{22}$$

By intuitive observations, we will then make the following approximation

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \eta(0^-, t) \, dt = h_s,$$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \eta^2(0^-, t) \, dt = h_s^2,$$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \eta^3(0^-, t) \, dt = h_s^3,$$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \eta(0^-, t) \, \eta_{1xx}(0^-, t) \, dt = 0,$$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \eta_{1x}^2(0^-, t) \, dt = 0.$$

By using the above approximation we will simplify equation (20) and obtained

$$\frac{m_s}{T_s} = -\lambda \, h_s + \frac{3}{4} \, h_s^2. \tag{23}$$

By using the same approximation we will simplify equation (22) and obtained

$$\frac{M_{hs}}{T_s} = -\lambda \, h_s^2 + h_s^3. \tag{24}$$

By dividing equation (24) by equation (23) we get the important relationship

$$\frac{M_{hs}}{m_s} = \frac{-\lambda \, h_s^2 + h_s^3}{-\lambda \, h_s + \frac{3}{4} \, h_s^2} = \frac{-\lambda \, h_s + h_s^2}{-\lambda + \frac{3}{4} \, h_s}. \tag{25}$$

But we know that the mass one soliton is given by

$$m_s = \int_{-\infty}^{\infty} \eta^{(k)} dx = 4 \left[\frac{a_s}{3} \right]^{\frac{1}{3}},$$

and the momentum of one soliton is given by

$$M_{hs} = \int_{-\infty}^{\infty} (\eta^{(k)})^2 dx = 8 \left[\frac{2}{3}(\lambda + s) \right]^{\frac{3}{2}} = 8 \left[\frac{a_s}{3} \right]^{\frac{3}{2}},$$

therefore

$$\frac{M_{hs}}{m_s} = \frac{8 \left[\frac{a_s}{3} \right]^{\frac{3}{2}}}{4 \left[\frac{a_s}{3} \right]^{\frac{1}{3}}} = \frac{2}{3} a_s. \tag{26}$$

By equating equation (26) to equation (25) we get the amplitude of forced solitons as given by

$$\begin{aligned}
\frac{2}{3} a_s &= \frac{-\lambda \, h_s + h_s^2}{-\lambda + \frac{3}{4} \, h_s} \\
a_s &= \frac{3}{2} \left[\frac{h_s(-\lambda + h_s)}{-\lambda + \frac{3}{4} \, h_s} \right] \\
a_s &= 2 \left[\frac{(h_d + \frac{4}{3}\lambda)(h_d + \frac{1}{3}\lambda)}{h_d} \right].
\end{aligned} \tag{27}$$

Again we will integrate equation (1) from $-\infty$ to x_D with respect to x, that will yield

$$
\int_{-\infty}^{x_D} \eta_t \, dx \quad + \quad \lambda \int_{-\infty}^{x_D} \eta_x \, dx - \frac{3}{2} \int_{-\infty}^{x_D} \eta \eta_x \, dx
$$
$$
- \quad \frac{1}{6} \int_{-\infty}^{x_D} \eta_{xxx} \, dx = \frac{\gamma}{2} \int_{-\infty}^{x_D} \delta_x(x) \, dx. \tag{28}
$$

Equation (28) then becomes

$$
\int_{-\infty}^{x_D} \eta_t \, dx + \lambda \Big[\eta\Big]_{-\infty}^{x_D} - \frac{3}{4}\Big[\eta^2\Big]_{-\infty}^{x_D} - \frac{1}{6}\Big[\eta_{xx}\Big]_{-\infty}^{x_D} = \frac{\gamma}{2}\Big[\delta(x)\Big]_{-\infty}^{x_D}. \tag{29}
$$

By substituting the approximation that we have made earlier; equation (29) will yield

$$
\frac{m_s}{T_s} + \lambda\, h_s - \frac{3}{4}\, h_s^2 \;\; = \;\; 0
$$
$$
\frac{m_s}{T_s} \;\; = \;\; \frac{3}{4}\, h_s^2 - \lambda\, h_s
$$
$$
T_s \;\; = \;\; \frac{m_s}{\frac{3}{4}\, h_s^2 - \lambda\, h_s}
$$
$$
T_s \;\; = \;\; \frac{4\big[\frac{a_s}{3}\big]^{\frac{1}{2}}}{\frac{3}{4}\, h_s^2 - \lambda\, h_s}
$$
$$
T_s \;\; = \;\; \frac{16}{3}\left[\frac{2(h_d + \frac{1}{3}\lambda)}{3h_d^3(h_d + \frac{4}{3}\lambda)}\right]^{\frac{1}{2}}. \tag{30}
$$

2.2   Profile of fKdV Solitons

From the above derivations, which is based upon the mass balance postulate that the upstream soliton mass comes solely from the downstream depression zone when time is sufficiently large, one can derive approximate expressions of the depression depth h_d, soliton amplitude a_s, soliton propagation speed s, and soliton generation period T_s, in terms of the control parameters γ and λ, Shen(2002) [6] given as

$$
h_d \;\; = \;\; \{\frac{3}{4}\gamma^2\}^{\frac{1}{3}} - \frac{2}{3}\lambda, \tag{31}
$$
$$
a_s \;\; = \;\; \frac{2(h_d + \frac{4}{3}\lambda)(h_d + \frac{1}{3}\lambda)}{h_d}, \tag{32}
$$
$$
s \;\; = \;\; \frac{a_s}{2} - \lambda, \tag{33}
$$
$$
T_s \;\; = \;\; \frac{16}{3}\left[\frac{2(h_d + \frac{1}{3}\lambda)}{3h_d^3(h_d + \frac{4}{3}\lambda)}\right]^{\frac{1}{2}}. \tag{34}
$$

These profiles are important features in the study and research on fKdV. With these new findings, we are able to provide another way or scheme to solve fKdV since the analytical solution of fKdV is still unknown.

2.3   Dirac-delta Forcing in fKdV Equation.

The forcing function in equation (1) given by $\frac{\gamma}{2}f'(x)$ which may due to the bottom topography of the fluid domain (such as a bump on the bottom of a two dimensional channel), or due to an external pressure on the free surface (such as the wind stress on the surface of an ocean). By taking $f'(x) = \delta_x(x)$ which is a Dirac-delta forcing as in Shen (1996), [9], $\gamma = 1$ and if we keep $\lambda = 0$ so as to remain in the transcritical region we are able to observe the generations of forced uniform solitons as given by Figure 2 which shows the 3D plot of the forced uniform solitons propagations. At a specific time $t = 10, t = 20$, $t = 30$ and $t = 40$ we observe that the solution to equation (1) was given by Figure 3, Figure 4, Figure 5 and Figure 6 respectively. The approximate solution of the nonlinear partial differential equation (1) with the depth of depression zone h_d, amplitude a_s, speed s and generation period T_s of the matured uniform forced solitons can be calculated from the above approximate expression. For $\lambda = 0$ and $\gamma = 1.0$ the profiles are:

$$h_d = 0.9086, \quad a_s = 1.8172, \quad s = 0.9086, \quad \text{and} \quad T_s = 5.0280.$$
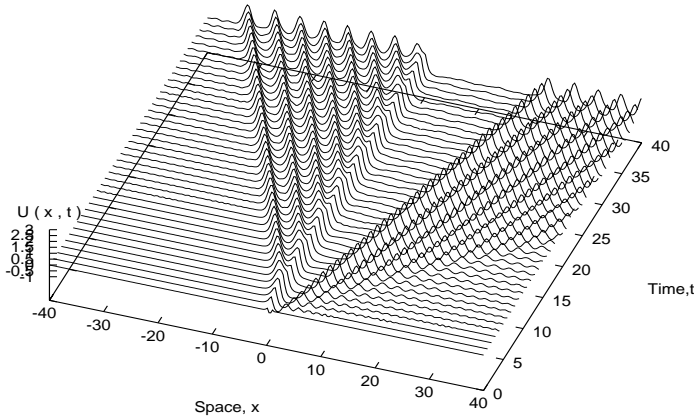

Figure 2: Uniform Solitons Generated by Dirac delta Forcing (3D Plot).

In Figure 3, when $t = 10 < 2T_s$ we can only observe one matured soliton and another soliton emerging and will be matured at $t = 10.056$.
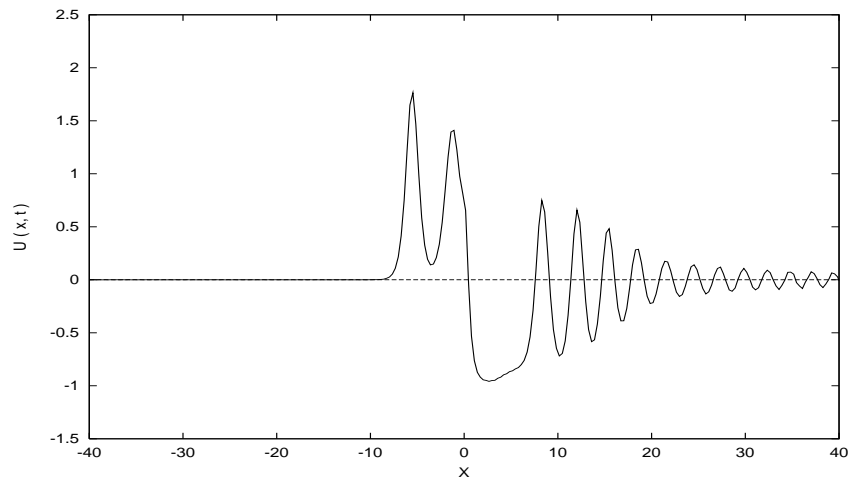
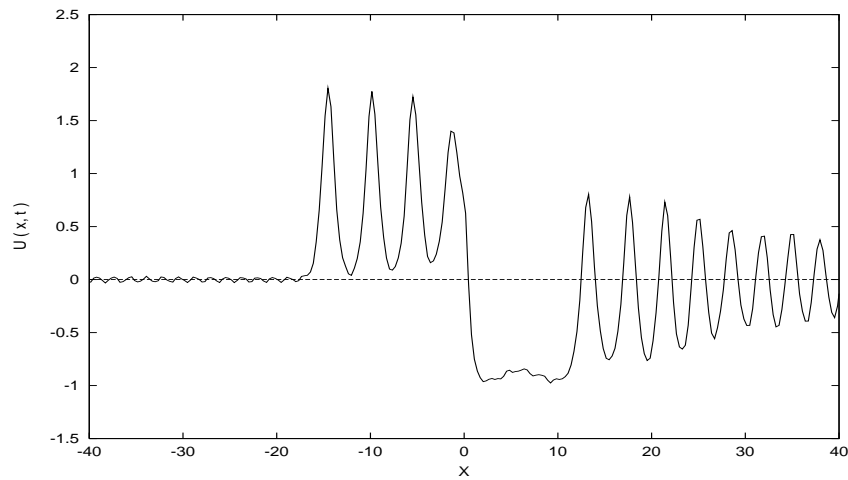Figure 3: Generations of Forced Solitons at $t = 10$ (2D Plot).

Figure 4: Generations of Forced Solitons at $t = 20$ (2D Plot).

In Figure 5, when $t = 30 < 6T_s$ we can only observe only 5 matured soliton and another soliton emerging and will be matured at $t = 30.168$.
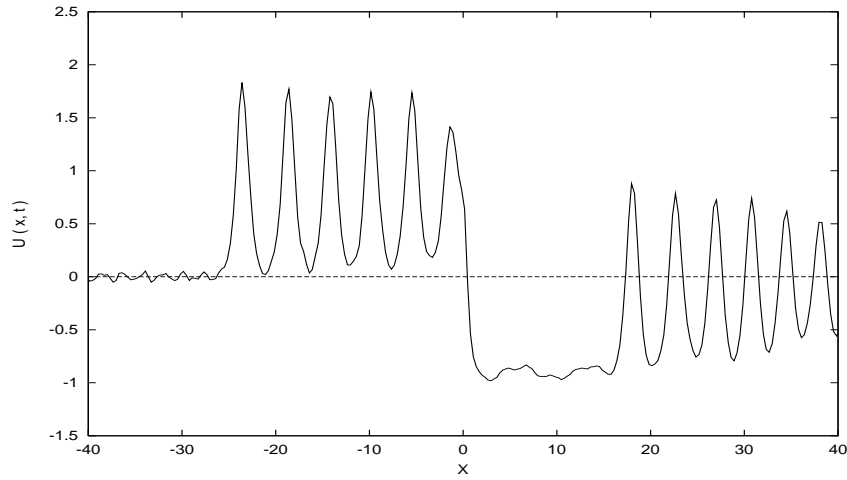
Figure 5: Generations of Forced Solitons at $t = 30$ (2D Plot).

In Figure 6, when $t = 40 < 8T_s$ we can only observe only 7 matured soliton and another soliton emerging and will be matured at $t = 40.224$.
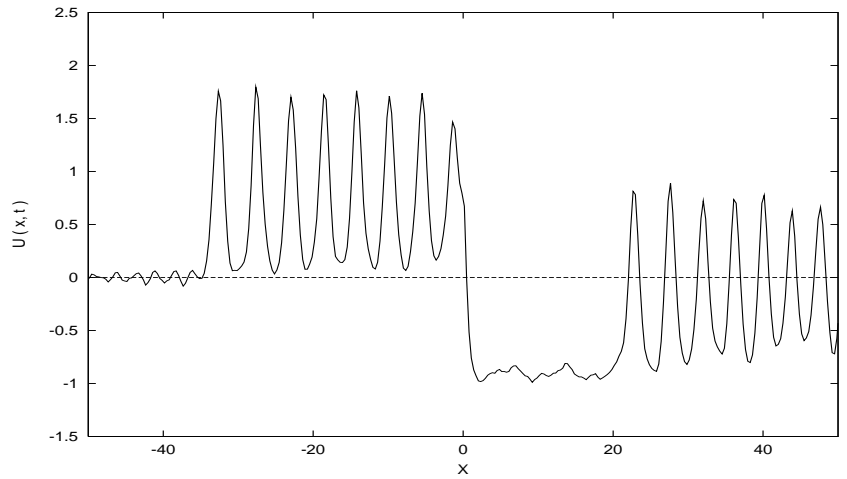

Figure 6: Generations of Forced Solitons at $t = 40$ (2D Plot).

3 Conclusion and Discussion

Approximate solution can be another scheme to solve fKdV equation. By knowing the various characteristics of those forced solitons generated we can be very sure of its nature. We also know the amplitude, speed, depth of the depression zone and the period of generation of forced uniform solitons. With known values of the control parameter λ

and γ we do generates forced uniform solitons and we know the profiles of each forced uniform solitons generated. With this new findings, we have another resource to confirm our research results with other schemes namely the numerical simulation or even with the analytical solution of fKdV if ever found.

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