AN INTEGRAL EQUATION METHOD FOR CONFORMAL MAPPING OF DOUBLY CONNECTED REGIONS VIA THE KERZMAN-STEIN AND THE NEUMANN KERNELS

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To my parents,

Thanks for the unfailing love and affection.

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ABSTRACT

An integral equation method based on the Kerzman-Stein and the Neumann kernels for conformal mapping of doubly connected regions onto an annulus is presented. The theoretical development is based on the boundary integral equations for conformal mapping of doubly connected regions derived by Murid and Razali (1999). However, the integral equations are not in the form of Fredholm integral equation and no numerical experiments are reported. If some information on the zero and singularity of the mapping function is known, then the integral equations can be reduced to the numerically tractable Fredholm integral equations involving the unknown inner radius. For numerical experiments, discretizing the integral equations lead to a system of non-linear equations. The system obtained is solved simultaneously using Newton's iterative method. Further modification of the integral equations of Murid and Razali (1999) has lead to an efficient and numerically tractable integral equations which involve the unknown inner radius. These integral equations are feasible for all doubly connected regions with smooth boundaries regardless of the information on the zeroes and singularities of the mapping functions. Discretizing the integral equations lead to an over determined system of non-linear equations which is solved using an optimization technique. Numerical implementations on some test regions are also presented.

ABSTRAK

Satu kaedah persamaan kamiran berdasarkan inti Kerzman-Stein dan Neumann untuk pemetaan mensebentuk bagi rantau berkait ganda dua keseluruh anulus dipersembahkan. Pembangunan teori berdasarkan persamaan kamiran sempadan bagi pemetaan konformal rantau berkait ganda dua yang dibangunkan oleh Murid and Razali (1999). Bagaimanapun, persamaan kamiran itu bukan merupakan persamaan kamiran Fredholm dan tiada kajian berangka dijalankan. Jika maklumat bagi pensifar dan pensingular bagi fungsi pemetaan diketahui, persamaan kamiran boleh diturunkan kepada persamaan kamiran Fredholm yang mudah diuruskan secara berangka yang melibatkan jejari dalam. Mendiskretkan persamaan kamiran menghasilkan sistem persamaan tak linear yang diselesaikan secara serentak menggunakan kaedah lelaran Newton. Usaha pengubahsuaian selanjutnya terhadap persamaan kamiran Murid and Razali (1999) membawa kepada persamaan kamiran yang efisien dan mudah diuruskan secara berangka yang melibatkan jejari dalam. Persamaan kamiran ini boleh dilaksanakan untuk semua rantau berkait ganda dua dengan sempadan licin tanpa memerlukan maklumat pensifar dan pensingular bagi fungsi pemetaan. Mendiskretkan persamaan kamiran tersebut menghasilkan sistem persamaan tak linear terlebih tentu yang diselesaikan menggunakan teknik pengoptimuman. Pelaksanaan berangka terhadap beberapa rantau ujikaji juga dipersembahkan.

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LIST OF SYMBOLS

A	-	Kerzman-Stein kernel
C	-	Set of complex numbers
$e, \exp$	-	Exponential($e \approx 2.718...$)
H	-	Cauchy kernel
i	-	$\sqrt{-1}$
Im	-	Imaginary part
N	-	Neumann kernel
R	-	Riemann mapping function
$\Re$	-	Set of real numbers
Re	-	Real
S	-	Szegö kernel
U	-	Unit disk
ϵ	-	Epsilon (small number $0 \leq \epsilon < 1$)
Γ	-	Curve (boundary of Ω)
Γ_0	-	Outer boundary of a doubly connected region
Γ_1	-	Inner boundary of a doubly connected region
π	-	Pi ($\pi \approx 3.142...$)
Ω	-	Connected region
$\in$	-	Component
$\subset$	-	Subset
∇	-	Gradient
∇^2	-	Laplace operator
$\sum$	-	Sum
$\int$	-	Integration

CHAPTER 1

INTRODUCTION

1.1 Background and Rationale

Exact solutions of boundary value problems for simple regions like disks and annuli can be determined with relative ease even when the boundary conditions are complicated. However, for regions with complex structure, solving a boundary value problem can be quite difficult, even for a simple problem such as the Dirichlet problem. One approach to solve this difficult problem is to conformally transform a given region onto a simpler region where they can be solved easily. Thus, if a Dirichlet problem can be solved for a region Ω, it can be also solved for all regions that are conformally equivalent to Ω. The success of this method rest on the ability to determine the one-to-one analytic function which performs the transformation. One of the most important classical theorem in conformal mapping is the Riemann mapping theorem which claims the existence and the uniqueness of conformal map that transforms any simply connected region onto a unit disk. Thus, the ideal standard region for simply connected regions is a unit disk. The Dirichlet problem on the unit disk can be solved by the Poisson's integral formula. The conformal map which handle such transformation is known as Riemann map, briefly R. One can extend the statement of the mapping function for the regions in the Riemann mapping theorem to the boundary by Osgood-Carathéodory theorem (Henrici, 1986, p. 346). Actually, a unit disk is not the only best standard region. We can select whichever region that renders the problem simplest: usually a washer, wedge, wall or upper-half plane (Saff and Snider, 2003, p. 371).

As compared to the simply connected regions, conformal mapping of multiply connected regions suffer from severe limitations. There is no exact equivalent of the Riemann mapping theorem that hold in multiply connected case. This implies that there is no guarantee that any two multiply connected regions of the same connectivity are conformally equivalent to each other. If we wish to map a doubly connected region Ω onto a standard region, such standard region likewise must also be a doubly connected region. This follows from the fact that conformal mapping preserves the order of connectivity. The circular annulus, $r_1 < |z| < r_2$ naturally recommend itself as a standard region. The outer radius of the annulus can be assumed to be equal to one without loss of generality. The inner radius (conformal radius), μ is not known in advance and have to be determined in the course of the numerical solution. Since μ is uniquely determined by Ω, then it follows that there is no single annulus that can be conformally equivalent to all doubly connected regions. Two annuli are only conformally equivalent to each other if and only if the ratio of two radii, $M = r_2/r_1$ are the same for both annulus. The quantity $M = r_2/r_1$ is known as the modulus of an annulus. For some discussion on multiply connected regions, see, e.g., Nehari (1952) and Kythe (1998).

Conformal mapping has been a familiar tool of science and engineering for generation. The practical limitation has always been that only for certain special regions are exact conformal maps known. While for the rest, they have to be computed numerically. Since 1960's, computer's speed have improved drastically and a mapping problem can now be solved numerically in seconds. The statement of Symm in 1966 that have been restated by Wegmann (2005): "When a conformal mapping, purporting to simplify solution of applied mathematics, can be obtained only by numerical means, it is often considered to have outlived its usefulness" is no longer true. Many methods have been proposed in the numerical approximation of the conformal mapping function such as expansion methods, integral equation methods, iterative methods, osculation methods, Cauchy-Riemann equation methods, charge simulation method, and methods of small parameter. See, e.g., Papamichael et al. (1986), Wegmann (1986), Kerzman and Trummer (1986), Henrici (1986), Razali et al. (1997) and Amano and Okano (1999). Interest in numerical conformal mapping began to grow in the late seventies and culminated in 1986 in Trefethen's collection of 15 articles (Trefethen, 1986).

The integral equation methods which deal with computing the boundary correspondence function, $\theta(t)$ has been regarded with great favor for solving numerical conformal mapping. This correspondence refer to a particular parametric representation of the boundary, Γ. See, e.g., Kerzman and Trummer (1984), Henrici (1986), Razali et al. (1997) and Aptekarev et al. (2004).

If a boundary correspondence function is known, the value of both the mapping function f and the inverse mapping function f^{-1} may be calculated by quadrature at arbitrary interior points of their regions of definition. By means of the Cauchy's integral formula, it can be shown that for simply connected case (Henrici, 1986, p. 380)

$$w = R(z) = \frac{1}{2\pi \mathrm{i}} \int_0^\beta \frac{e^{i\theta(t)} z'(t)}{z(t) - z} \, dt, \quad z \in \Omega, \quad z(t) \in \Gamma$$

and

$$z = R^{-1}(w) = \frac{1}{2\pi}\int_0^{\beta} \frac{z(t)\theta'(t)}{1 - e^{-\mathrm{i}\theta(t)}w}\,dt, \quad w \in U, \quad z(t) \in \Gamma.$$

Hence the Riemann mapping functions R and R^{-1} are uniquely determined by the function θ and θ'. This fact is greatly utilized in numerical conformal mapping. Various classical and modern integral equation which represent the boundary correspondence function $\theta(t)$ and $\theta'(t)$ have been derived such as Warschawski and Gerschgorin integral equations. Typically, the boundary is discretized at n point, so that the integral equation reduces to an algebraic system of linear equations.

The boundary correspondence function $\theta(t)$ can also be generalized to the doubly connected case that maps the region bounded by Γ_0 and Γ_1 to an annulus $A := \mu < |w| < 1$. This correspondence refer to a particular parametric representation of the outer and inner boundaries, $\Gamma = \Gamma_0 \cup \Gamma_1$. If the boundary correspondence functions $\theta_0(t)$ and $\theta_1(t)$ are known, both $f(z)$ and $f^{-1}(w)$ are calculated easily for arbitrary $z \in \Omega$ and $w \in A$, for instance by Cauchy's integral formula. In the following formulas, the strong singularity of Cauchy's integral formula for $|w|$ near 1 or μ can be weakened by the integration by parts (Henrici,1986, p. 462):

$$f(z) = \frac{1}{2\pi\mathrm{i}}\int_0^{\beta_0} e^{\mathrm{i}\theta_0(t)}\frac{z_0'(t)}{z_0(t)}\,dt - \frac{1}{2\pi\mathrm{i}}\int_0^{\beta_0} e^{\mathrm{i}\theta_0(t)}\mathrm{Log}\left(1 - \frac{z}{z_0(t)}\right)\theta_0'(t)\,dt$$
$$+ \frac{1}{2\pi\mathrm{i}}\int_0^{\beta_0} e^{\mathrm{i}\theta_1(t)}\mathrm{Log}\left(1 - \frac{z_1(t)}{z}\right)\theta_1'(t)\,dt, \quad z_0(t) \in \Gamma_0,\ z_1(t) \in \Gamma_1.$$

For the inverse mapping function, we have

$$z = f^{-1}(w) = \frac{1}{2\pi}\int_0^{\beta_0} z_0(t)\theta_0'(t)\,dt - \frac{1}{2\pi\mathrm{i}}\int_0^{\beta_0} z_0'(t)\mathrm{Log}\left(1 - \frac{w}{e^{\mathrm{i}\theta_0(t)}}\right)\,dt$$
$$- \frac{1}{2\pi\mathrm{i}\mu}\int_0^{\beta_0} z_1'(t)\mathrm{Log}\left(1 - \frac{e^{\mathrm{i}\theta_1(t)}}{w}\right)\,dt, \quad z_0(t) \in \Gamma_0,\ z_1(t) \in \Gamma_1.$$

One method for the construction of the Riemann mapping function which can be used for a simply connected region is by means of reproducing kernel function. At present, two useful types of reproducing kernel function are the Szegö kernel function (briefly S) and the Bergman kernel function (briefly B). The relationship between the two kernels are given by (see, e.g., Razali et al. (1997))

$$B(z,a) = 4\pi S(z,a)^2. \tag{1.1}$$

For solving the conformal mapping problem, it is sufficient to compute the Szegö kernel or the Bergman kernel functions due to the fact that there are classical relations between these two kernels and the Riemann mapping function which are given by the following two equations:

$$R(z) = \frac{1}{\mathrm{i}} T(z) \frac{S(z,a)^2}{|S(z,a)|^2}, \qquad z \in \Gamma,$$
$$R(z) = \frac{1}{\mathrm{i}} T(z) \frac{B(z,a)}{|B(z,a)|}, \qquad z \in \Gamma.$$

An integral equation of the second kind that expressed the Szegö kernel as the solution is first introduced by Kerzman and Trummer (1986) using operator-theoretic approach. Henrici (1986) gave a markedly different derivation of the Kerzman-Stein-Trummer integral equation based on a function-theoretic approach. The discovery of the Kerzman-Stein-Trummer integral equation, briefly KST integral equation, for computing the Szegö kernel later leads to the formulation of an integral equation for the Bergman kernel as given by Razali et al. (1997). Both integral equations can be used effectively for numerical conformal mapping of simply connected regions.

By using a boundary relationship satisfied by a function analytic in a doubly connected region, Murid and Razali (1999) extended the construction to a doubly connected region and obtained a boundary integral equation for conformal mapping of doubly connected regions. Special realization of this boundary integral equation are the integral equations for conformal mapping of doubly connected regions via the Kerzman-Stein and the Neumann kernels. However, the integral equations are not in the form of Fredholm integral equations and no numerical experiments are reported. In this research, we shall analyse these integral equations and figure out ways to overcome their drawback.

1.2 Problem Statement

Our research problem is to formulate new integral equations for conformal mapping of doubly connected regions with smooth boundaries via the Kerzman-Stein and the Neumann kernels which are feasible for numerical purposes.

1.3 Research Objectives

The objectives of this research are:

1. To formulate new, numerically tractable integral equations for conformal mapping of doubly connected regions via the Kerzman-Stein and the Neumann kernels.
2. To use the integral equations to solve numerically the problem of conformal mapping of doubly connected regions onto an annulus.

1.4 Scope of the Study

The research will focus on the theoretical and numerical computation of the doubly connected regions onto an annulus. The theoretical development will be based on the integral equations for conformal mapping of doubly connected regions via the Kerzman-Stein and the Neumann kernels derived by Murid and Razali (1999). The drawbacks of the integral equations are that they are not in the form of Fredholm integral equations and no numerical experiments are reported in that paper. The aim of this research is to obtain new, numerically tractable integral equations for conformal mapping of doubly connected regions via the Kerzman-Stein and the Neumann kernels.

1.5 Outline of Thesis

The thesis is organized into six chapters. The introductory Chapter 1 details some discussion on the background and rationale of research, description to the problem, objectives of research, scope of the study and chapter organization.

Chapter 2 gives an overview of methods for conformal mapping, in particular of doubly connected regions. We discuss some theories of the Riemann mapping function as well as the conformal mapping of multiply connected regions. We also present some exact conformal mapping of doubly connected regions for certain special regions like frame of limacon, elliptic frame, frame of Cassini's oval and circular frame. Some numerical methods that have been proposed in the literature for the numerical evaluation for conformal mapping of doubly connected regions are also presented in the last section of Chapter 2. The boundary integral equation for conformal mapping of doubly connected regions derived by Murid and Razali (1999) is also presented.

In Chapter 3, we introduce the integral equations for conformal mapping of doubly connected regions via the Kerzman-Stein and the Neumann kernel that are the special realizations of the boundary integral equation derived by Murid and Razali (1999). The drawbacks of these integral equations are that they contained the unknown inner radius, μ and not in the form of numerically tractable Fredholm integral equations. In this chapter, we show that if some information on the zeroes and singularities of the mapping function are known, then the integral equations via the Kerzman-Stein and the Neumann kernel can be reduced to the numerically tractable Fredholm integral equations. Numerical experiments on some test regions are also presented.

In Chapter 4, we show how the integral equation for conformal mapping of doubly connected regions via the kerzman-Stein kernel studied in Chapter 3 can be further modified that totally avoid any prior knowledge on the zeroes and singularities of a mapping function. Numerical experiments on some test regions are also presented.

In Chapter 5, the approach in Chapter 4 is applied to develop an integral equation for conformal mapping of doubly connected regions via the Neumann kernel. Numerical experiments on some test regions are also reported.

In Chapter 6, we give some conclusions of this study and some suggestions for further study.

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