ABSTRACT

This research determines solutions of the exterior Neumann problem in multiply connected regions by using the method of boundary integral equations. The method depends on reducing the boundary value problem in question to an integral equation on the boundary of the domain of the problem, and then solves this integral equation. Our approach in this research is to convert the exterior Neumann problem into the exterior Riemann-Hilbert problem. The exterior Riemann-Hilbert problem is then solved using a uniquely solvable Fredholm integral equation on the boundary of the region. The kernel of this integral equation is the Neumann kernel. Once this equation is solved, the auxiliary function and the solution of the exterior Neumann problem can be obtained. As an examination of the proposed method, some numerical examples for some different test regions are presented. These examples include comparison between the numerical results and the exact solutions. Numerical examples reveal that the present method offers an effective solution technique for the exterior Riemann-Hilbert problems when the boundaries are sufficiently smooth.

ABSTRAK

Penyelidikan ini menyelesaikan masalah Neumann luaran atas satah terkait berganda dengan menggunakan kaedah persamaan kamiran. Kaedah ini bergantung kepada penurunan masalah nilai sempadan ke persamaan kamiran atas sempadan rantau dan seterusnya menyelesaikan persamaan kamiran tersebut. Pendekatan yang digunakan dalam penyelidikan ini adalah menukarkan masalah Neumann luaran kepada masalah Riemann-Hilbert luaran. Kemudian, masalah Riemann-Hilbert luaran akan dijelmakan kepada satu persamaan kamiran Fredholm yang mempunyai penyelesaian unik dalam rantau tersebut. Inti bagi persamaan kamiran ini adalah inti Neumann. Apabila persamaan ini diselesaikan, jawapan untuk masalah Neumann luaran boleh dicari. Untuk mengkaji kaedah yang dipersembahkan, beberapa contoh berangka melibatkan beberapa rantau terpilih disampaikan. Perbandingan berangka juga diberi antara keputusan berangka dengan penyelesaian tepat. Contoh berangka ini memperlihatkan bahawa kaedah yang dipersembahkan memberi teknik penyelesaian yang efektif untuk masalah Riemann-Hilbert luaran apabila sempadan tidak mempunyai penjuru.

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CHAPTER 1

RESEARCH FRAMEWORK

1.1 Introduction

The problem of finding a function which is harmonic in a specified domain and which satisfies prescribed conditions on the boundary of that domain is abound in applied mathematics. If the values of the function are prescribed along the boundary, the problem is known as a boundary value problem of the first kind, or a Dirichlet problem. If the values of the normal derivatives of the function are prescribed on the boundary, then the boundary value problem is of the second kind, or a Neumann problem.

The Neumann problem is a class of fundamental boundary value problems for analytic functions. It is a boundary value problem for determining a harmonic function, $u(x, y)$ interior or exterior to a region with prescribed values of its normal derivatives, $\frac{\partial u}{\partial n}$ on the boundary.

In the simplest form, the Neumann problem consists in finding a function $u(x, y)$ which satisfying the following conditions:

i. u is harmonic in Ω.

ii. u is continuous and differentiable in Ω and Γ.

iii. $\Delta u(z) = 0$ for all z in Ω.

iv. For u on the boundary,

$$\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\eta(t)} = \gamma(t), \qquad \eta(t) \in \Gamma,$$

where $\frac{\partial}{\partial \mathbf{n}}$ denotes differentiation in the direction of the exterior normal. This condition is known as a Neumann condition (Asmar, 2002).

The Neumann problem is often solved by conformal mapping for arbitrary simply connected region. The basic technique is to transform a given boundary value problem in the *xy* plane into a simpler one in the *uv* plane where they can be solved easily. The desired answer can be obtained by transforming back to the original region.

Besides conformal mapping, there are still many techniques from various mathematics fields for solving the Neumann problem such as finite difference method, finite element method, iterative method, collocation method and boundary integral equation method. The Neumann problem can be used in many applications such as heat problems in an insulated plate, potential of flow around airfoil and electrostatic potential in a cylinder.

The purpose of this research is to consider the boundary integral equation methods for solving the exterior Neumann problems on a multiply connected region. This approach has the advantage of reducing the dimensionality of the problem.

1.2 Background of the Problem

A function that solves Laplace's equation is called a harmonic function, or sometimes called potential function in physics and engineering. The problem that we are interested about is classified as a boundary value problem and it asks for a harmonic function on a defined region and satisfies a boundary condition related to the normal derivative of this function on the boundary on that region. Such problem is called a Neumann problem (after the German mathematician Carl Gottfried Neumann (1832-1925)) and sometimes referred to as a Dirichlet problem of the second kind (Asmar,2002).

The boundary integral equation method is a classical method for solving the Neumann problem. The classical boundary integral equations for the Neumann problem are derived by representing the solutions of Neumann problems as the potential of single layer. Due to the non-uniqueness of the Neumann problem, the boundary integral method leads to non-uniquely solvable integral equation. However, it is still possible to overcome non-uniqueness by imposing additional conditions which are consistent with the nature of the problem. Adding the constraints to the original equation yielding a modified uniquely solvable integral equation which kernel is slightly different form the Neumann kernel [Atkinson (1997), Henrici (1986)]. Extra calculations are needed for determining the boundary values of the solutions of the Neumann problems from the solutions of the integral equation. Kulkarni et al. (2004) has solved the problem in a bounded doubly connected region using a numerical method called the Boundary Walk Method.

Nasser (2007) has reduced the interior and exterior Neumann problems to equivalent Dirichlet problems by using Cauchy-Riemann equations and it is uniquely solvable. Then, the boundary integral equation involving generalized Neumann problem kernel is derived for the Dirichlet problem. Ummu Tasnim (2009) has reduced the Neumann problem on a simply connected region to Riemann-Hilbert problem. The

Riemann-Hilbert problem is then formulated as a boundary integral equation with a Neumann kernel which is uniquely solvable. Later, Ejaily (2009) extended Ummu Tasnim's work by formulated a new boundary integral equation with Neumann kernel for solving the interior Neumann problem in multiply connected regions with smooth boundaries. He has reduced the Neumann problem into the equivalent Riemann-Hilbert problem from which an integral equation is constructed. Recently, the previous research by Azlina (2009) has reduced the exterior Neumann problem in a simply connected region to exterior Riemann-Hilbert problem by using Cauchy-Riemann equations. This leads to an integral equation with the Neumann kernel. But there is no discussion on the reduction of exterior Neumann problem on a multiply connected region to the exterior Riemann-Hilbert problem given in Azlina (2009).

1.3 Statement of Problem

The question now arises whether it is possible to reduce the exterior Neumann problem on a multiply connected region to the exterior Riemann-Hilbert problem using Cauchy-Riemann equations. The next question is whether it is possible to derive a uniquely solvable integral equation related to the exterior Riemann-Hilbert problem on multiply connected region.

1.4 Objectives of the Study

This study embarks on the following objectives:

i. To reduce the exterior Neumann problem on a multiply connected region to the exterior Riemann-Hilbert problem.

ii. To derive the boundary integral equation for the exterior Riemann-Hilbert problem on a multiply connected region.
iii. To determine the solvability of the formulated integral equation.
iv. To provide a numerical technique for solving the boundary integral equation using Mathematica.

1.5 Scope of the Study

In this research, we are mainly concern on the exterior Neumann problem on multiply connected planar regions with smooth boundaries. We wish to solve the Neumann problem using boundary integral equation method without conformal mapping.

1.6 Significance of the Study

The purpose of the study is to develop a new integral equation for solving the exterior Neumann problem on a multiply connected region. The method is based on recent investigation by Azlina (2009) for the exterior Neumann problem on a simply connected region and on the interplay of Riemann-Hilbert problems and Fredholm integral equations with generalized Neumann kernel [Wegmann et al. (2005), Wegmann and Nasser (2008)]. This approach will enrich the numerical procedure for solving exterior Neumann problem on a multiply connected region and enhance the numerical effectiveness of solving it.