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AN INTEGRAL EQUATION METHOD FOR SOLVING NEUMANN PROBLEMS ON SIMPLY AND MULTIPLY CONNECTED REGIONS WITH SMOOTH BOUNDARIES

(KAEDAH PERSAMAAN KAMIRAN UNTUK PENYELESAIAN MASALAH NEUMANN ATAS RANTAU TERKAIT RINGKAS DAN BERGANDA DENGAN SEMPADAN LICIN)

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ABSTRACT

(*Keywords: Laplace's equation, Dirichlet problem, Neumann problem, multiply connected region, boundary integral equation, generalized Neumann kernel*)

This research presents several new boundary integral equations for the solution of Laplace's equation with the Neumann boundary condition on both bounded and unbounded multiply connected regions. The integral equations are uniquely solvable Fredholm integral equations of the second kind with the generalized Neumann kernel. The complete discussion of the solvability of the integral equations is also presented. Numerical results obtained show the efficiency of the proposed method when the boundaries of the regions are sufficiently smooth.

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ABSTRAK

(*Katakunci: Persamaan Laplace, masalah Dirichlet, masalah Neumann, rantai terkait berganda, persamaan kamiran, inti Neumann teritlak*)

Penyelidikan ini menghasilkan beberapa persamaan kamiran baru untuk penyelesaian persamaan Laplace dengan syarat sempadan Neumann atas rantau terkait berganda yang terbatas dan tak terbatas. Persamaan kamiran ini merupakan persamaan kamiran Fredholm jenis kedua dengan inti Neumann teritlak yang memiliki penyelesaian unik. Perbincangan mengenai kebolehselesaian persamaan kamiran ini turut disampaikan. Keputusan berangka yang diperoleh menunjukkan keberkesanan kaedah yang dipersembahkan jika sempadan rantau adalah licin secukupnya.

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CHAPTER 1

RESEARCH FRAMEWORK

1.1 General Introduction

A popular source of integral equations has been the study of elliptic partial differential equations. It is also well known from books on the equations of mathematical physics that the basic boundary value problems for the Laplace equation are solved by means of the so-called potentials of simple and double layers (Gakhov, 1966). There are two of them; the Dirichlet problem and the Neumann problem. Given a boundary value problem for an elliptic partial differential equation over region D, the problem can often be reformulated as an equivalent integral equation over the boundary of D. Such a reformulation is called a boundary integral equation (Atkinson, 1997). As an example of such a reformulation, Carl Neumann investigated the solvability of some boundary integral equation reformulations for Laplace's equation, thereby also obtaining solvability results for Laplace's equation.

Various boundary integral equation reformulations have long been used as means of solving Laplace's equation numerically, although this approach has been less popular than the use of finite difference and finite element methods. Since 1970, there has been a significant increase in the popularity of using boundary integral equations to solve Laplace's equation and many other elliptic equations, including the biharmonic equation, the Helmholtz equation, the equations of linear elasticity and the equations for Stokes' fluid flow (Atkinson, 1997).

Solving the Neumann problem by the boundary integral equation method is one of the classical methods. The classical boundary integral equations for the Neumann problems are the two Fredholm integral equations of second kind with the Neumann kernel. The solutions of the Neumann problems are represented as the potential of a single layer as the way of deriving these boundary integral equations (Nasser, 2007). In general, the Neumann kernel usually appears in the integral equations related to the Dirichlet problem, the Neumann problem and conformal mappings (Ismail, 2007).

However, the integral equation for the interior Neumann problem is not uniquely solvable since the lack of unique solvability for the Neumann problem itself. The simplest way to deal with the lack of uniqueness in solving the integral equation is to introduce an additional condition (Atkinson, 1997) which lead to a unique solution. There are other ways of converting integral equation to a uniquely solvable equation. By using Kelvin transform, the interior Neumann problem can be converted to an equivalent exterior problem. This seems to be a very practical approach in solving interior Neumann problems, but it does not appear to have been used much in the past.

Recently, Nasser (2007) has developed two uniquely solvable second kind Fredholm integral equations with the generalized Neumann kernel that can be used to solve the interior and exterior Neumann problems on simply connected regions with smooth boundaries.

1.2 Background of the Problem

A Neumann problem is a boundary value problem for determining a harmonic function $u(x, y)$ interior or exterior to a region with prescribed values of its normal derivative $\partial u/\partial n$ on the boundary.

Applications of Neumann problems abound in classical mathematical physics. Some examples are heat problems in an insulated plate, electrostatic potential in a cylinder and potential of flow around airfoil. If the region is a disk, exact solution formula for the Neumann problem is known and the formula is called Dini's formula. For arbitrary simply connected region, the solution formula requires conformal mapping.

A more direct approach that avoid conformal mapping is the boundary integral equation method for solving the Neumann problem. Recently, Nasser (2007) has developed two integral equations with the generalized Neumann kernel that can provide the boundary values of the solution of the Neumann problem. Nasser's method is based on an earlier works by Murid and Nasser (2003) and Wegmann et al. (2005) related to the Riemann problem and Dirichlet problem.

1.3 Statement of the Problem

Through the previous research by Nasser (2007), the interior and exterior for Neumann problems are reduced to equivalent Dirichlet problems by using Cauchy-Riemann equations and it is uniquely solvable. Then, boundary integral equations are derived for the Dirichlet problems.

The question now arises whether it is possible to derive integral equations for Neumann problems based on Murid and Nasser (2003) with the reduction to Riemann-Hilbert problems using Cauchy-Riemann equations. This research has answered the question in affirmative.

1.4 Objectives of the Research

The objectives of this research are to:

i. study the Neumann problem and the Riemann-Hilbert problem as well as integral equations for Riemann-Hilbert problem on multiply connected regions.
ii. reduce the interior and exterior Neumann problems to the equivalent Riemann-Hilbert problems.
iii. derive the boundary integral equations related to the interior and exterior Riemann-Hilbert problems.
iv. determine the solvability of the attained integral equations related to the Riemann-Hilbert problems.
v. perform numerical calculations for solving the boundary integral equations using softwares such as MATHEMATICA or MATLAB.

1.5 Importance of the Research

Knowledge on complex analysis in general and boundary integral equations in particular is still growing. There have been several studies on boundary integral equations with Neumann kernel related to Neumann problem. This research has developed non-singular integral equations with continuous kernel associated to Neumann problem on multiply connected regions with smooth boundaries. Furthermore, the analysis of solvability for these integral equations are determined as well.

This approach has enriched the integral equation method for solving Neumann problem and enhances the numerical effectiveness of solving it. Thus, the integral equation for Neumann problem of this proposed research will assist scientists and engineers of our nation and abroad working with mathematical models involving Neumann problem.

1.6 Scope of the Research

This research is mainly on the theoretical reduction of the Neumann problem to Riemann-Hilbert problem. The Neumann problem is then solved numerically using the integral equation related to Riemann-Hilbert problem. We are mainly concerned on the interior and exterior Neumann problems on multiply connected regions with smooth boundaries.

1.7 Outline of Report

This project consists of seven chapters. The introductory Chapter 1 details some discussion on the background of the problem, problem statement, objectives of research, importance of the research, scope of the study and chapters organization.

Chapter 2 presents some auxiliary materials related to the Neumann problem, the Riemann-Hilbert problems as well as integral equation for Riemann-Hilbert problems. In this chapter, we reduce the interior Neumann problem into the interior Riemann-Hilbert problem and construct the boundary integral equation for solving it. We discuss the question on how to treat the integral equations numerically. Some numerical examples are presented to show the effectiveness of the method.

Chapter 3 focuses on the development of a numerical method for the exterior Neumann problem in a simply connected smooth region. Firstly, we reduce the exterior Neumann problem to the exterior Riemann-Hilbert problem. Then, the boundary integral equation for the Neumann problem is derived based on the exterior Riemann-Hilbert problem. Numerical implementations of the derived integral equation are also presented.

In Chapter 4, we extend the results of Chapter 2 to reduce the Neumann problem to the Riemann-Hilbert problem in multiply connected region, and then derive an integral equation with the

Neumann kernel related to the Riemann-Hilbert problem (briefly, RH problem). This integral equation is the Fredholm integral equation of the second kind. Solvability of the integral equation is also discussed. Numerical experiments on some test regions are also reported.

Chapter 5 deals with the reduction of exterior Neumann problem on a multiply connected region to the exterior Riemann-Hilbert problem. Thus this chapter extends the results of Chapter 3. We show how to reduce the exterior Neumann problem on multiply connected region into the exterior Riemann-Hilbert problem and derive the boundary integral equation for solving it. Then, we provide a numerical technique for solving the boundary integral equation and present some numerical examples on several test regions.

Finally the concluding Chapter 6 contains a summary of findings and achievements.

CHAPTER 2

AN INTEGRAL EQUATION METHOD FOR SOLVING NEUMANN PROBLEMS ON SIMPLY CONNECTED SMOOTH REGIONS

2.1 Introduction

A Neumann problem is a boundary value problem of determining a harmonic function interior or exterior to a region with prescribed values of its normal derivative on the boundary. Applications of Neumann problems abound in classical mathematical physics. Some examples are heat problems in an insulated plate, electrostatic potential in a cylinder, potential flow around airfoil. If the region is a disk, exact solution formula for the Neumann problem is known and the formula is called Dini's formula. For arbitrary simply connected region, the solution formula requires conformal mapping. A more direct approach that avoid conformal mapping is the boundary integral equation method for solving Neumann problem.

The boundary integral equation method is one of the classical methods for solving the Neumann problem, see, for example, the books by Atkinson (1997) and Henrici (1986). Some classical boundary integral equations for the Neumann problems are the Fredholm integral equations of second kind with the Neumann kernel. These integral equations are derived by representing the solutions of the Neumann problems as the potential of a single layer. However, the integral equation for the interior Neumann problem is not uniquely solvable since the lack of unique solvability for the Neumann problem itself. The simplest way to deal with the lack of uniqueness in solving the integral equation is to introduce additional conditions (Atkinson, 1997).

Recently, Nasser (2007) proposes a new method to solve the interior and the exterior Neumann problems in simply connected regions with smooth boundaries. The method is based on two uniquely solvable Fredholm integral equations of the second kind with the generalized Neumann kernel.

This chapter is organized as follows: Section 2.2 presents some auxiliary materials related to the Neumann problem, the Riemann-Hilbert problems as well as integral equation for Riemann-Hilbert problems. In Section 2.3, we reduce the interior Neumann problem into the interior Riemann-Hilbert problem and construct the boundary integral equation for solving it. We will discuss the question on how to treat the integral equations numerically in Section 2.4. Some numerical examples are presented in Section 2.5. In Section 2.6, a short conclusion is given.

2.2 Auxiliary Material

Let Ω be a bounded simply connected Jordan region with $0 \in \Omega$ (see Figure 2.1). The boundary $\Gamma = \partial \Omega$ is assumed to have a positively oriented parameterization $\eta(t)$ where $\eta(t)$ is a 2π-periodic twice continuously differentiable function with $\dot{\eta}(t) = d\eta / dt \neq 0$. The parameter t need not be the arc length parameter. The exterior of Γ is denoted by Ω^{-}. For a fixed α with $0 < \alpha < 1$, the Hölder space H^{α} consists of all 2π-periodic real functions which are uniformly Hölder continuous with exponent α. It becomes a Banach space when provided with the usual

Hölder norm. A Hölder continuous function $\hat{h}$ on Γ can be interpreted via $h(t)=\hat{h}(\eta(t))$ as a Hölder continuous function *h* of the parameter *t* and vice versa.

Let $A(t)$ be a continuous differentiable 2π-periodic function with $A\neq 0$. We define two real kernels *N* and *M* by

$$M(\tau,t)=\frac{1}{\pi}\operatorname{Re}\left(\frac{A(\tau)}{A(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(\tau)}\right), \tag{2.1}$$

$$N(\tau,t)=\frac{1}{\pi}\operatorname{Im}\left(\frac{A(\tau)}{A(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(\tau)}\right). \tag{2.2}$$

The kernel $N(\tau,t)$ is called a *generalized Neumann kernel* formed with A and η. When $A=1$, the kernel *N* is the Neumann kernel which arises frequently in the integral equations for potential theory and conformal mapping.

Lemma 2.1 (Wegmann et al., 2005)

a) *The kernel* $N(\tau,t)$ *is continuous with*

$$N(t,t)=\frac{1}{\pi}\operatorname{Im}\left(\frac{1}{2}\frac{\ddot{\eta}(t)}{\dot{\eta}(t)}-\frac{\dot{A}(t)}{A(t)}\right). \tag{2.3}$$

b) *The kernel* $M(\tau,t)$ *has the representation*

$$M(\tau,t)=-\frac{1}{2\pi}\cot\frac{\tau-t}{2}+M_1(\tau,t), \tag{2.4}$$

with the continuous kernel M_1 *which takes on the diagonal the values*

$$M_1(t,t)=\frac{1}{\pi}\operatorname{Re}\left(\frac{1}{2}\frac{\ddot{\eta}(t)}{\dot{\eta}(t)}-\frac{\dot{A}(t)}{A(t)}\right). \tag{2.5}$$

Let $\mathcal{N}$ and $\mathcal{M}_1$ be the Fredholm integral operators associate with the continuous kernels *N* and M_1, i.e.,

$$\left(\mathcal{N}\mu\right)(\tau) = \int_{0}^{2\pi} N(\tau,t)\,\mu(t)\,dt, \tag{2.6}$$

$$\left(\mathcal{M}_1\mu\right)(\tau) = \int_{0}^{2\pi} M_1(\tau,t)\,\mu(t)\,dt. \tag{2.7}$$

Let also $\mathcal{M}$ and $\mathcal{K}$ be the singular integral operators

$$\left(\mathcal{M}\mu\right)(\tau) = \int_{0}^{2\pi} M(\tau,t)\,\mu(t)\,dt, \tag{2.8}$$

$$\left(\mathcal{K}\mu\right)(\tau) = \frac{1}{2\pi}\int_{0}^{2\pi} \mu(t)\cot\frac{\tau-t}{2}\,dt. \tag{2.9}$$

The integrals in (2.8) and (2.9) are principal value integrals. The operator $\mathcal{K}$ is known as the Hilbert transform. The operators $\mathcal{N}$, $\mathcal{M}$, $\mathcal{M}_1$ and $\mathcal{K}$ are bounded in H^{α} and map H^{α} into H^{α}. It follows from (2.4) that

$$\mathcal{M} = \mathcal{M}_1 - \mathcal{K}. \tag{2.10}$$

Lemma 2.2 (Wegmann et al., 2005)

a) Let N be the generalized Neumann kernel formed with $A = 1$ *and* η*. Then* $\lambda = -1$ *is not an eigenvalue of N.*

b) Let N be the generalized Neumann kernel formed with $A = \eta$ *and* η*. Then* $\lambda = 1$ *is not an eigenvalue of N.*

2.2.1 The Neumann Problem

Interior Neumann problem. Let **n** be the exterior normal to Γ and let $\gamma \in H^{\alpha}$ be a given function such that

$$\int_{0}^{2\pi} \gamma(\tau) \left| \dot{\eta}(\tau) \right| d\tau = 0 . \tag{2.11}$$

Find the function *u* harmonic in Ω, Hölder continuous on Γ and on the boundary Γ, *u* satisfies the boundary condition (see Figure 2.1)

$$\left. \frac{\partial u}{\partial \mathbf{n}} \right|_{\eta(t)} = \gamma(t), \quad \eta(t) \in \Gamma \ . \tag{2.12}$$

The interior Neumann problem is uniquely solvable up to an additive real constant (Atkinson, 1997). This arbitrary real constant is specified by assuming $u(\eta(0)) = 0$.

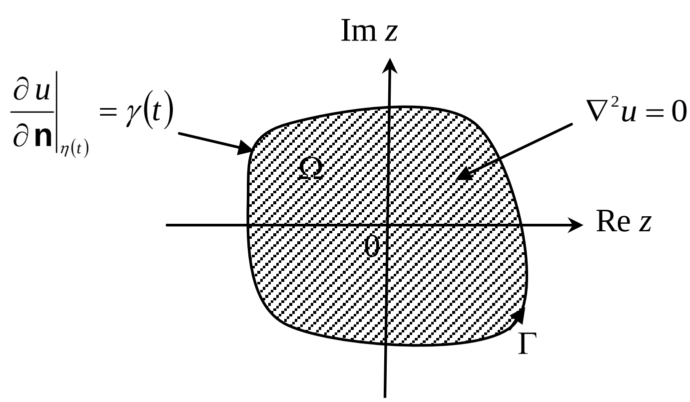

Figure 2.1: A Neumann problem in region Ω.

Lemma 2.3 (Atkinson, 1997)

The interior Neumann problem with the condition $u(\eta(0)) = 0$ *is uniquely solvable.*

2.2.2 The Riemann-Hilbert Problem

Interior Riemann-Hilbert problem. Let $C \in H^{\alpha}$ and A be given functions. Find a function F analytic in Ω, continuous on the closure $\overline{\Omega}$, such that the boundary values F^{+} from inside Γ satisfy

$$a(t)u^{+}(t) - b(t)v^{+}(t) = C(t). \tag{2.13}$$

Another frequently used form of writing the boundary condition is

$$\operatorname{Re}\left[\, A(t)F^{+}(\eta(t)) \,\right]\Big|_{\eta(t)} = C(t), \quad \eta(t) \in \Gamma, \quad A(t) = a(t) + ib(t). \tag{2.14}$$

If $C(t) \neq 0$, then this equation is a non-homogeneous boundary condition, while if $C(t) = 0$, we have a homogeneous boundary condition, i.e.,

$$\operatorname{Re}\left[\, A(t)F^{+}(\eta(t)) \,\right]\Big|_{\eta(t)} = 0, \quad \eta(t) \in \Gamma. \tag{2.15}$$

Equation (2.13) is also equivalent to

$$F^{+}(\eta(t)) = -\frac{\overline{A(t)}}{A(t)}\overline{F^{+}(\eta(t))} + \frac{2C(t)}{A(t)}, \tag{2.16}$$

where

$$A(t) = a(t) + ib(t). \tag{2.17}$$

So, we call equation (2.14) as the non-homogeneous interior Riemann-Hilbert problem and equation (2.15) as the homogeneous interior Riemann-Hilbert problem.

The solvability of the Riemann-Hilbert problem in a region Ω depends upon the geometry of the simply connected region Ω with smooth boundary as well as upon the index of the function $A(t)$ with respect to the boundary $\Gamma = \partial\Omega$ which is denoted by

$$\kappa = \text{ind}_\Gamma(A). \tag{2.18}$$

It is also regarded as the index of the Riemann-Hilbert problem (Gakhov, 1966).

Suppose that Γ is a smooth Jordan curve and $A(t)$ is a continuous non-vanishing function given on Γ. The index of the function $A(t)$ with respect to the curve Γ is defined to be the increment of its argument in traversing the curve Γ in the positive direction, divided by 2π. Then, the index of $A(t)$ can be written in the form

$$\kappa = \text{ind}_\Gamma(A) = \frac{1}{2\pi}[\arg A(t)]_\Gamma . \tag{2.19}$$

The index can be expressed by the integral

$$\kappa = \text{ind}_\Gamma(A) = \frac{1}{2\pi}\int_\Gamma d \arg A(t) = \frac{1}{2\pi i}\int_\Gamma d \ln A(t), \tag{2.20}$$

where the integral is understood in the sense of the Stieltjes (Gakhov, 1966). If the function $A(t)$ is continuously differentiable on Γ, then

$$\kappa = \text{ind}_\Gamma(A) = \frac{1}{2\pi i}\int_\Gamma d \ln A(t) = \frac{1}{2\pi i}\int_\Gamma \frac{\dot{A}(t)}{A(t)}\, dt . \tag{2.21}$$

Since Γ is closed and $A(t)$ is a non-vanishing continuous function on Γ, the index $\kappa = \text{ind}_\Gamma(A)$ is an integer.

2.2.3 Integral Equation for Interior Riemann-Hilbert Problem

Theorem 2.1 (Wegmann et al., 2005)

If F is the solution of the interior Riemann-Hilbert problem (2.14) *with boundary values*

$$A(t)F^{+}(\eta(t)) = C(t) + i\mu(t), \tag{2.22}$$

then the imaginary part μ *in* (2.22) *satisfies the integral equation*

$$\mu - \mathcal{N}\mu = -\mathcal{M}C\,. \tag{2.23}$$

Theorem 2.2 (Wegmann et al., 2005)

a) *If* $\kappa \le 0$ *then the integral equation*

$$\mu - \mathcal{N}\mu = \Psi\,, \tag{2.24}$$

has a solution if and only if $\Psi \in R^{-}$,where

$$R^{+} = \left\{ C \in H^{\alpha} : C(t) = \operatorname{Re}\left[\ A(t)G^{-}(\eta(t))\ \right],\ G \text{ analytic in } \Omega^{-},\ \ G(\infty) = 0 \right\}.$$

b) *If* $\kappa > 0$ *then the integral equation* (2.24) *has a unique solution for any* $\Psi \in H^{\alpha}$.

c) *Equation* (2.23) *is solvable for each* $C \in H^{\alpha}$.

2.3 Reduction of the Neumann Problem to the Riemann-Hilbert Problem

Suppose that u is a solution of the interior Neumann problem. Since u is harmonic function in Ω, then u has a harmonic conjugate v in Ω. Then $f = u + iv$ is analytic in $\Omega \cup \Gamma$ with derivative

$$f' = u_x + iv_x \ . \tag{2.25}$$

The directional derivative of f^+ in the direction of the outer unit normal vector to the path Γ is given by (Husin, 2009)

$$\left.\frac{\partial f^+(\eta)}{\partial \mathbf{n}}\right|_{\eta(t)} = \left.\left(\frac{\partial u}{\partial \mathbf{n}} + i\frac{\partial v}{\partial \mathbf{n}}\right)\right|_{\eta(t)} . \tag{2.26}$$

Assuming $\eta(t) = x(t) + iy(t) \in \Gamma$ and applying $\dfrac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}$ and

$\mathbf{n} = \dfrac{y'(t)}{|\dot{\eta}(t)|}\vec{i} + \dfrac{-x'(t)}{|\dot{\eta}(t)|}\vec{j} = \mathbf{n}_x\vec{i} + \mathbf{n}_y\vec{j}$, we obtain

$$\left.\frac{\partial f^+(\eta)}{\partial \mathbf{n}}\right|_{\eta(t)} = \left.\left(u_x + iv_x\right)\mathbf{n}_x + i\left(v_y - iu_y\right)\mathbf{n}_y\right|_{\eta(t)} . \tag{2.27}$$

Applying (2.25) and Cauchy-Riemann equations, we get

$$\left.\frac{\partial f^+(\eta)}{\partial \mathbf{n}}\right|_{\eta(t)} = \left. f^{+\prime}(\eta)\left(\mathbf{n}_x + i\mathbf{n}_y\right)\right|_{\eta(t)} . \tag{2.28}$$

Taking the real parts of both sides and applying (2.12), the above equation becomes

$$\gamma(t) = \left.\mathrm{Re}\left[\ \mathbf{n}\ f^{+\prime}(\eta)\ \right]\right|_{\eta(t)} . \tag{2.29}$$

After substituting $\mathbf{n} = \dfrac{-i\dot{\eta}(t)}{|\dot{\eta}(t)|}$ into (2.29), we get

$$\gamma(t)|\dot{\eta}(t)| = \mathrm{Re}\left[-i\dot{\eta}(t) f^{+\prime}(\eta(t)) \right]\Big|_{\eta(t)} \quad . \tag{2.30}$$

Letting $g^{+} = -i\, f^{+\prime}$ and $\psi(t) = \gamma(t)|\dot{\eta}(t)|$ we obtain

$$\mathrm{Re}\left[\dot{\eta}(t) g^{+}(\eta(t)) \right]\Big|_{\eta(t)} = \psi(t), \tag{2.31}$$

which is the interior Riemann-Hilbert problem. Comparison of (2.31) with (2.14) yields $A(t) = \dot{\eta}(t)$, $F^{+}(\eta(t)) = g^{+}(\eta(t))$, and $C(t) = \psi(t)$.

2.3.1 Integral Equation for Solving Interior Neumann Problem

By Theorem 2.1, if $g^{+}(\eta(t))$ is the solution of the interior Riemann–Hilbert problem (2.31) with boundary values

$$\dot{\eta}(t) g^{+}(\eta(t)) = \psi(t) + i\mu(t), \tag{2.32}$$

then the imaginary part μ in (2.32) satisfies the integral equation

$$\mu - \mathcal{N}\mu = -\mathcal{M}\psi \,, \tag{2.33}$$

i.e.

$$\mu(\tau) - \int_0^{2\pi} N(\tau,t)\mu(t)\,dt = -\int_0^{2\pi} M(\tau,t)\psi(t)\,dt. \tag{2.34}$$

The solvability of the Riemann-Hilbert problem (2.31) is determined by the index of the function $A=\dot{\eta}(t)$ which is defined as the winding number of A with respect to 0. Applying (2.21) with $A=\dot{\eta}(t)$, we get

$$\kappa=\frac{1}{2\pi i}\int_{\Gamma}\frac{1}{\dot{\eta}(t)}d\dot{\eta}(t). \tag{2.35}$$

By means of the Cauchy Integral Formula, we get $\kappa=1$.

Thus, by Theorem 2.2, integral equation (2.33) has a unique solution for $-M\psi\in H^{\alpha}$ and integral equation (2.33) is solvable for each $C\in H^{\alpha}$.

2.4 Numerical Implementation

Since the functions $A=\dot{\eta}(t)$ and $\eta(t)$ are 2π-periodic, the integral operator $\mathcal{N}$ in (2.6) can be best discretized on an equidistant grid by the trapezoidal rule, i.e., the integral operator $\mathcal{N}$ is discretized by the Nyström method (Atkinson, 1997).

Define the *n* equidistant collocation points t_i by

$$t_i=(i-1)\frac{2\pi}{n},\quad i=1,2,...,n. \tag{2.36}$$

Then, using the Nyström method with the trapezoidal rule to discretize the integral equation (2.33), we obtain the linear system

$$\mu_n(t_i)-\frac{2\pi}{n}\sum_{j=1}^{n}N(t_i,t_j)\mu_n(t_j)=-(\mathcal{M}\psi)(t_i), \tag{2.37}$$

for $i=1,2,...,n$ where μ_n is an approximation to μ. Note that the kernels *N* and *M* in (2.37) are formed with $A=\dot{\eta}$.

Defining the matrix $Q = [Q_{ij}]$ and vector $\boldsymbol{x} = [x_i]$ and $\boldsymbol{y} = [y_i]$ by

$$Q_{ij} = \frac{2\pi}{n} N\left(t_i, t_j \right), \;\; x_i = \mu_n\left(t_i\right), \;\; y_i = (\mathcal{M}\psi)\left(t_i\right),$$

equation (2.37) can be rewritten as an n by n system

$$\left(I - Q\right)\boldsymbol{x} = -\boldsymbol{y}. \tag{2.38}$$

For the calculation on the right-hand side of (2.37), we first calculate

$$\psi\left(t_i \right) = \gamma\left(t_i \right) |\, \dot{\eta}\left(t_i \right) |, \;\; i = 1,\; 2, \ldots, n. \tag{2.39}$$

Then, by using (2.10), we have

$$\left(\mathcal{M}\psi\right)\left(t_i \right) = \left(\mathcal{M}_1\psi\right)\left(t_i \right) - \left(\mathcal{K}\psi\right)\left(t_i \right) \tag{2.40}$$

for $i = 1,\; 2, \ldots, n$. The values of $\left(\mathcal{K}\psi\right)\left(t_i \right)$ will be calculated directly by using MATHEMATICA package, i. e., **CauchyPrincipalValue**, while the values of $\left(\mathcal{M}_1\psi\right)\left(t_i \right)$ will be calculated by using the trapezoidal rule, i. e.,

$$\left(\mathcal{M}_1\psi\right)\left(t_i \right) = \frac{2\pi}{n} \sum_{j=1}^{n} M_1\left(t_i, t_j \right)\psi\left(t_j \right), \tag{2.41}$$

for $i = 1,\; 2, \ldots, n$. Finally, (2.32) implies

$$-i\dot{\eta}\left(t_i \right) f^{+}{}'\left(\eta\left(t_i \right)\right) = \psi\left(t_i \right) + i\mu_n\left(t_i \right).$$

Clearly the equation gives the solution of the Neumann problem since $f^{+}\left(\eta\right) = u + iv$. By obtaining f', we can obtain the function f by the Fundamental Theorem of Complex Integration which in turn gives the solution u of the Neumann problems.

2.5 Examples

Example 2.1

For our examples, we use two boundary curves: an ellipse and the oval of Cassini. For ellipse, the boundary has the parametrization (see Figure 2.2).

$$\Gamma_1 : \eta(t) = \cos t + i5\sin t\,, \quad 0 \le t \le 2\pi\,,$$

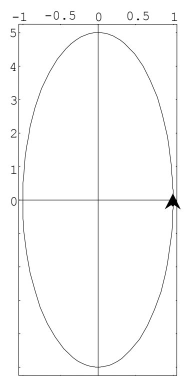

Figure 2.2: An ellipse for Example 2.1

We choose a unique solution of the Neumann problem with condition $u(\eta(0)) = 0$ as

$$u(z) = e^{x}\cos y - e\,, \quad z = x + iy \in \Omega\,. \tag{2.42}$$

For the above Neumann problem, the function $\gamma(t)$ is given by

$$\gamma(t) = \left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\eta(t)}. \tag{2.43}$$

i. e.,

$$\gamma(t) = e^{\cos t}\cos(5\sin t)\left[\frac{5\cos t}{\sqrt{\sin^2 t + 25\cos^2 t}}\right] - e^{\cos t}\sin(5\sin t)\left[\frac{\sin t}{\sqrt{\sin^2 t + 25\cos^2 t}}\right].$$

We now give a verification of (2.31).

$$\psi(t)=\gamma(t)|\,\dot{\eta}(t)\,|=5e^{\cos t}\cos t\cos\left(5\sin t\right)-e^{\cos t}\sin t\sin\left(5\sin t\right) \ .$$

Now, we consider the left-hand side of equation (2.31) for the same region. We define the analytic function as

$$f\left(z\right)=f\left(x,y\right)=u(x,y)+iv\left(x,y\right),$$

where $v(x,y)=e^{x}\mathbf{sin}y$ is a harmonic conjugate of $u(x,y)=e^{x}\cos y-e$

$$f^{+}\left(\eta(t)\right)=e^{\cos t}\cos\left(5\sin t\right)-e+ie^{\cos t}\sin\left(5\sin t\right). \tag{2.44}$$

Thus

$$f^{+}{'}\left(\eta(t)\right)=u_{x}+iv_{x}\big|_{\eta(t)}=e^{\cos t}\cos\left(5\sin t\right)+ie^{\cos t}\sin\left(5\sin t\right). \tag{2.45}$$

Multiplying this result with $\dot{\eta}(t)\left(-i\right)$, we have

$$\dot{\eta}(t)\left(-i\right)f^{+}{'}\left(\eta(t)\right)=\left[\,-\sin t+i5\cos t\,\right]\left(e^{\cos t}\sin\left(5\sin t\right)-ie^{\cos t}\cos\left(5\sin t\right)\right).$$

Therefore, the left-hand side of equation (2.31) is

$$\mathrm{Re}\left[\ \dot{\eta}(t)(-i)f^{+}{}'(\eta(t))\ \right]=5e^{\cos t}\cos t\cos(5\sin t)-e^{\cos t}\sin t\sin(5\sin t),$$

which is equal to the right-hand side of (2.31). This completes the verification.

Example 2.2

For the oval of Cassini, the boundary parametrization is

$$\Gamma_2 : \eta(t) = R(t)e^{it}, \quad 0 \le t \le 2\pi,$$

where $R(t) = 2.5 + 2\cos 2t$ (see Figure 2.3).

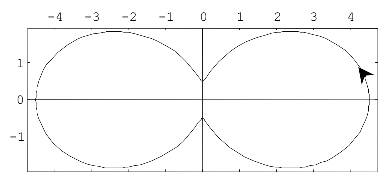

Figure 2.3: An oval of Cassini for Example 2.2

We again choose the unique solution of the Neumann problem with condition $u(\eta(0)) = 0$ as

$$u(z) = e^{x}\cos y - e, \quad z = x + iy \in \Omega. \tag{2.46}$$

For this problem, the unit tangent vector $T(t) = \dfrac{\dot{\eta}(t)}{|\dot{\eta}(t)|}$, i. e.,

$$T(t) = \frac{\dot{\eta}(t)}{|\dot{\eta}(t)|} = \frac{-R(t)\sin t + R'(t)\cos t + i\,[R(t)\cos t + R'(t)\sin t]}{|-R(t)\sin t + R'(t)\cos t + i\,[R(t)\cos t + R'(t)\sin t]|}.$$

For the above Neumann problem, the function $\gamma(t)$ is given by,

$$\gamma(t) = \frac{\partial u}{\partial \mathbf{n}} = \frac{u_x y'(t) - u_y x'(t)}{|\dot{\eta}(t)|}, \tag{2.47}$$

i.e.,

$$\gamma(t) = e^{R(t)\cos t} \cos\left(R(t)\sin t\right)\left[\frac{R(t)\cos t + R'(t)\sin t}{\left| -R(t)\sin t + R'(t)\cos t + i\,[R(t)\cos t + R'(t)\sin t]\ \right|}\right] - e^{R(t)\cos t} \sin\left(R(t)\sin t\right)\left[\frac{R(t)\sin t - R'(t)\cos t}{\left| -R(t)\sin t + R'(t)\cos t + i\,[R(t)\cos t + R'(t)\sin t] \right|}\right].$$

We now give a verification of (2.31). The right-hand side of equation (2.31) for the second region is

$$\psi(t) = \gamma(t)\,|\,\dot{\eta}(t)\,| = e^{R(t)\cos t} \cos\left(R(t)\sin t\right)\left(R(t)\cos t + R'(t)\sin t\right) - e^{R(t)\cos t} \sin\left(R(t)\sin t\right)\left(R(t)\sin t - R'(t)\cos t\right).$$

Now, we consider the left-hand side of equation (2.31) for the same region. Thus

$$f^{+}{}'(\eta(t)) = u_x + iv_x\big|_{\eta(t)} = e^{R(t)\cos t} \cos\left(R(t)\sin t\right) + ie^{R(t)\cos t} \sin\left(R(t)\sin t\right). \tag{2.48}$$

Multiplying the result with $\dot{\eta}(t)(-i)$, we have

$$\dot{\eta}(t)(-i)f^{+}{}'(\eta(t)) = \left(-R(t)\sin t + R'(t)\cos t + i\left[\,R(t)\cos t + R'(t)\sin t\,\right]\right) \left(-ie^{R(t)\cos t} \cos\left(R(t)\sin t\right) + e^{R(t)\cos t} \sin\left(R(t)\sin t\right)\right).$$

Therefore, the left-hand side of equation (2.31) is

$$\mathrm{Re}\left[\ \dot{\eta}(t)(-i)f^{+}{}'(\eta(t))\ \right]=e^{R(t)\cos t}\cos\left(R(t)\sin t\right)\left(R(t)\cos t+R'(t)\sin t\right)-$$
$$e^{R(t)\cos t}\sin\left(R(t)\sin t\right)\left(R(t)\sin t-R'(t)\cos t\right),$$

which is equal to the right-hand side (2.31). This completes the verification.

In Table 2.1 we calculate the sup-norm error $\left\| f_n^{+}{}' - f^{+}{}' \right\|_\infty$ between the approximate boundary values of $f_n^{+}{}'$ and the exact boundary values of $f^{+}{}'$ at the node points for Examples 2.1 and 2.2. We also calculate the sup-norm error $\left\| u_{x_n} - u_x \right\|_\infty$ between the approximate boundary values of u_{x_n} and the exact boundary values of u_x at the node points. With the same reasoning, Table 2.2 presents the sup-norm errors $\left\| f_n^{+} - f^{+} \right\|_\infty$ and $\left\| u_n - u \right\|_\infty$ for Examples 2.1 and 2.2.

Table 2.1: Numerical results for Examples 2.1 and 2.2

	$\left\| f_n^{+}{}' - f^{+}{}' \right\|_\infty$		$\left\| u_{x_n} - u_x \right\|_\infty$	
n	Γ_1	Γ_2	Γ_1	Γ_2
32	1.14 (-05)	7.40 (-02)	3.56 (-06)	7.40 (-02)
64	9.17 (-11)	1.18 (-03)	7.49 (-12)	1.18 (-03)
128	-	5.73 (-07)	-	5.73 (-07)
256	-	9.21 (-10)	-	4.59 (-10)

Table 2.2: Numerical results for Example 2.1 and 2.2

	$\| f_n^+ - f^+ \|_\infty$		$\| u_n - u \|_\infty$	
n	Γ_1	Γ_2	Γ_1	Γ_2
32	1.41 (-00)	-	9.49 (-02)	-
64	1.01 (-01)	6.45 (-01)	3.06 (-03)	6.07 (-01)
128	6.81 (-03)	4.50 (-02)	4.39 (-04)	1.84 (-02)
256	5.10 (-04)	4.76 (-03)	1.56 (-04)	4.31 (-03)
512	1.59 (-04)	1.40 (-03)	4.28 (-05)	1.37 (-03)

2.6 Conclusion

This chapter has presented a new integral equation for solving the interior Neumann problem in a simply connected region. The approach used in this paper to reduce the interior Neumann problem to an interior RH problem has been used before in (Henrici,p. 281, 1986). But the RH problem is solved in (Henrici, 1986) using a non-uniquely solvable integral equation (Henrici, Equation (15.9-8), 1986). However, in this research, the RH problem is solved using a new uniquely solvable integral equation. Thus the results of this research have significant advantages over the results given in Henrici (1986).

CHAPTER 3

AN INTEGRAL EQUATION METHOD FOR SOLVING EXTERIOR NEUMANN PROBLEMS ON SIMPLY CONNECTED SMOOTH REGIONS

3.1 Introduction

In Chapter 2, the interior Neumann problem is reduced to equivalent Riemann-Hilbert problem by using Cauchy-Riemann equations. The boundary integral equation is then derived for the Riemann-Hilbert problem based on an earlier work by Nasser (2007).

This chapter will focus on the development of a numerical method for the exterior Neumann problem in a simply connected smooth region. Firstly, the exterior Neumann problem will be reduced to the exterior Riemann-Hilbert problem. Then, the boundary integral equation for the Neumann problem will be derived based on the exterior Riemann-Hilbert problem.

The organization of this chapter is as follows. We present some auxiliary materials in Section 3.2. In Section 3.3, we show how to construct our new integral equation for the exterior Neumann problem based on the exterior Riemann-Hilbert problem. In Section 3.4 we discuss the numerical implementations of the derived integral equation. Finally, a short conclusion is given in Section 3.5.

3.2 Auxiliary Material

Let Ω be a bounded simply connected Jordan region bounded by Γ and let the exterior of Γ be denoted by Ω^{-} with $0 \in \Omega$ and ∞ belongs to Ω^{-}. The boundary $\Gamma := \partial\Omega$ is assumed to have a positively oriented parametrization $\eta(t)$ where $\eta(t)$ is a $2\pi-$periodic twice continuously differentiable function with $\dot{\eta}(t) = \dfrac{d\eta}{dt} \neq 0$. The parameter t need not be the arc length parameter. Let γ be a real-valued function defined on $\partial\Omega$.

Definition 3.1 (Nasser, 2007)

The Exterior Neumann Problem. Let $\mathbf{n}$ be the exterior normal to Γ and let $\gamma \in H^{\alpha}$ be a given function such that

$$\int_0^{2\pi} \gamma(t)\,|\,\dot{\eta}(t)\,|\,d\tau = 0\,. \tag{3.1}$$

Find the function u harmonic in Ω^{-}, Hölder continuous on Γ and satisfies on the boundary condition (see Figure 3.1)

$$\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\eta(t)} = \gamma(t), \qquad \eta(t) \in \Gamma. \tag{3.2}$$

The function u is also required to satisfy the additional condition

$$u(z) = O(|\,z\,|)^{-1} \qquad \text{as} \qquad z \to \infty\,. \tag{3.3}$$

Lemma 3.1 (Atkinson, 1997)
The exterior Neumann problem (3.2) is uniquely solvable.

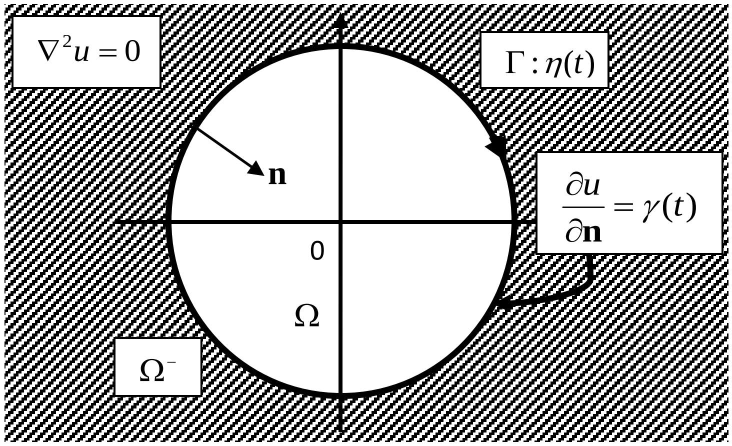

Figure 3.1: The exterior Neumann Problem

3.2.1 Definition of Normal Derivative

The complex unit tangent vector $T(t)$ is defined by

$$T(t) = \frac{\dot{\eta}(t)}{|\dot{\eta}(t)|} \quad , \tag{3.4}$$

where $\eta(t)$ is a complex parametrization of Γ. We call $T(t)$ the tangent directional derivative to Γ at $\eta(t)$. We define the unit normal vector $\mathbf{n}$ to the curve at t to be the vector that is perpendicular to $T(t)$ and has the same direction as $T'(t)$. By rotating the tangent T clockwise by $\frac{\pi}{2}$, we obtain

$$\mathbf{n} = \frac{e^{-i\frac{\pi}{2}}\dot{\eta}(t)}{|\dot{\eta}(t)|} = \frac{-i\dot{\eta}(t)}{|\dot{\eta}(t)|} \quad . \tag{3.5}$$

The directional derivative of $u(x, y)$ in the direction of the outward unit normal to the path at point $\eta(t)$ is denoted by

$$\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}, \tag{3.6}$$

where $\nabla u(x.y)=u_x\vec{i}+u_y\vec{j}$. If $\eta(t)=x(t)+iy(t)$ is the parametrization of Γ, then $\dot{\eta}(t)=x'(t)+iy'(t)$. Therefore, (3.5) becomes

$$\mathbf{n}=\frac{-i(x'(t)+iy'(t))}{|\dot{\eta}(t)|}=\frac{y'(t)}{|\dot{\eta}(t)|}\vec{i}+\frac{-x'(t)}{|\dot{\eta}(t)|}\vec{j}=\mathbf{n}_x\vec{i}+\mathbf{n}_y\vec{j}\,. \tag{3.7}$$

Substituting (3.7) into (3.6), we obtain

$$\frac{\partial u}{\partial \mathbf{n}}=\frac{u_x y'(t)-u_y x'(t)}{|\dot{\eta}(t)|}\,. \tag{3.8}$$

3.2.2 The Exterior Riemann-Hilbert (RH) Problem

Definition 3.2 (Zamzamir et al.,2009)

The Exterior RH Problem. Given functions A and C, it is required to find a function g analytic in Ω^- and continuous on the closure $\overline{\Omega^-}$ with $g(\infty)=0$ such that the boundary values g^- satisfy

$$\operatorname{Re}[A(t)g^-(\eta(t))]=C(t), \qquad \eta(t)\in\Gamma\,. \tag{3.9}$$

The boundary condition (3.9) can be written in the equivalent form

$$g^-(\eta(t))=-\frac{\overline{A(t)}}{A(t)}\overline{g^-(\eta(t))}+\frac{2C(t)}{A(t)}, \qquad \eta(t)\in\Gamma\,. \tag{3.10}$$

The homogenous boundary condition of the exterior RH problem is given by

$$\mathrm{Re}[A(t)g^{-}(\eta(t))] = 0, \qquad \eta(t) \in \Gamma \ . \tag{3.11}$$

The solvability of the RH problem is determined by the index κ of the function A. The index of the function A is defined as the winding number of A with respect to zero. If the function $A(t)$ is continuously differentiable on Γ, then

$$\kappa = \mathrm{ind}_{\Gamma}(A) = \frac{1}{2\pi i}\int_{\Gamma} d \ln A(t) = \frac{1}{2\pi i}\int_{\Gamma} \frac{\dot{A}(t)}{A(t)} dt \ . \tag{3.12}$$

The number of arbitrary real constants in the solutions of the homogenous RH problems, i.e. $\dim(S^{\pm})$ and the number of conditions on the function C so that the non-homogenous RH problems are solvable, i.e. $\mathrm{codim}(R^{\pm})$ are given in term of the index κ as in the following theorem from Wegmann et al. (2005).

Theorem 3.1.

The codimensions of the spaces $R^{\pm}$ and the dimensions of the spaces $S^{\pm}$ are given by the formulas

$$\mathrm{codim}(R^{\pm}) = \max(0, \pm 2\kappa \mp 1),$$
$$\dim(S^{\pm}) = \max(0, \mp 2\kappa \pm 1).$$

3.2.3 Integral Operators

Let $A(t)$ be a continuously differentiable $2\pi-$periodic function with $A \neq 0$. We define two real functions N and M by

$$N(\tau,t) := \frac{1}{\pi}\operatorname{Im}\left(\frac{A(\tau)}{A(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(\tau)}\right),$$

$$M(\tau,t) := \frac{1}{\pi}\operatorname{Re}\left(\frac{A(\tau)}{A(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(\tau)}\right).$$

We also define the kernels U and V (when $A=1$ for the kernels N and M) as

$$V(\tau,t) := \frac{1}{\pi}\operatorname{Im}\left(\frac{\dot{\eta}(t)}{\eta(t)-\eta(\tau)}\right),$$

$$U(\tau,t) := \frac{1}{\pi}\operatorname{Re}\left(\frac{\dot{\eta}(t)}{\eta(t)-\eta(\tau)}\right).$$

We then define the kernels U^* and V^* (the adjoint kernels of U and V) as

$$V^*(\tau,t) := \frac{1}{\pi}\operatorname{Im}\left(\frac{\dot{\eta}(\tau)}{\eta(\tau)-\eta(t)}\right),$$

$$U^*(\tau,t) := \frac{1}{\pi}\operatorname{Re}\left(\frac{\dot{\eta}(\tau)}{\eta(\tau)-\eta(t)}\right).$$

Lemma 3.2 (Wegmann et al., 2005).

(a) The kernel $N(\tau,t)$ is continuous with $N(\tau,t)=\frac{1}{\pi}\operatorname{Im}\left(\frac{1}{2}\frac{\ddot{\eta}(t)}{\dot{\eta}(t)}-\frac{\dot{A}(t)}{A(t)}\right)$.

(b) The kernel $M(\tau,t)$ has representation $M(\tau,t)=\frac{1}{2\pi}\cot\frac{\tau-t}{2}+M_1(\tau,t)$ with a continuous kernel M_1 which takes on the diagonal the values $M_1(\tau,t):=\frac{1}{\pi}\operatorname{Re}\left(\frac{1}{2}\frac{\ddot{\eta}(t)}{\dot{\eta}(t)}-\frac{\dot{A}(t)}{A(t)}\right)$.

3.2.4 Integral Equation for the Exterior Riemann-Hilbert Problem

For a given function C, $\mu\in H^{\alpha}$, let the function $\phi(z)$ be defined by

$$\phi(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{C(\eta)+i\mu(\eta)}{A(\eta)}\frac{d\eta}{\eta-z}, \qquad z\notin\Gamma. \tag{3.13}$$

Based on the application of Plemelj's formula, a boundary integral equation with generalized Neumann kernel has been derived for exterior RH problem by Wegmann et al. (2005), as in the following theorem.

Theorem 3.2

If $g(z)$ is a solution of the exterior problem (3.9) with boundary values

$$A(t)g^{-}(\eta(t))=C(t)+i\mu(t) \tag{3.14}$$

then the imaginary part μ in (3.14) satisfies the integral equation

$$\mu + \mathbf{N}\mu = \mathbf{M}C. \tag{3.15}$$

By Theorem 3.2, if $g(\eta(t))$ is the solution of the exterior Riemann-Hilbert problem with boundary values

$$\dot{\eta}(t)g^{-}(\eta(t)) = \varphi(t) + i\mu(t), \tag{3.16}$$

then the imaginary part μ in (3.16) satisfies the integral equation

$$\mu + \mathbf{N}\mu = \mathbf{M}\varphi, \tag{3.17}$$

i.e.

$$\mu(\tau) + \int_0^{2\pi} N(\tau,t)\mu(t)dt = \int_0^{2\pi} M(\tau,t)\varphi(t)dt. \tag{3.18}$$

The next theorem represents the solvability of the integral equation (3.15).

Theorem 3.3 (Zamzamir et al., 2009)

If $\kappa \le 0$, then the integral equation (3.15) is uniquely solvable. If $\kappa > 0$, then the integral equation (3.15) is non-uniquely solvable.

3.3 Modification of the Exterior Neumann Problem

3.3.1 Reduction of the Exterior Neumann Problem to the RH Problem

Suppose that u is the solution of the exterior Neumann problem and v is a harmonic conjugate of u in Ω^-. The function $f(z) = u(x,y) + iv(x,y)$ is analytic in $\Omega^- \cup \Gamma$ where $\Gamma : \eta(t) = x(t) + iy(t) = x(t)\vec{i} + y(t)\vec{j}$ if and only if the partial derivatives of u and v are continuous and satisfy the Cauchy-Riemann equations. The directional derivative of f in the direction of the outer unit normal vector to the path Γ is given by

$$\left.\frac{\partial f(\eta)}{\partial \mathbf{n}}\right|_{\eta(t)} = \left.\left(\frac{\partial u}{\partial \mathbf{n}} + i\frac{\partial v}{\partial \mathbf{n}}\right)\right|_{\eta(t)}. \tag{3.19}$$

Using the concept of normal derivative and Cauchy-Riemann equation, we obtain

$$\left.\frac{\partial f(\eta)}{\partial \mathbf{n}}\right|_{\eta(t)} = f'(\eta)(\mathbf{n}_x + i\mathbf{n}_y)\big|_{\eta(t)}. \tag{3.20}$$

Therefore, we can write (3.2) as the real part of (3.20),

$$\mathrm{Re}[f'(\eta)(\mathbf{n}_x + i\mathbf{n}_y)\big|_{\eta(t)}] = \gamma(t). \tag{3.21}$$

Substituting (3.7) and (3.5) into (3.21), we get

$$\mathrm{Re}\left[\mathbf{n} f'(\eta)\big|_{\eta(t)}\right] = \mathrm{Re}\left[\frac{-i\dot{\eta}(t)}{|\dot{\eta}(t)|} f'(\eta)\right] = \gamma(t). \tag{3.22}$$

Letting $g^-(\eta(t)) = -if'(\eta(t))$ and $\varphi(t) = \gamma(t)|\dot{\eta}(t)|$, we obtain

$$\mathrm{Re}[\dot{\eta}(t)g^{-}(\eta(t))]=\varphi(t) \tag{3.23}$$

which is the exterior RH problem as defined in Section 3.2.2. Comparison of (3.23) with (3.9) yields $A(t)=\dot{\eta}(t)$ and $C(t)=\varphi(t)$.

By Theorem 3.2, if $g(z)$ is a solution of the exterior problem (3.23) with boundary values $\dot{\eta}(t)g^{-}(\eta(t))=\varphi(t)+i\mu(t)$, then the imaginary part μ satisfies the integral equation

$$\mu+\mathbf{N}\mu=\mathbf{M}\varphi \ . \tag{3.24}$$

Applying (3.12) with $A(t)=\dot{\eta}(t)$, we obtain $\kappa=1$. From Theorem 3.1, we obtain $\mathrm{codim}(R^{-})=0$ which means that the non-homogenous exterior RH problem is solvable and $\dim(S^{-})=1$ implies that the solution of the exterior RH problem is not unique. Also from Theorem 3.3, we conclude that the integral equation (3.24) is non-uniquely solvable. The approach to overcome the non-uniqueness is discussed in the following section.

3.3.2 Modified Integral Equation for the Exterior RH Problem

With $A(t)=\dot{\eta}(t)$, the kernel N becomes $N(\tau,t)=-V^{*}(\tau,t)$ while the kernel M becomes $M(\tau,t)=-U^{*}(\tau,t)$. Therefore, the non-uniquely solvable integral equation (3.24) become

$$\mu-\mathbf{V}^{*}\mu=-\mathbf{U}^{*}\varphi \ . \tag{3.25}$$

Recall from Section 3.1 that the function f is analytic on $\Omega^{-}\cup\Gamma$. Then at each point z in that domain when $z_0=0$, $f(z)$ can be represented by the following Laurent series expansion such that $f(\infty)=0$:

$$f(z) = \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots \tag{3.26}$$

Differentiate once respect to z and multiply with $-iz$, we obtain

$$-zif'(z) = -\frac{c_1}{z} - \frac{2c_2}{z^2} - \frac{3c_3}{z^3} - \cdots = F(z) \tag{3.27}$$

Therefore, $g^-(\eta(t)) = \dfrac{F(z)}{z}$ since $g^-(\eta(t)) = -if'(\eta(t))$. By means of Cauchy Integral Formula, we obtain

$$\int_\Gamma g^-(\eta)d\eta = 0. \tag{3.28}$$

Notice that, $g^-(\eta(t)) = \dfrac{\varphi(t) + i\mu(t)}{\eta(t)}$, so (3.28) becomes

$$\int_0^{2\pi} \varphi(t)dt + i\int_0^{2\pi} \mu(t)dt = 0. \tag{3.29}$$

Note that, the exterior Neumann problem need to satisfy (3.1). With $\varphi(t) = \gamma(t)|\dot{\eta}(t)|$, (3.1) becomes $\int_0^{2\pi} \varphi(t)dt = 0$ which is the additional condition for the exterior RH problem which the right-hand side of the RH problem needs to satisfy. Thus (3.29) becomes

$$\int_0^{2\pi} \mu(t)dt = 0\,. \tag{3.30}$$

Let the kernel $\xi(\tau,t)$ be defined as $\xi(\tau,t) = \dfrac{1}{2\pi}$ and let the operator $\mathbf{J}$ be defined by

$$\mathbf{J}\mu(t) = \int_0^{2\pi} \xi(\tau,t)\mu(t)dt = \int_0^{2\pi} \frac{1}{2\pi}\mu(t)dt\,. \tag{3.31}$$

Therefore, we can write (3.30) as (3.31) where $\mathbf{J}\mu(t) = \frac{1}{2\pi}\int_0^{2\pi}\mu(t)dt = 0$. Adding this integral equation with our integral equation (3.25) yields the new integral equation

$$\mu - \mathbf{V}^*\mu + \mathbf{J}\mu = -\mathbf{U}^*\varphi\,. \tag{3.32}$$

According to Mikhlin (1957), this new integral equation (3.32) is uniquely solvable. The proof of the solvability of the integral equation can be followed from the paper written by Atkinson (1967).

From Lemma 3.2, we can write the kernel of the right-hand side of (3.32) as $-U^*(\tau,t) = M(\tau,t)$ which is singular since it is unbounded when $\tau = t$. To remove the difficulty, we will use the function $B(\tau,t)$ defined in Wegmann et al. (2005), with $A(t) = \dot{\eta}(t)$ where

$$B(\tau,t) = \begin{cases} \frac{1}{\pi}\operatorname{Re}\left(\frac{\eta(\tau)\varphi(t) - \eta(t)\varphi(\tau)}{\eta(t)-\eta(\tau)}\right), & \text{for} \quad \tau \neq t, \\ \frac{1}{\pi}\left[\dot{\varphi}(t) - \varphi(t)\operatorname{Re}\left(\frac{\eta(\tau)}{\eta(t)}\right)\right], & \text{for} \quad \tau = t. \end{cases} \tag{3.33}$$

Therefore, our uniquely solvable integral equation (3.32) becomes

$$\mu - \mathbf{V}^*\mu + \mathbf{J}\mu = \mathbf{B}\phi \tag{3.34}$$

where $\mathbf{B}\phi = \int_0^{2\pi} B(\tau,t)dt.$

By obtaining μ, the Cauchy integral formula implies that the function $f'(z)$ can be calculated for $z \in \Omega^-$ by

$$f'(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f'(\eta)}{\eta - z}d\eta = \frac{1}{2\pi i}\int_{0}^{2\pi}\frac{\varphi(t)+i\mu(t)}{-i\eta(t)}\frac{\dot{\eta}(t)dt}{\eta(t)-z} \tag{3.35}$$

and the function $f(z)$ can be expressed as an anti-derivative function of $f'(z)$ for $z \in \Omega^-$. So we have

$$f(z) = \frac{1}{2\pi i}\int_{0}^{2\pi}\frac{\varphi(t)+i\mu(t)}{-i\eta(t)}\log\left(1-\frac{\eta(t)}{z}\right)\dot{\eta}(t)dt. \tag{3.36}$$

3.4 Numerical Implementations of the Boundary Integral Equation

Since the function $A(t) = \dot{\eta}(t)$ and $\eta(t)$ are $2\pi-\text{periodic}$, the integrals in the integral equation (3.34) can be best discretized by the Nyström method with the trapezoidal rule as the quadrature rule (Atkinson, 1997). Let n be a given integer and define the n equidistant collocation points t_k by

$$t_k = (k-1)\frac{2\pi}{n}, \qquad k = 1, 2, \ldots, n. \tag{3.37}$$

Then, using the Nyström method with the trapezoidal rule to discretized the integral equation (3.34), we obtain the linear system

$$\mu_n(t_j) - \frac{2\pi}{n}\sum_{k=1}^{n}V^*(t_j,t_k)\mu_n(t_k) + \frac{2\pi}{n}\sum_{k=1}^{n}J(t_j,t_k)\mu_n(t_k) = \frac{2\pi}{n}\sum_{k=1}^{n}B(t_j,t_k) \tag{3.38}$$

with $j = 1, 2, \cdots, n$ and μ_n is an approximation to μ.

Let $\mathbf{I}$ be the $n\times n$ identity matrix. Also let $\mathbf{I}_{11}$ and $\mathbf{I}_1$ be the $n\times n$ matrix and $n\times 1$ vector whose elements are all unity. Define the matrices $\mathbb{V}=[\mathbb{V}_{kj}], \mathbb{J}=[\mathbb{J}_{kj}], \mathbb{B}=[\mathbb{B}_{kj}]$ and vectors $\mathbb{x}=[\mathbb{x}_k], \mathbb{y}=[\mathbb{y}_k]$ by

$$\mathbb{V}_{kj}=\frac{2\pi}{n}\mathbf{V}^*(t_k,t_j)=\frac{2\pi}{n}\begin{cases}\frac{1}{\pi}\operatorname{Im}\left(\frac{\dot{\eta}(t_k)}{\eta(t_k)-\eta(t_j)}\right), & t_k\neq t_j\\ \frac{1}{2\pi}\left(\frac{\ddot{\eta}(t_j)}{\dot{\eta}(t_j)}\right), & t_k=t_j\end{cases}, \tag{3.39}$$

$$\mathbb{J}_{kj}=\frac{2\pi}{n}\mathbf{J}(t_k,t_j)\mathbf{I}_{11}=\frac{2\pi}{n}\left(\frac{1}{2\pi}\right)\mathbf{I}_{11}, \tag{3.40}$$

$$\mathbb{B}_{kj}=\frac{2\pi}{n}\mathbf{B}(t_k,t_j)=\frac{2\pi}{n}\begin{cases}\frac{1}{\pi}\operatorname{Re}\left(\frac{\dot{\eta}(t_k)\varphi(t_j)-\dot{\eta}(t_j)\varphi(t_k)}{\eta(t_j)-\eta(t_k)}\right), & t_k\neq t_j\\ \frac{1}{\pi}\left(\dot{\varphi}(t_j)-\varphi(t_j)\operatorname{Re}\left(\frac{\ddot{\eta}(t_j)}{\dot{\eta}(t_j)}\right)\right), & t_k=t_j\end{cases}, \tag{3.41}$$

$$\mathbb{x}_k=\mu_n(t_k), \tag{3.42}$$

$$\mathbb{y}_k=\mathbb{B}_{kj}\mathbf{I}_1 . \tag{3.43}$$

Hence, the application of Nyström method to the uniquely solvable integral equations (3.34) leads to the following n by n linear system

$$(\mathbf{I}-\mathbf{V}+\mathbf{J})\mathbf{x}=\mathbf{y}. \tag{3.44}$$

By solving the linear system (3.44), we obtain $\mu_n(t_k)$ for $k=1,2,\dots,n$. Then the approximate solution $\mu_n(t)$ can be calculated for all $t\in[0,2\pi]$ using the Nyström interpolating formula, i.e., the approximation $\mu_n(t)$ of the integral equation (3.34) is given by

$$\mu_n(t)=\frac{2\pi}{n}\sum_{j=1}^{n}B(t,t_j)+\frac{2\pi}{n}\sum_{j=1}^{n}V^*(t,t_j)\mu_n(t_j)-\frac{2\pi}{n}\sum_{j=1}^{n}\left(\frac{1}{2\pi}\mu_n(t_j)\right). \tag{3.45}$$

By obtaining μ ,(3.35) and (3.36) implies that $f'(z)$ and $f(z)$ can be calculated for $z\in\Omega^-$.

3.4.1 Examples

For our examples, we use three boundary curves: an ellipse, the oval of Cassini, and an "amoeba". These examples were also used in Nasser (2007). For the ellipse (see Figure 3.2), the boundary has parametrization

$$\Gamma_1 : \eta(t) = \cos t + i \sin t, \qquad 0 \le t \le 2\pi \ . \tag{3.46}$$

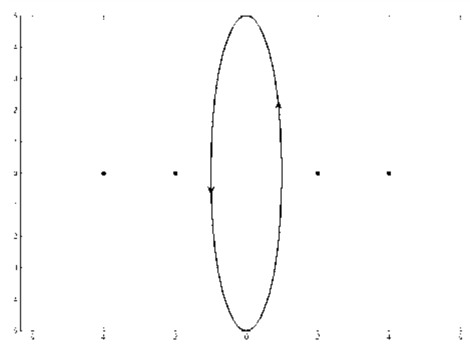

Figure 3.2: The Curve Γ_1 and the Exterior Test Points.

For the oval of Cassini (see Figure 3.3), the boundary parametrization

$$\Gamma_2 : \eta(t) = R(t)e^{it}, \qquad 0 \le t \le 2\pi \qquad \text{where} \qquad R(t) = 2.5 + 2\cos 2t\,. \tag{3.47}$$

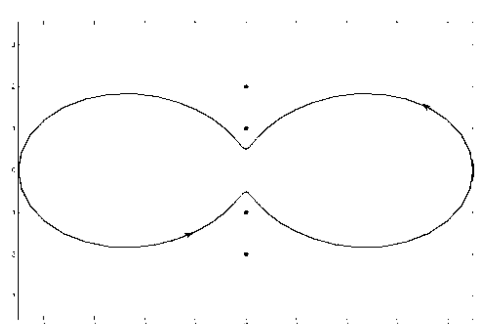

Figure 3.3: The Curve Γ_2 and the Exterior Test Points.

For the “amoeba” (see Figure 3.4), the boundary parametrization is

$$\Gamma_3 : \eta(t) = R(t)e^{it}, \qquad 0 \le t \le 2\pi \qquad \text{where} \qquad R(t) = e^{\cos t}\cos^2 2t + e^{\sin t}\sin^2 2t . \qquad (3.48)$$

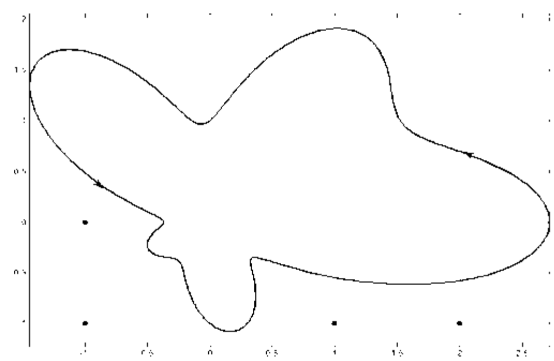

Figure 3.4: The Curve Γ_3 and the Exterior Test Points.

The linear system (3.44) is then solved using the MATLAB “\” operator that makes use of the Gauss elimination method. The maximum error norm $\|f'(z) - f_n'(z)\|_\infty$ between the exact values of $f'(z)$ and the approximate value of $f_n'(z)$ on the boundary is presented in Table 3.1. The absolute error $|f(z) - f_n(z)|$ at four test points z outside Γ for the exterior Neumann problem is listed in Tables 3.2 – 3.4. The numerical results are presented for various values of n where n is the number of node points given in (3.37).

Table 3.1: The Error $\left\| f'(z) - f_n'(z) \right\|_\infty$ for the Exterior Neumann Problem on the Boundaries Γ_1, Γ_2, and Γ_3.

n	Γ_1	Γ_2	Γ_3
16	6.64(-02)	7.29(-01)	4.59(-01)
32	3.71(-03)	1.28(-01)	2.27(-01)
64	6.34(-06)	3.35(-03)	3.55(-02)
128	1.50(-11)	2.00(-06)	3.71(-04)
256	2.66(-15)	6.46(-13)	1.54(-08)
512	2.94(-15)	5.87(-14)	6.14(-14)

Table 3.2: The Error $\left| f(z) - f_n(z) \right|$ for the Exterior Neumann Problem on the Boundary Γ_1.

n	$z = -4$	$z = -2$	$z = 2$	$z = 4$
16	8.45(-02)	1.42(-01)	1.42(-01)	8.45(-02)
32	3.94(-03)	6.70(-03)	6.70(-03)	3.94(-03)
64	6.10(-06)	1.04(-05)	1.04(-05)	6.10(-06)
128	1.41(-11)	2.40(-11)	2.40(-11)	1.41(-11)
256	2.52(-16)	5.63(-16)	6.18(-16)	2.80(-16)

Table 3.3: The Error $\left| f(z) - f_n(z) \right|$ for the Exterior Neumann Problem on the Boundary Γ_2.

n	$z = -2i$	$z = -i$	$z = i$	$z = 2i$
16	4.25(-03)	7.88(-02)	7.88(-02)	4.25(-03)
32	8.30(-04)	3.20(-03)	3.20(-03)	8.30(-04)
64	1.28(-05)	1.85(-05)	1.85(-05)	1.28(-05)
128	4.09(-09)	4.11(-10)	4.11(-10)	4.10(-09)
256	8.88(-16)	3.55(-15)	3.78(-15)	10.00(-16)

Table 3.4: The Error $|f(z) - f_n(z)|$ for the Exterior Neumann Problem on the Boundary Γ_3.

n	$z = -1$	$z = -1 - i$	$z = 1 - i$	$z = 2 - i$
32	6.68(-03)	7.64(-03)	7.75(-03)	5.14(-03)
64	1.67(-04)	1.33(-04)	1.64(-04)	1.35(-04)
128	4.32(-07)	3.28(-07)	5.80(-07)	2.27(-07)
256	1.52(-11)	1.35(-11)	1.87(-11)	4.23(-12)
512	6.92(-16)	6.75(-16)	2.29(-16)	2.36(-16)

3.5 Conclusion

In this chapter, we discuss mainly on numerical solution of the exterior Neumann problem using boundary integral equation method. Unlike the classical methods for solving the Neumann problem which require the availability of a conformal mapping from the problem domain to a ‘simpler’ one, the present method can be used to solve the Neumann problem in its original domain. Furthermore, the presented method has the advantage that it needs less numerical operations and is easier to program.

CHAPTER 4

A BOUNDARY INTEGRAL EQUATION FOR THE INTERIOR NEUMANN PROBLEM ON BOUNDED MULTIPLY CONNECTED REGION

4.1 Introduction

Neumann problem is classified as a boundary value problem associated with Laplace's equation and Neumann boundary condition. Different types of Neumann problems occur naturally in some fields like electrostatics, fluid flow, heat flow and elasticity.

Boundary integral equation method is one of the common methods for solving Neumann problem. This method reduces the task to solve an integral equation only on the boundary of the region, thus reducing the dimension of the Neumann problem by one. Numerical treatment is usually needed to solve the resulting integral equation. One example of such approach is given in Nasser (2007) where the Neumann problem is reduced to a Dirichlet problem from which an integral equation is constructed. In this chapter, we extend the results of Chapter 2 to reduce the Neumann problem to the Riemann-Hilbert problem in multiply connected region, and then derive an integral equation with the Neumann kernel related to the Riemann-Hilbert problem (briefly, RH problem). This integral equation is the Fredholm integral equation of the second kind.

4.2 Auxiliary Material

Let Ω be a multiply connected region in the complex plane as shown in Figure 4.1. We assume that each boundary Γ_i has a parameterization $\eta_i(t),\ t \in I_i$ which is a complex periodic function with period 2π, where $I_i = [0, 2\pi]$ is the parametric interval for each η_i. The parameterization η_i also need to be twice continuously differentiable with $\dot{\eta}_i(t) = \dfrac{d\eta_i(t)}{dt} \neq 0$.

We consider the parametric interval I of the parameterization $\eta(t)$ of the whole boundary Γ as a disjoint union of the intervals I_i, where $\eta(t)$ is defined as

$$\eta(t) = \begin{cases} \eta_0(t), & t \in I_0 \\ \eta_1(t), & t \in I_1 \\ \quad \vdots & \\ \eta_m(t), & t \in I_m \end{cases}$$

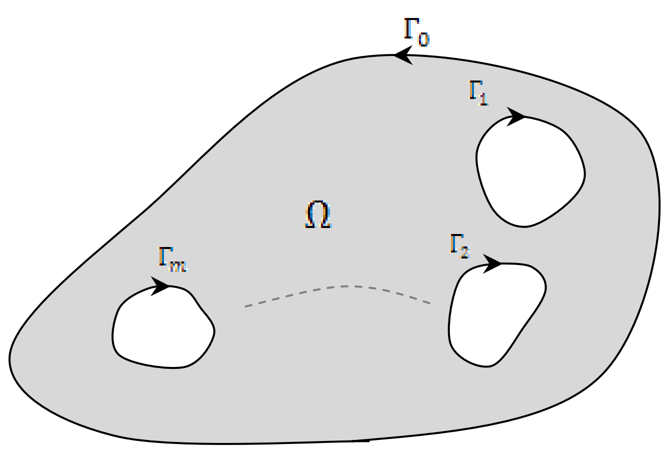

Figure 4.1: Bounded multiply connected region.

4.2.1 Neumann Kernels

Let $A(t)$ be a nonzero twice continuously differentiable periodic function with period 2π, and define the following kernels (Nasser, 2007)

$$M(s,t) = \frac{1}{\pi}\operatorname{Re}\left(\frac{A(s)}{A(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(s)}\right), \tag{4.1}$$

$$N(s,t) = \frac{1}{\pi}\operatorname{Im}\left(\frac{A(s)}{A(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(s)}\right). \tag{4.2}$$

The kernel N is called the generalized Neumann kernel which is continuous with

$N(t,t) = \frac{1}{\pi}\operatorname{Im}\left(\frac{1}{2}\frac{\ddot{\eta}(t)}{\dot{\eta}(t)} - \frac{\dot{A}(t)}{A(t)}\right)$. With $A(t)=1$, (4.1) and (4.2) reduce respectively to

$$U(s,t) = \frac{1}{\pi}\operatorname{Re}\left(\frac{\dot{\eta}(t)}{\eta(t)-\eta(s)}\right), \tag{4.3}$$

$$V(s,t) = \frac{1}{\pi}\operatorname{Im}\left(\frac{\dot{\eta}(t)}{\eta(t)-\eta(s)}\right). \tag{4.4}$$

The kernel V is the classical Neumann kernel. The adjoint kernels of the above kernels, respectively denoted by M^*, N^*, U^*, and V^* are defined as

$$M^*(s,t) = M(t,s) = \frac{1}{\pi}\operatorname{Re}\left(\frac{A(t)}{A(s)}\frac{\dot{\eta}(s)}{\eta(s)-\eta(t)}\right),$$
$$N^*(s,t) = N(t,s) = \frac{1}{\pi}\operatorname{Im}\left(\frac{A(t)}{A(s)}\frac{\dot{\eta}(s)}{\eta(s)-\eta(t)}\right),$$
$$U^*(s,t) = U(t,s) = \frac{1}{\pi}\operatorname{Re}\left(\frac{\dot{\eta}(s)}{\eta(s)-\eta(t)}\right),$$
$$V^*(s,t) = V(t,s) = \frac{1}{\pi}\operatorname{Im}\left(\frac{\dot{\eta}(s)}{\eta(s)-\eta(t)}\right).$$

4.2.2 The Neumann Problem

Consider the multiply connected region Ω with smooth boundary. The Neumann problem is to find a harmonic function $u(x,y)$ defined on Ω such that at each point of the boundary Γ, the directional derivative of u in the outward normal direction $\mathbf{n}$ is equal to a function γ defined on this boundary. In other words, the solution u must satisfy the following conditions

$$\nabla^2 u(z) = 0, \qquad z = x + iy \in \Omega, \tag{4.5}$$

$$\frac{\partial u(\eta(t))}{\partial \mathbf{n}} = \gamma(t), \qquad \eta(t) \in \Gamma, \tag{4.6}$$

where $\gamma(t)$ is a known real continuous function defined on Γ. The sufficient conditions for the existences and uniqueness of the solution are given by

$$\int_{\Gamma} \gamma(t)|\dot{\eta}(t)|dt = 0, \qquad u(\sigma) = 0, \tag{4.7}$$

where σ is a fixed point in Ω.

The condition (4.7) is sometimes called the compatibility condition. Indeed, since u is harmonic in Ω, this implies that

$$\int_{\Omega} (\nabla^2 u) dz = 0.$$

By using the divergence theorem, we get

$$\int_{\Gamma} \frac{\partial u}{\partial \mathbf{n}} ds = 0$$

which is just (4.7).

With the condition (4.7), a solution to (4.5) and (4.6) is guaranteed. However, the solution is not unique. Indeed, by adding any constant to any particular solution u we still obtain a solution. Therefore, the function u is required to satisfy the additional condition which can be written as

$$u(\sigma) = 0,$$

where σ is a fixed point in Ω.

4.2.3 The Riemann-Hilbert problem

Let Ω be the multiply connected region as described before, and let $A(t) = a(t) + ib(t)$ be a non-zero complex function which is twice continuously differentiable periodic function with period 2π, and let $\phi(t)$ and $\psi(t)$ be real Hölder continuously periodic functions with period 2π. The RH problem consists of finding a function $g = u + iv$ that is analytic in Ω, continuous in its closure $\overline{\Omega}$ and has boundary values $g^+ = u^+ + iv^+$ on Γ which satisfy

$$a(t)u^+(t) - b(t)v^+(t) = \phi(t), \tag{4.8}$$

where $+$ in the superscript denotes the one-sided limit from the positive (or left) side of Γ. Equation (4.8) is called the Riemann condition which can also be written as

$$\mathrm{Re}[A(t)g^+(\eta(t))] = \phi(t). \tag{4.9}$$

If we let $\mathrm{Im}[A(t)g^+(\eta(t))] = \psi(t)$, then the Riemann condition (4.9) is the real part of the equation $A(t)g^+(\eta(t)) = \phi(t) + i\psi(t).$

Now, we define the adjoint RH problem as

$$\mathrm{Re}[\widetilde{A}(t)g^+(\eta(t))] = \phi(t), \tag{4.10}$$

where $\tilde{A}(t)=\dfrac{\dot{\eta}(t)}{A(t)}$ is the adjoint function of A.

The solvability of the RH problem depends on the index of the function A, which is denoted by κ and it can be written as $\kappa=\operatorname{ind}(A)=\dfrac{1}{2\pi}\arg A(t)\Big|_{\Gamma}$. We define the range space of the RH problems and the solution space of the homogeneous RH problems as follows (Wegmann and Nasser, 2008):

$$R^{+}=\{\phi\in H:\phi=\operatorname{Re}[Ag^{+}]\},\qquad S^{+}=\{\phi\in H:\phi=Ag^{+}\},\qquad g \text{ analytic in } \Omega,$$
$$\tilde{R}^{+}=\{\phi\in H:\phi=\operatorname{Re}[\tilde{A}g^{+}]\},\qquad \tilde{S}^{+}=\{\phi\in H:\phi=\tilde{A}g^{+}\},\qquad g \text{ analytic in } \Omega,$$

where H is the space of real Hölder continuous functions on Γ.

Theorem 4.1 (Wegmann and Nasser, 2008)

The solvability of the RH problem is connected with the solution space of the homogeneous adjoint problem by the relations

$$R^{+}=(\tilde{S}^{+})^{\perp},\qquad \tilde{R}^{+}=(S^{+})^{\perp},$$

where $\perp$ denotes the orthogonal complement space.

Theorem 4.2 (Wegmann and Nasser, 2008)

The number of linearly independent solutions of the homogeneous RH problem and its adjoint are connected by the formula

$$\dim(S^{+})-\dim(\tilde{S}^{+})=1-m-2\kappa.$$

4.2.4 Integral Equation for the RH Problem

There is a close relation between the RH problem and the integral equation with the generalized Neumann kernel.

Theorem 4.3 (Wegmann and Nasser, 2008)
If g *is a solution of the RH problem* (4.9) *with boundary values* $Ag^{+} = \phi(t) + i\psi(t)$, *then the imaginary part* ψ *satisfies the integral equation*

$$\psi - \mathbf{N}\psi = -\mathbf{M}\phi, \tag{4.11}$$

where the operators $\mathbf{N}$ *and* $\mathbf{M}$ *are defined as*

$$(\mathbf{N}\psi)(s) = \int_{I} N(s,t)\psi(t)dt, \qquad (\mathbf{M}\phi)(s) = \int_{I} M(s,t)\phi(t)dt.$$

The solvability of this integral equation also depends on the index of the function A and the connectivity of Ω. This is clarified in the following theorem.

Theorem 4.4 (Wegmann and Nasser, 2008)
The number of linearly independent solutions of the homogeneous integral equations with operator $\mathbf{I} \pm \mathbf{N}$ *is given by*

$$\dim(\mathrm{Null}(\mathbf{I}+\mathbf{N})) = \max(0, 2\kappa_0 - 1) + \sum_{i=1}^{m} \max(0,\, 2\kappa_i + 1),$$
$$\dim(\mathrm{Null}(\mathbf{I}-\mathbf{N})) = \max(0, -2\kappa_0 + 1) + \sum_{i=1}^{m} \max(0, -2\kappa_i - 1),$$

where $\mathbf{I}$ *is the identity operator.*

By this theorem and the Fredholm alternative theorem we can determine the solvability of the integral equation (4.11).

4.3 A Boundary Integral Equation for the Neumann Problem

4.3.1 Reduction of the Neumann Problem to the RH Problem

Let $u(z)$ be the solution of the Neumann problem on the multiply connected region Ω. Hence we can write u as a real part of an analytic function f defined on $\Omega \cup \Gamma$, i.e. $f(z) = u(z) + iv(z),$ where v is the harmonic conjugate of u. Then the Neumann problem can be reduced to the form (Alejaily, 2009)

$$\mathrm{Re}[\dot{\eta}(t)(-if^{+}{}'(\eta))] = |\dot{\eta}(t)|\,\gamma(t). \tag{4.12}$$

Equation (4.12) is the RH problem (4.9) with

$$A(t) = \dot{\eta}(t), \qquad g^{+}(\eta(t)) = -if^{+}{}'(\eta(t)), \qquad \phi(t) = |\dot{\eta}(t)|\gamma(t).$$

The method of reduction has also been used in Husin, (2009), but limited to the case of interior Neumann problem on a simply connected region. With $A(t) = \dot{\eta}(t)$, the integral equation (4.11) related to the RH problem (4.12) becomes

$$\psi(s) + (\mathbf{v}^{*}\psi)(s) = (\mathbf{u}^{*}\phi)(s), \tag{4.13}$$

where the operators $\mathbf{u}^{*}$ and $\mathbf{v}^{*}$ are defined as

$$(\mathbf{v}^{*}\psi)(s) = \int_{I} V^{*}(s,t)\psi(t)dt, \qquad (\mathbf{u}^{*}\phi)(s) = \int_{I} U^{*}(s,t)\phi(t)dt,$$

and $U^{*}(s,t) = U(t,s)$ and $V^{*}(s,t) = V(t,s)$ are the adjoint kernels of U and V respectively.

4.3.2 Solvability of the RH Problem and the Derived Integral Equation

Since our RH problem (4.12) is the RH problem (4.9) with $A(t) = \dot{\eta}(t)$, the index for our multiply connected region $\kappa = \mathrm{ind}(\dot{\eta}(t)) = \sum_{j=0}^{m} \kappa_j = 1 - m$. By applying Theorem 4.1 and Theorem 4.2 we get (Alejaily, 2009)

$$\mathrm{codim}(\mathrm{R}^+) = \dim(\tilde{S}^+) = 1, \qquad \dim(S^+) = m,$$

where $\mathrm{codim}(\mathrm{R}^+)$ represents the number of conditions for the right-hand side such that the RH problem (4.9) is solvable. With regard to the integral equation (4.13), Theorem 4.4 implies that

$$\dim(\mathit{Null}(\mathbf{I} + \mathbf{v}^*)) = \dim(\mathit{Null}(\mathbf{I} - \mathbf{N})) = m.$$

This means that (4.13) is solvable subjected to m conditions on ψ. So, we have non-uniquely solvable RH problem (4.12) which gives rise to a non-uniquely solvable integral equation (4.13). The value $\mathrm{codim}(\mathrm{R}^+) = 1$ means that the RH problem is solvable if and only if the right-hand side ϕ satisfies a certain condition. This condition is the same as the solvability condition of Neumann problem.

Now, we show how to obtain a unique solution of the integral equation (4.13), which will give a unique solution of the RH problem (4.12). This means we have to impose m conditions on the function ψ to get a unique solution to the integral equation (4.13). Let us define the kernel $J(s,t)$ for $(s,t) \in I_i \times I_j$ such that (Atkinson, 1967)

$$J(s,t) = \begin{cases} \dfrac{1}{2\pi}, & s,t \in I_i, \quad i = 1, 2, \ldots, m, \\ 0, & s \in I_i, \quad t \in I_j, \quad i \neq j, \quad i, j = 0,1,2,\ldots,m. \end{cases}$$

This means that $J(s,t)$ is equal to 1 when $\eta(s)$ and $\eta(t)$ belong to same boundary Γ_i except the boundary Γ_0 and equal to 0 otherwise. Then the integral operator $\mathbf{J}$ defined as $(\mathbf{J}\mu)(s) = \int_I J(s,t)\mu(t)dt$ satisfies (Alejaily, 2009)

$$(\mathbf{J}\psi)(s) = \int_{I_k} J(s,t)\psi(t)dt = 0 . \tag{4.14}$$

For $k = 1,2,\cdots,m$, this provides the m conditions for the function ψ. By adding (4.14) to our integral equation (4.13), we get

$$\psi(s) + (\mathbf{v}^*\psi)(s) + (\mathbf{J}\psi)(s) = (\mathbf{u}^*\phi)(s), \tag{4.15}$$

which is uniquely solvable.

4.4 Numerical Implementation

We apply the Nyström method with the trapezoidal rule to discretize our integral equation on an equidistant grid, where each interval $I_i = [0, 2\pi]$ is subdivided into n steps of size $h = 2\pi/n$. Since $i = 0,1,\ldots,m$, this leads to a system of $r = (m+1)n$ equations in r unknowns. Our choice of the trapezoidal rule was due to the periodicity of the functions A and η, where this method is very accurate for periodic functions (Davies and Rabinowitz, 1984). So, the operators $\mathbf{v}^*$ and $\mathbf{u}^*$ tend to be best described on an equidistant grid by the trapezoidal rule. Then, we get the following linear system of equations

$$(I_\gamma + B + C)\Psi = D\Phi , \tag{4.16}$$

where I_γ is the identity matrix of dimension r while B,C and D are matrices of size $r \times r$, derived from the discretization of the operators $\mathbf{v}^*, \mathbf{J}$ and $\mathbf{u}^*$ respectively. Ψ and Φ are $r \times 1$ vectors that approximate the values of functions ψ and ϕ respectively at the collection points.

To solve the system (4.16), we have used the method of Gaussian elimination of MATLAB. Since (4.15) is uniquely solvable, for n sufficiently large, the system of linear equations (4.16) also has a unique solution (Atkinson, 1997).

4.5 Examples

In this section, we consider two examples of test regions to examine our method. The sample problems are such that the analytic solutions are known. This allows us to compare our numerical results with the exact solutions.

Example 4.1

In our first example we consider a doubly connected region, Ω_1 as shown in Figure 4.2. The boundaries of this region are parameterized by the functions

$$\Gamma_0 : \eta_0(t) = 3\cos t + i5\sin t, \quad \Gamma_1 : \eta_1(t) = \cos t - i\sin t.$$

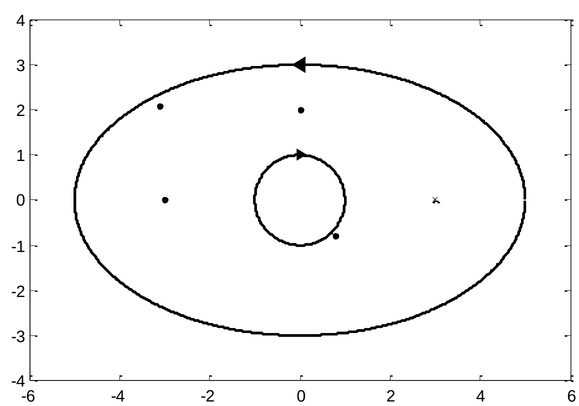

Figure 4.2: The test region Ω_1 for Example 4.1.

We choose the function

$$f(z) = e^z - e^\sigma, \qquad \sigma = \alpha + i\beta$$

which is analytic in Ω_1. Then the function

$$u(z) = e^x \cos y - e^\alpha \cos\beta$$

solves the Neumann problem uniquely in this region with the boundary condition

$$\gamma(t) = \left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\eta(t)} = \frac{1}{|\eta(t)|}\operatorname{Re}[-if^{+\prime}(\eta(t))\dot{\eta}(t)]$$

and the additional condition $u(\sigma) = 0$.

We describe the error by the infinity-norm error $\|u(z) - u_n(z)\|_\infty$ where $u_n(z)$ is the numerical approximations of $u(z)$. We choose four test points with $\sigma = 3$. The results are shown in Table 4.1.

Table 4.1: The error $\|u(z) - u_n(z)\|_\infty$ for Example 4.1.

z	$n=16$	$n=32$	$n=64$	$n=128$
$2i$	1.167992e-02	1.484597e-04	8.333704e-08	2.131628e-14
$-3.1+i2.1$	7.973386e-03	4.840204e-04	1.849295e-05	3.889558e-07
-3	5.088192e-03	1.344374e-08	2.486900e-14	5.684342e-14
$0.8-i0.8$	4.313050e-04	6.790590e-05	4.156548e-07	5.763212e-11

Example 4.2

Let Ω_2 be a multiply connected region of connectivity five as shown in Figure 4.3. The boundaries of Ω_2 are parameterized by the functions

$$\Gamma_0 : \eta_0(t) = (9 + \cos 6t)e^{it}$$
$$\Gamma_1 : \eta_1(t) = (3 + \cos 3t)e^{-it}$$
$$\Gamma_2 : \eta_2(t) = (3 + i5)e^{-it}$$
$$\Gamma_3 : \eta_3(t) = -6 + e^{-it}$$
$$\Gamma_4 : \eta_4(t) = (3 - i5)e^{-it}$$

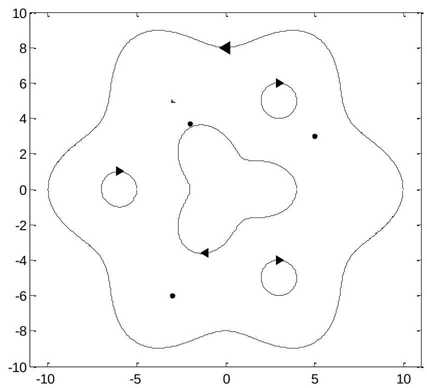

Figure 4.3: The test region Ω_2 for Example 4.2.

We choose the function

$$f(z) = z^2 - \sigma z, \qquad \sigma = \alpha + i\beta$$

which is analytic in Ω_2, Then the function $u(z) = x^2 - \alpha x + \beta y - y^2$ is harmonic in this region and satisfies the Neumann problem with the boundary condition $\gamma(t) = \dfrac{1}{|\dot{\eta}(t)|}\mathrm{Re}[-if^{+}{}'(\eta(t))\dot{\eta}(t)]$ and the additional condition $u(\sigma) = 0$. We choose three test points and the results are shown in Table 4.2.

Table 4.2: The error $\|u(z)-u_n(z)\|_\infty$ for Example 4.2.

z	$n=32$	$n=64$	$n=128$	$n=512$
$5+3i$	1.011974e-03	7.123924e-06	5.137224e-12	1.065814e-13
$-2+3.7i$	7.561762e-02	9.193517e-03	2.055304e-04	4.258816e-13
$-3-6i$	4.090850e-03	9.002238e-08	4.263256e-14	0.000000e+00

In the numerical results, the solutions of the Neumann problem at the interior points are calculated from the boundary values of ψ by means of the formula (Alejaily, 2009)

$$u(z)=\mathrm{Re}\left[\frac{-1}{2\pi i}\int_I(-\psi(t)+i|\dot{\eta}(t)|\gamma(t))\ln\left(1-\frac{z-\sigma}{\eta(t)-\sigma}\right)dt\right].$$

4.6 Conclusion

This chapter has formulated a new boundary integral equation for solving the Neumann problem on multiply connected regions with smooth boundaries. The idea of formulation of this integral equation is firstly to reduce the Neumann problem into the equivalent RH problem from which an integral equation is constructed. We have solved this integral equation numerically using Nyström method with the trapezoidal rule. Once we got the solution ψ, the solution of the Neumann problem is within our reach. The results of the numerical examples show the efficiency of our approach. However, the accuracy of values of $u(z)$ for z near to the boundary is not as good as for z far from the boundary. This can be treated by increasing the number of nodes on the boundary or using the method proposed in Helsing and Ojala (2008).

CHAPTER 5

A BOUNDARY INTEGRAL EQUATION FOR THE EXTERIOR NEUMANN PROBLEM ON MULTIPLY CONNECTED REGION

5.1 Introduction

Chapter 4 is on extended of Chapter 2 by formulating a new boundary integral equation with Neumann kernel for solving the interior Neumann problem on multiply connected regions with smooth boundaries. Chapter 4 has reduced the Neumann problem into the equivalent Riemann-Hilbert problem from which an integral equation is constructed. The previous Chapter 3 has reduced the exterior Neumann problem on a simply connected region to exterior Riemann-Hilbert problem by using Cauchy-Riemann equations. This leads to an integral equation with the Neumann kernel. This chapter deals with the reduction of exterior Neumann problem on a multiply connected region to the exterior Riemann-Hilbert problem. Thus this chapter extends the results of Chapter 3.

This chapter is organized as follows: Section 5.2 presents some auxiliary materials related to exterior Neumann and Riemann-Hilbert problems as well as integral equation for Riemann-Hilbert problems. In Section 5.3, we reduce the exterior Neumann problem on multiply connected region into the exterior Riemann-Hilbert problem and derive the boundary integral equation for solving it. Then, in Section 5.4, we provide a numerical technique for solving the boundary integral equation by using Mathematica. Some numerical examples are presented in Section 5.5. A short conclusion is given in Section 5.6.

5.2 Auxiliary Material

Let Ω be a bounded multiply connected Jordan region bounded by Γ and let the exterior of Γ be denoted by Ω^{-} with $0 \in \Omega$ and ∞ belongs to Ω^{-}. The boundary $\Gamma = \partial\Omega$ is assumed to have a positively oriented parameterization $\eta(t)$ where $\eta(t)$ is a 2π -periodic twice continuously differentiable function with $\dot{\eta}(t) = \dfrac{d\eta}{dt} \neq 0$. The parameter t need not be the arc length parameter (Atkinson, 1997). Figure 5.1 shows the problem with multiply connected region.

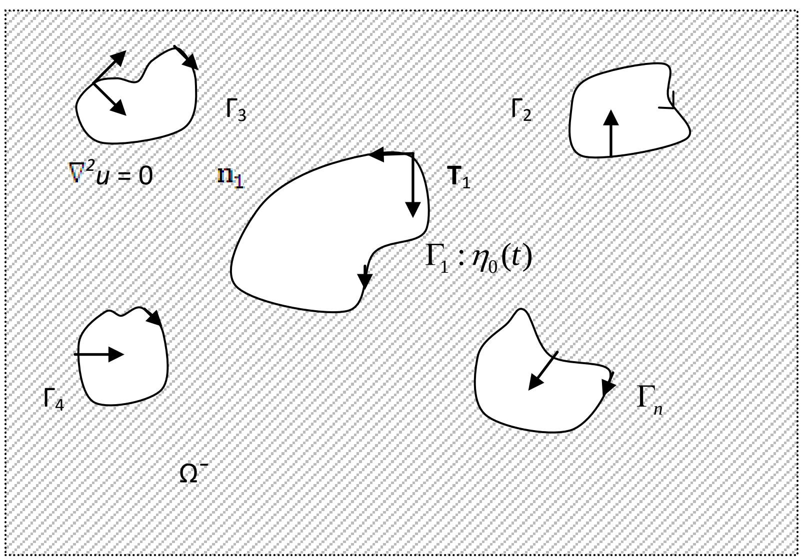

Figure 5.1: Unbounded multiply connected region

5.2.1 Boundary Integral Equation for Solving Exterior Neumann Problem

Denifition 5.1 (Atkinson,1997)

Let H^{α} represents a Hölder space which consists all 2π-periodic real functions which are uniformly Holder continuous with exponent α. Let **n** *be the exterior normal to Γ and let $\gamma \in H^{\alpha}$ be a given function such that*

$$\int_{0}^{2\pi} \gamma(\tau)\left|\dot{\eta}(t)\right| d\tau = 0 . \tag{5.1}$$

The exterior Neumann problem consists in finding the function u harmonic in $\mathbf{\Omega}^{-}$, Hölder continuous on Γ and satisfies the boundary condition

$$\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\eta(t)} = \gamma(t), \qquad \eta(t) \in \Gamma. \tag{5.2}$$

The function u is also required to satisfy the additional condition

$$u(z) = O(|z|)^{-1} \qquad \text{as} \qquad z \to \infty . \tag{5.3}$$

Theorem 5.1 (Atkinson, 1997)

The exterior Neumann problem (5.2) *is uniquely solvable.*

According to Atkinson (1997), *u* satisfies an integral equation in the form of

$$u(\eta) - \frac{1}{\pi}\int_{\Gamma} u(t)\frac{\partial}{\partial n_t}\left[\text{Log}\frac{1}{|t-\eta|}\right]|dt| = -\frac{1}{\pi}\int_{\Gamma}\gamma(t)\left[\text{Log}\frac{1}{|t-\eta|}\right]|dt|, \in \Gamma. \tag{5.4}$$

This equation is uniquely solvable and is considered a practical approach to solving the exterior Neumann problem. The details regarding the numerical solution of the exterior Neumann problem using this approach can be referred to Atkinson (1997).

Nasser (2007) has also developed a uniquely solvable second kind Fredholm integral equation with the generalized Neumann kernel that can be used to solve the exterior Neumann problems on simply connected regions with smooth boundaries. Suppose that u is the solution of the exterior Neumann problem and v is a harmonic conjugate of u in Ω^{-}. Suppose $\tilde{\gamma}+i\tilde{\mu}$ is a boundary value of a function g analytic in Ω, i.e.,

$$g^{-}(\eta(t))=\tilde{\gamma}(t)+i\tilde{\mu}(t), \tag{5.5}$$

where

$$\tilde{\gamma}(t)=u(\eta(t)), \qquad \tilde{\mu}(t)=v(\eta(t)), \qquad 0\le t\le 2\pi$$

with $g(z)=\tilde{c}+O(z^{-1})$ near ∞ with a real constant $\tilde{c}$. Since

$$u(z)=\operatorname{Re} g(z)\to 0 \text{ when } z\to\infty$$

hence $\tilde{c}=0$. It is shown in Nasser (2007), that the boundary values of g can be written as

$$g^{-}(\eta)=-\mu+i\varphi+iv(\eta(0)). \tag{5.6}$$

where $\varphi(t)=\int_0^t \gamma(t)|\dot{\eta}(t)|d\tau$ and μ satisfies the integral equation

$$\mu+\mathbf{N}\mu=\mathbf{M}\varphi \tag{5.7}$$

Since $g(\infty)=0$ then by Cauchy integral formula (Gakhov, 1966), the function $g(z)$ for $z\in\Omega^{-}$ is in the form

$$g(z)=-\frac{1}{2\pi i}\int_{\Gamma}\frac{-\mu(t)+\varphi(t)}{\eta-z}d\eta\,. \tag{5.8}$$

Furthermore, the function g satisfies

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{g^{-}(\eta)}{\eta}d\eta=0\,.$$

Hence, by (5.7),

$$v(\eta(0))=-\frac{1}{2\pi i}\int_{\Gamma}\frac{\varphi+i\mu}{\eta}d\eta\,. \tag{5.9}$$

Then, the unique solution of the exterior Neumann problem is given in $\Omega^{-}\cup\Gamma$ by

$$u(z)=\operatorname{Re} g(z)\,. \tag{5.10}$$

The previous Chapter 3 has reduced the exterior Neumann problem on a simply connected region to the equivalent exterior Riemann-Hilbert problem from which an integral equation is constructed. The derived integral equation is in the form

$$\mu-\mathbf{V}^{*}\mu+\mathbf{J}\mu=\mathbf{B}\,, \tag{5.11}$$

where $\mathbf{J}\mu(\tau)=\dfrac{1}{2\pi}\int_{0}^{2\pi}\mu(t)dt=0$ and

$$N(\tau,t)=-V^{*}(\tau,\mathrm{t})=\begin{cases}\dfrac{1}{\pi}\operatorname{Im}\left(\dfrac{\dot{\eta}(\tau)}{\eta(t)-\eta(\tau)}\right), & t\neq\tau,\\[2ex] -\dfrac{1}{2\pi}\operatorname{Im}\left(\dfrac{\ddot{\eta}(t)}{\dot{\eta}(t)}\right), & t=\tau.\end{cases}$$

The unique solution of the exterior Neumann problem on a simply connected region is given for unbounded **Ω** in Chapter 3 as

$$u(z)=\operatorname{Re}[f(z)], \tag{5.12}$$

where

$$f(z)=\frac{1}{2\pi i}\int_{0}^{2\pi}\frac{\varphi(t)+i\mu(t)}{-i\dot{\eta}(t)}\log\left(1-\frac{\eta(t)}{z}\right)\dot{\eta}(t)dt\,. \tag{5.13}$$

The exterior Neumann problem has a unique solution u in $\mathbf{\Omega}^{-}$ which is not necessary a real part of a single-valued analytic function *f*. However, there exist *m* real constants $a_1,a_2,\ldots,a_m$, uniquely determined by γ, such that u can be written as the real part of

$$f(z)=F(z)+\sum_{j=1}^{m}a_j\log(z-\mathrm{z}_j)\,, \tag{5.14}$$

where F is a single-valued analytic function in $\mathbf{\Omega}^{-}$ (Mikhlin (1957), Muskhelishvili (1953)). The real constants are chosen such that F is a single-valued analytic function in $\mathbf{\Omega}^{-}$, i.e.,

$$\int_{\Gamma_j}F'(\eta)d\eta=0, \qquad j=1,2,\ldots,m. \tag{5.15}$$

It can be shown that $a_k = -\frac{1}{2\pi i}\int\limits_{\Gamma_k} f'(\eta)d\eta$. Indeed, we have

$$F(z) = f(z) - \sum_{j=1}^{\infty} a_j \log(z - z_j).$$

Differentiate the whole equation, we get

$$F'(z) = f'(z) - \sum_{j=1}^{\infty} a_j \frac{1}{z - z_j}.$$

Then, integrate the equation we obtain

$$\int\limits_{\Gamma_k} F'(\eta)d\eta = \int\limits_{\Gamma_k} f'(\eta)d\eta - \sum_{j=1}^{\infty} a_j \int\limits_{\Gamma_k} \frac{1}{\eta - z_j} d\eta.$$

By the single-valuedness property,

$$\int\limits_{\Gamma_k} F'(\eta)d\eta = 0.$$

Hence we get

$$0 = \int\limits_{\Gamma_k} f'(\eta)d\eta + 2\pi i a_k,$$

and finally we obtain

$$a_k = -\frac{1}{2\pi i}\int\limits_{\Gamma_k} f'(\eta)d\eta.$$

For Ω^-, the constants $a_1, a_2, \ldots, a_m$ satisfy

$$\sum_{j=1}^{m} a_j = -\sum_{j=1}^{m} \frac{1}{2\pi i}\int\limits_{\Gamma_j} f'(\eta)d\eta = -\frac{1}{2\pi i}\int\limits_{\Gamma} f'(\eta)d\eta = 0. \tag{5.16}$$

In view of (5.3), we can assume without loss the generality that $f(\infty) = 0$.

5.2.2 Boundary Integral Equation for Solving Exterior Riemann-Hilbert Problem

Suppose that Ω^{-} is the exterior of Γ such that the ∞ of the coordinate system belong to Ω^{-}. If a function $g(z)$ is defined in a region containing Γ, then the limiting values of the function $g(z)$ when the point z tends to the point $\eta \in \Gamma$ from the exterior of Γ will be denoted by $g^{-}(z)$. Let $A(t)$ be a complex continuously differentiable 2π-periodic function with $A \neq 0$. With C, $\mu \in H^{\alpha}$, let the function $\Phi(z)$ be defined by

$$\Phi(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{C(\eta)+i\mu(\eta)}{A(\eta)} \frac{d\eta}{\eta - z}, \qquad z \notin \Gamma. \tag{5.17}$$

Then $\Phi(z)$ is analytic in Ω as well as in Ω^{-} and the boundary values Φ^{+} from inside and Φ^{-} from outside belong to H^{α} and can be calculated by Plemelj's formula (see Atkinson (1967), Asmar (2002))

$$\Phi^{\pm}(\xi) = \pm\frac{1}{2}\frac{C(\xi)+i\mu(\xi)}{A(\xi)} + \mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma} \frac{C(\eta)+i\mu(\eta)}{A(\eta)} \frac{d\eta}{-z}, \qquad \xi \in \Gamma. \tag{5.18}$$

The integral (5.18) is a Cauchy principal value integral. The boundary values satisfy the jump relation

$$A\Phi^{+} - A\Phi^{-} = C + i\mu. \tag{5.19}$$

Definition 5.2 (Henrici (1986), Exterior Riemann-Hilbert Problem)

Given a function A and C, it is required to find a function g analytic in Ω^{-} *and continuous on the closure* $\overline{\Omega^{-}}$ *with* $g(\infty) = 0$ *such that the boundary values* g^{-} *satisfy*

$$\mathrm{Re}[A(t)g^{-}(\eta(t))] = C(t), \qquad \eta(t) \in \Gamma. \tag{5.20}$$

The boundary condition (5.20) *can be written in the equivalent form*

$$g^{-}(t) = -\frac{\overline{A(t)}}{A(t)}\overline{g^{-}(\eta(t))} + \frac{\overline{2C(t)}}{A(t)}, \qquad \eta(t) \in \Gamma. \tag{5.21}$$

The homogeneous boundary condition of the exterior Riemann-Hilbert problem is given by

$$\mathrm{Re}[A(t)g^{-}(\eta(t))]=0, \qquad \eta(t)\in\Gamma, \tag{5.22}$$

where $C(t) = 0$.

The solvability of the Riemann-Hilbert problem is determined by the index κ of the function A. The index of the function A is defined as winding number of A with respect to zero (see Gakhov (1966), Murid and Nasser (2009)), i.e.,

$$\kappa=\mathrm{ind}(A)=\frac{1}{2\pi}\arg(A)\Big|_{0}^{2\pi}. \tag{5.23}$$

We define the range space of the exterior Riemann-Hilbert problem, i.e., the space of function C for which the exterior Riemann-Hilbert problem is solvable,

$$R^{-}=\{C\in H^{\alpha}:C(t)=\mathrm{Re}[A(t)g^{-}(\eta(t))], g \text{ analytic in } \Omega^{-}, g(\infty)=0\}. \tag{5.24}$$

We defined the solution space of the homogenous exterior Riemann-Hilbert problem as

$$S^{-}=\{C\in H^{\alpha}:C(t)=A(t)g^{-}(\eta(t)), g \text{ analytic in } \Omega^{-}, g(\infty)=0\}. \tag{5.25}$$

Then the number of arbitrary real constants in the solutions of the homogenous exterior Riemann-Hilbert problems, i.e. dim (S^{-}) and the number of conditions on the function C so that the non-homogenous exterior Riemann-Hilbert problems are solvable, i.e. codim(S^{-}) are given in term of the index κ as in the following theorem.

Theorem 5.2 [Gakhov (1966), Murid and Nasser (2009)]

The codimension of the space R^{-} *and the dimension of the space* S^{-} *are given by the formulas*

$$\mathrm{codim}\ (R^{-}) = \max(0, -2\kappa + 1), \tag{5.26}$$

$$\dim\ (S^{-}) = \max(0, 2\kappa - 1). \tag{5.27}$$

Now, let $A(t)$ be continuously differentiable 2π-periodic function with $A\neq 0$. We define two real functions N and M by

$$N(\tau,t)=\frac{1}{\pi}\operatorname{Im}\left(\frac{A(\tau)}{A(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(\tau)}\right), \tag{5.28}$$

$$M(\tau,t)=\frac{1}{\pi}\operatorname{Re}\left(\frac{A(\tau)}{A(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(\tau)}\right). \tag{5.29}$$

The kernel $N(\tau,t)$ is called the generalized Neumann kernel form with A and η (Wegmann et al., 2005).

5.2.3 The Riemann-Hilbert Problem

We consider the same region as mentioned before and let A be a continuously differentiable complex function on Γ with $A \neq 0$. We assume that A is given in parametric form $A(s)$ such that $A(s)$ is continuously differentiable for all $s \in J$ (Nasser, 2009). With $\varphi,\psi \in H$, we define the function

$$\Phi(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{\varphi+i\psi}{A}\frac{d\eta}{\eta-z}, \qquad z \notin \Gamma . \tag{5.30}$$

Lemma 5.1 (Wegmann et al., 2005)

The boundary values of the function Φ can be represented by

$$2\operatorname{Re}[A(s)\Phi^{-}(\eta(s))]=-\varphi+\mathbf{N}\varphi+\mathbf{M}\psi, \tag{5.31}$$

$$2\operatorname{Im}[A(s)\Phi^{-}((s))]=-\psi+\mathbf{N}\psi-\varphi. \tag{5.32}$$

With assumptions about the Hölder continuity of the functions A, φ and ψ, the boundary functions Φ^{-} is Hölder continuous on Γ. The operators $\mathbf{N}$ and $\mathbf{M}$ map H^{α} into H^{α}. Both operators are bounded in H^{α}.

Suppose that G^- is an unbounded multiply connected region of connectivity *m* bounded by $\Gamma = \partial G^- = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m$ where the curves $\Gamma_j, j = 1,2,\ldots,m$, are simple non-intersecting smooth clockwise oriented closed curves (see Figure 5.1). The function Φ is an analytic function in G^- with $\Phi(\infty) = 0$. The boundary values Φ^+ from inside G^- and Φ^- from outside G^- are Hölder continuous on Γ and can be calculated by Plemelj's formulas

$$\Phi^{\pm}(\eta(s)) = \pm\frac{1}{2}\frac{\varphi(s)+i\psi(s)}{A(s)} + \frac{1}{2\pi i}\int_J \frac{\varphi(t)+i\psi(t)}{A(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(s)}dt \tag{5.33}$$

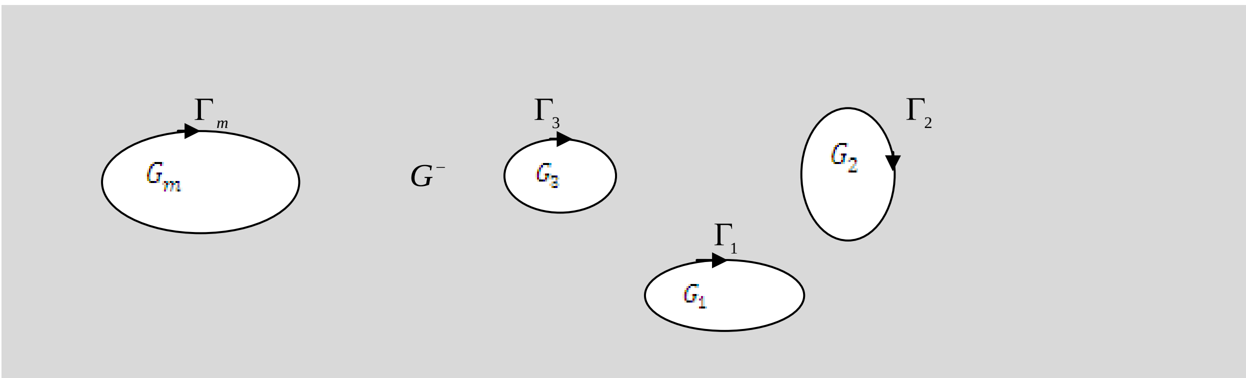

Figure 5.2: An unbounded multiply connected region *G* of connectivity

The boundary values satisfy the jump relation

$$A(s)\Phi^{+}(\eta(s)) - A(s)\Phi^{-}(\eta(s)) = \varphi(s) + i\psi(s). \tag{5.34}$$

Assume a function g analytic in G^-, continuous on $G^- \cup \Gamma$, such that the boundary values g^- satisfy on Γ

$$\mathrm{Re}[A(s)g^{-}(\eta(s))] = \varphi(s) \tag{5.35}$$

and

$$\mathrm{Im}[A(s)g^{-}(\eta(s))] = \psi(s).$$

We define also the range space R^- as the space of all real functions $\gamma \in H$ for which the Riemann-Hilbert problems (briefly, RH problems) are solvable and the space S^- as the spaces of the boundary values of solutions of the homogeneous RH problem, i.e. (Nasser, 2009),

$$R^{-} = \{\gamma \in H : \gamma = \mathrm{Re}[Ag^{-}], g \text{ analytic in } G^{-}, g(\infty) = 0\}, \tag{5.36}$$

$$S^{-} = \{\gamma \in H : \gamma = Ag^{-}, g \text{ analytic in } G^{-}, g(\infty) = 0\}. \tag{5.37}$$

Theorem 5.3 (Nasser, 2009)

If g *is a solution of the RH problem* (5.35) *with boundary values*

$$Ag^{-} = \varphi(t) + i\psi(t) \tag{5.38}$$

then the imaginary part ψ *in* (5.38) *satisfies the integral equation*

$$\psi - \mathbf{N}\psi = -\mathbf{M}\varphi \tag{5.39}$$

where the operators $\mathbf{N}$ *and* $\mathbf{M}$ *are defined as*

$$(\mathbf{N}\psi)(s) = \int_I N(s,t)\psi(t)dt,$$
$$(\mathbf{M}\varphi)(s) = \int_I M(s,t)\varphi(t)dt.$$

5.2.4 The Solvability of the Riemann-Hilbert Problem

The solvability on unbounded multiply connected regions can be deduced from the solvability on bounded regions by means of Moebius transform (Wegmann, 2001). Let z_0 be a fixed point in G^{-}, say $z_0 \in G_m$. The unbounded region G^{-} is transformed by means of the Moebius transform

$$\Psi(z) = \frac{1}{z - z_0} \tag{5.40}$$

onto a bounded multiply connected region $\hat{G} = \Psi(G)$ of connectivity *m*. The Moebius transform Ψ transforms also the bounded exterior region $G_1 \cup G_2 \cup \cdots \cup G_m$ onto an unbounded region $\hat{G}^-$ exterior to the boundary $\hat{\Gamma} = \partial\hat{G} = \Psi(\Gamma)$. The boundary $\hat{\Gamma}$ is given by $\hat{\Gamma} = \hat{\Gamma}_0 \cup \hat{\Gamma}_1 \cup \cdots \cup \hat{\Gamma}_{m-1}$ where $\hat{\Gamma}_0 = \Psi(\Gamma_m)$ is the outer curve and is counterclockwise oriented; and the other curves $\hat{\Gamma}_j = \Psi(\Gamma_j), j = 1,2,\ldots,m-1$, are clockwise oriented and are inside $\hat{\Gamma}_0$. The curve $\hat{\Gamma}$ is parameterized by (Nasser, 2009)

$$\zeta(s) = \frac{1}{\eta(s) - z_0}, \qquad s \in J. \tag{5.41}$$

Let the function $\hat{A}$ be defined by

$$\hat{A}(s) = \zeta(s)A(s), \qquad s \in J. \tag{5.42}$$

According Nasser (2009), for a given $\varphi \in H$, a function $\hat{g}$ analytic in $\hat{G}^-$ with $\hat{g}(\infty) = 0$, continuous on $\hat{G} \cup \hat{\Gamma}$, such that the boundary values $\hat{g}^-$ satisfy on $\hat{\Gamma}$

$$\mathrm{Re}[\hat{A}(s)\hat{g}^-(\zeta(s))] = \varphi(s). \tag{5.43}$$

We define also the spaces

$$\hat{R}^- = \{\gamma \in H : \gamma = \mathrm{Re}[\hat{A}\hat{g}^-], \hat{g} \text{ analytic in } \hat{G}^-\}, \tag{5.44}$$

$$\hat{S}^- = \{\gamma \in H : \gamma = \hat{A}\hat{g}^-, \hat{g} \text{ analytic in } \hat{G}^-\}. \tag{5.45}$$

Lemma 5.2 (Nasser, 2009)

The spaces $R^-, S^-, \hat{R}^-$ and $\hat{S}^-$ *satisfy*

$$\hat{R}^- = R^-, \qquad \hat{S}^- = S^-. \tag{5.46}$$

The index κ of the function A on the whole boundary curve Γ is the sum

$$\kappa = \sum_{j=1}^{m} \kappa_j, \tag{5.47}$$

where the κ_j of the function A on the curve Γ_j is defined as the winding number of A with respect to 0, i.e.

$$\kappa_j = \frac{1}{2\pi} \arg(A)\Big|_{\Gamma_j}, \qquad j = 1,2,\ldots,m. \tag{5.48}$$

Theorem 5.4 (Wegmann and Nasser, 2008)

The dimension of the space S^- and the codimension of the space R^- are determined by the index of A as follows:

a) If $\kappa \geq 0,$ then

$$\dim(S^-) = 0, \qquad \operatorname{co}\dim(R^-) = 2\kappa + m. \tag{5.49}$$

b) If $\kappa \leq -m$, then

$$\dim(S^-) = -2\kappa - m, \qquad \operatorname{co}\dim(R^-) = 0. \tag{5.50}$$

c) *If $-m+1 \leq \kappa \leq -1$, then*

$$-2\kappa - m \leq \dim(S^-) \leq -\kappa, \qquad 2\kappa + m \leq co\dim(R^-) \leq m + \kappa. \tag{5.51}$$

Theorem 5.5 (Wegmann and Nasser, 2008)

The dimensions of the null-spaces of the operators $I \pm \mathbf{N}$ are given by

$$\dim(\mathrm{Null}(I + \mathbf{N})) = \sum_{j=1}^{m} \max(0, 2\kappa_j + 1), \tag{5.52}$$

$$\dim(Null(I - \mathbf{N})) = \sum_{j=1}^{m} \max(0, -2\kappa_j - 1). \tag{5.53}$$

5.3 Modification of the Exterior Neumann Problem

5.3.1 Reduction of the Exterior Neumann Problem to the Exterior Riemann-Hilbert Problem

Let $u(z)$ be the solution of the exterior Neumann problem on the multiply connected region Ω^-. Hence we can write *u* as a real part of an analytic function *f* defined on $\Omega^- \cup \Gamma$, i.e.

$$f(z) = u(z) + iv(z),$$

where v is the harmonic conjugate of u. The derivative of f with respect to z is given by

$$f'(z) = u_x(z) + iv_x(z),$$

where $u_x = \dfrac{\partial u}{\partial x}$ and $u_y = \dfrac{\partial u}{\partial y}$.

The boundary values of f and f' are

$$\begin{aligned} f^-(\eta) &= u(\eta) + iv(\eta), \\ f^-{}'(\eta) &= u_x(\eta) + iv_x(\eta). \end{aligned}$$

The normal derivative of f^- is given by

$$\frac{\partial f^-(\eta)}{\partial \mathbf{n}} = \frac{\partial u(\eta)}{\partial \mathbf{n}} + i\frac{\partial v(\eta)}{\partial \mathbf{n}}. \tag{5.54}$$

From vector calculus, we can write

$$\begin{aligned} \frac{\partial u}{\partial \mathbf{n}} &= (u_x\mathbf{i} + u_y\mathbf{j}) \cdot \mathbf{n}, \\ \frac{\partial v}{\partial \mathbf{n}} &= (v_x\mathbf{i} + u_y\mathbf{j}) \cdot \mathbf{n} \end{aligned}$$

If

$$\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j},$$

where n_x, n_y are the components of the outward vector, then (5.54) becomes

$$\frac{\partial f^-(\eta)}{\partial \mathbf{n}} = (u_x\mathbf{i} + u_y\mathbf{j}) \cdot (n_x\mathbf{i} + n_y\mathbf{j}) + i(v_x\mathbf{i} + v_y\mathbf{j}) \cdot (n_x\mathbf{i} + n_y\mathbf{j}).$$

After some arrangement, we get

$$\frac{\partial u(\eta)}{\partial \mathbf{n}} + i\frac{\partial v(\eta)}{\partial \mathbf{n}} = \mathbf{n}f^-{}'(\eta).$$

Equating real parts on both sides we get

$$\frac{\partial u(\eta)}{\partial \mathbf{n}} = \gamma(t) = \mathrm{Re}\left[\frac{-i\dot{\eta}(t)}{|\dot{\eta}(t)|}\right] f^{-}{}'(\eta)$$

which is equivalent to

$$\mathrm{Re}[\dot{\eta}(t)(-if^{-}{}'(\eta))] = |\dot{\eta}(t)|\gamma(t). \tag{5.55}$$

Equation (5.55) is the RH problem (5.35) with

$$A(t) = \dot{\eta}(t), \qquad g^{-}(\eta(t)) = -if^{-}{}'(\eta(t)), \qquad \varphi(t) = |\dot{\eta}(t)|\gamma(t). \tag{5.56}$$

The above method of reduction is only limited to the case of exterior Neumann problem on a simply connected region.

5.3.2 Integral Equation Related to the Exterior Riemann-Hilbert Problem

In this section, we will show the way to construct an integral equation related to the RH problem (5.55) based on Theorem 5.3. With $A(t) = \dot{\eta}(t)$, then

$$N(s,t) = \frac{1}{\pi}\mathrm{Im}\left(\frac{\dot{\eta}(s)}{\dot{\eta}(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(s)}\right) = \frac{-1}{\pi}\mathrm{Im}\left(\frac{\dot{\eta}(s)}{\eta(s)-\eta(t)}\right) = -V^{*}(s,t),$$
$$M(s,t) = \frac{1}{\pi}\mathrm{Re}\left(\frac{\dot{\eta}(s)}{\dot{\eta}(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(s)}\right) = -\frac{1}{\pi}\mathrm{Re}\left(\frac{\dot{\eta}(s)}{\eta(s)-\eta(t)}\right) = -U^{*}(s,t),$$

where U^{*} and V^{*} are the adjoint kernels of U and V respectively. Then the integral equation (5.39) related to the RH problem (5.55) becomes

$$\psi + \mathbf{v}^{*}\psi = \mathbf{u}^{*}\varphi \tag{5.57}$$

where the operators $\mathbf{u}^{*}$ and $\mathbf{v}^{*}$ are defined as

$$(\mathbf{v}^{*}\psi)(s) = \int_I V^{*}(s,t)\psi(t)dt,$$
$$(\mathbf{u}^{*}\varphi)(s) = \int_I U^{*}(s,t)\varphi(t)dt.$$

5.3.3 The Solvability of the Exterior Riemann-Hilbert Problem

Since our RH problem (5.55) is the RH problem (5.35) with $A(t) = \dot{\eta}(t)$, the index $\kappa = \text{ind}(\dot{\eta}(t))$ for exterior multiply connected region Ω implies

$$\kappa_j = -1, \qquad j = 1,2,3,\ldots,m.$$

Hence, we have

$$\kappa = \sum_{j=1}^{m} \kappa_j = -m.$$

For the doubly connected regions, we have $m = 2$ and

$$\kappa_1 = -1, \quad \kappa_2 = -1.$$

Hence,

$$\kappa = \kappa_1 + \kappa_2 = -2.$$

From Theorem 3.2, we obtain

$$\text{codim}(R^{-}) = 0, \tag{5.58}$$

$$\dim(S^{-}) = 2. \tag{5.59}$$

Equation (5.58) means that there is no condition for φ in (5.55), so that the non-homogeneous exterior Riemann-Hilbert problem is solvable. Equation (5.59) implies that, there are two real constants in the solution of the homogeneous exterior Riemann-Hilbert problem so that the solution of the exterior Riemann-Hilbert problem is not unique.

To overcome the non-uniqueness, we need to impose additional constraint which can be embedded into integral equation that will yield a uniquely solvable integral equation.

5.3.4 Modified Integral Equation for the Exterior Riemann-Hilbert Problem

Recall from Chapter 3 Section 3.2.3 with $A(t) = \dot{\eta}(t)$, the generalized Neumann kernel N becomes

$$N(s,t) = \begin{cases} -\dfrac{1}{\pi}\operatorname{Im}\left(\dfrac{\dot{\eta}(s)}{\eta(s)-\eta(t)}\right), & t \neq s \\ \dfrac{1}{2\pi}\operatorname{Im}\left(\dfrac{\ddot{\eta}(t)}{\dot{\eta}(t)}\right), & t = s \end{cases} \tag{5.60}$$

$$= -V^{*}(s,t), \tag{5.61}$$

while the kernel M becomes

$$M(s,t) = -\frac{1}{\pi}\operatorname{Re}\left(\frac{\dot{\eta}(s)}{\eta(s)-\eta(t)}\right) \tag{5.62}$$

$$= -U^{*}(s,t). \tag{5.63}$$

So, our non-uniquely solvable integral equation (5.39) becomes (5.57) and can be rewritten in the form below,

$$\psi(s) - \mathbf{N}\psi(t) = -\mathbf{M}\varphi(t),$$

$$\psi(s) - \int_I N(s,t)\psi(t)dt = -\int_I M(s,t)\varphi(t)dt,$$

$$\psi(s) - \int_I -\frac{1}{\pi}\left[\operatorname{Im}\left(\frac{\dot{\eta}(s)}{\eta(s)-\eta(t)}\right)\right]\psi(t)dt = -\int_I -\frac{1}{\pi}\left[\operatorname{Re}\left(\frac{\dot{\eta}(s)}{\eta(s)-\eta(t)}\right)\right]\varphi(t)dt,$$

$$\psi(s)+\int_I \frac{1}{\pi}\left[\operatorname{Im}\left(\frac{\dot{\eta}(s)}{\eta(s)-\eta(t)}\right)\right]\psi(t)dt=\int_I \frac{1}{\pi}\left[\operatorname{Re}\left(\frac{\dot{\eta}(s)}{\eta(s)-\eta(t)}\right)\right]\varphi(t)dt. \tag{5.64}$$

For $s \in I_1$, equation (5.64) becomes

$$\psi_1(s)+\int_0^{2\pi}\frac{1}{\pi}\left[\operatorname{Im}\left(\frac{\dot{\eta}_1(s)}{\eta_1(s)-\eta_1(t)}\right)\right]\psi_1(t)dt+\int_0^{2\pi}\frac{1}{\pi}\left[\operatorname{Im}\left(\frac{\dot{\eta}_1(s)}{\eta_1(s)-\eta_2(t)}\right)\right]\psi_2(t)$$

$$=\int_0^{2\pi}\frac{1}{\pi}\left[\operatorname{Re}\left(\frac{\dot{\eta}_1(s)}{\eta_1(s)-\eta_1(t)}\right)\right]\varphi_1(t)dt+\int_0^{2\pi}\frac{1}{\pi}\left[\operatorname{Re}\left(\frac{\dot{\eta}_1(s)}{\eta_1(s)-\eta_2(t)}\right)\right]\varphi_2(t)dt. \tag{5.65}$$

For $s \in I_2$, equation (5.64) becomes

$$\psi_2(s)+\int_0^{2\pi}\frac{1}{\pi}\left[\operatorname{Im}\left(\frac{\dot{\eta}_2(s)}{\eta_2(s)-\eta_1(t)}\right)\right]\psi_1(t)dt+\int_0^{2\pi}\frac{1}{\pi}\left[\operatorname{Im}\left(\frac{\dot{\eta}_2(s)}{\eta_2(s)-\eta_2(t)}\right)\right]\psi_2(t)$$

$$=\int_0^{2\pi}\frac{1}{\pi}\left[\operatorname{Re}\left(\frac{\dot{\eta}_2(s)}{\eta_2(s)-\eta_1(t)}\right)\right]\varphi_1(t)dt+\int_0^{2\pi}\left[\operatorname{Re}\left(\frac{\dot{\eta}_2(s)}{\eta_2(s)-\eta_2(t)}\right)\right]\varphi_2(t)dt. \tag{5.66}$$

Recall from Section 5.3.1, that the function *f* is analytic on $\Omega^- \cup \Gamma$. Then at each point *z* in that domain when $z_0 = 0$, $f(z)$ can be represented by the following Laurent series expansion such that $f(\infty)=0$:

$$f(z)=\frac{c_1}{z}+\frac{c_2}{z^2}+\frac{c_3}{z^3}+\cdots \tag{5.67}$$

Differentiate once with respect to *z* and multiply with –*iz* we will obtain

$$-zif'(z)=-\frac{c_1}{z}-\frac{2c_2}{z^2}-\frac{3c_3}{z^3}-\cdots=F(z) \tag{5.68}$$

Since $g^-(\eta(t))=-if^{-}{}'(\eta(t))$,

$$g^-(z)=\frac{F(z)}{z}. \tag{5.69}$$

By means of Cauchy-Formula and the fact from Gakhov (1966, p.24),

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{F(\eta)}{\eta}d\eta = F(\infty) = 0, \tag{5.70}$$

$$\frac{1}{2\pi i}\int_{\Gamma} g^{-}(\eta)d\eta = 0, \tag{5.71}$$

$$\int_{\Gamma} g^{-}(\eta)d\eta = 0. \tag{5.72}$$

From (5.38), $g^{-}(\eta(t)) = \dfrac{\varphi(t)+i\psi(t)}{\dot{\eta}(t)}$ where $A(t) = \dot{\eta}(t)$, so (5.72) becomes

$$\int_{\Gamma}\frac{\varphi(t)+i\psi(t)}{\dot{\eta}(t)}\dot{\eta}(t)dt = 0, \tag{5.73}$$

$$\int_{0}^{2\pi}\varphi(t)dt + i\int_{0}^{2\pi}\psi(t)dt = 0. \tag{5.74}$$

Note that, the exterior Neumann problem need to satisfy (4.7) in Chapter 4. From (5.55),

$$\varphi(t) = |\dot{\eta}(t)|\gamma(t).$$

Therefore, (4.7) becomes

$$\int_{0}^{2\pi}\varphi(t)dt = 0. \tag{5.75}$$

We consider (5.75) as the additional condition for the exterior Riemann-Hilbert problem which the right-hand side of the Riemann-Hilbert problem (5.55) need to satisfy. Thus from (5.75), (5.74) becomes

$$\int_{0}^{2\pi}\psi(t)dt = 0. \tag{5.76}$$

Let us define the kernel $J(s,t)$ for $(s,t)\in I_i \times I_j$ such that

$$J(s,t) = \begin{cases} \dfrac{1}{2\pi}, & s,t\in I_i, \quad i = 1,2,\ldots,m \\ 0, & s\in I_i, \quad t\in I_j, \quad i\neq j, \quad i,j = 1,2,\ldots,m. \end{cases}$$

This means that $J(s,t)$ is equal to 1 when $\eta(s),\eta(t)$ belong to same boundary Γ_i and equal to 0 otherwise. Then we define the operator $\mathbf{J}$ by

$$(\mathbf{J}\psi)(s) = \int_{I} J(s,t)\psi(t)dt. \tag{5.77}$$

Adding (5.77) into (5.57) yields the new integral equation

$$\psi + \mathbf{v}^{*}\psi + \mathbf{J}\psi = \mathbf{u}^{*}\varphi \tag{5.78}$$

which is uniquely solvable.

5.3.5 Modifying the Singular Integral Operator

Consider the right-hand side of (5.78),

$$\mathbf{u}^{*}\varphi = \int_I U^{*}(s,t)\varphi(t)dt = \frac{1}{\pi}\int_I \mathrm{Re}\left(\frac{\dot{\eta}(s)}{\eta(s)-\eta(t)}\right)\varphi(t)dt. \tag{5.79}$$

Using the fact that

$$\frac{1}{\pi}\int_I \mathrm{Re}\left(\frac{\dot{\eta}(t)}{\eta(t)-\eta(s)}\right)dt = 0, \tag{5.80}$$

(5.79) becomes

$$\begin{aligned}
\int_I U^{*}(s,t)\varphi(t)dt &= \frac{1}{\pi}\int_I \mathrm{Re}\left(\frac{\dot{\eta}(s)}{\eta(s)-\eta(t)}\right)\varphi(t)dt + \varphi(s)\left[\frac{1}{\pi}\int_I \mathrm{Re}\left(\frac{\dot{\eta}(t)}{\eta(t)-\eta(s)}\right)dt\right] \\
&= -\frac{1}{\pi}\int_I \mathrm{Re}\left(\frac{\dot{\eta}(s)}{\eta(t)-\eta(s)}\right)\varphi(t)dt + \varphi(s)\left[\frac{1}{\pi}\int_I \mathrm{Re}\left(\frac{\dot{\eta}(t)}{\eta(t)-\eta(s)}\right)dt\right] \\
&= -\left\{\frac{1}{\pi}\int_I \mathrm{Re}\left(\frac{\dot{\eta}(s)}{\eta(t)-\eta(s)}\right)\varphi(t)dt - \varphi(s)\left[\frac{1}{\pi}\int_I \mathrm{Re}\left(\frac{\dot{\eta}(t)}{\eta(t)-\eta(s)}\right)dt\right]\right\} \\
&= -\left\{\frac{1}{\pi}\int_I \mathrm{Re}\left[\dot{\eta}(s)\dot{\eta}(t)\left(\frac{\dfrac{\varphi(t)}{\dot{\eta}(t)} - \dfrac{\varphi(s)}{\dot{\eta}(s)}}{\eta(t)-\eta(s)}\right)\right]dt\right\} \\
&= -\int_I B(s,t)dt,
\end{aligned}$$

where

$$B(s,t) = \frac{1}{\pi}\mathrm{Re}\left[\frac{\dot{\eta}(s)\varphi(t) - \dot{\eta}(t)\varphi(s)}{\eta(t) - \eta(s)}\right], \tag{5.81}$$

for $s \neq t$. We now show that $B(s,t)$ also exists for $s = t$.

Letting $\zeta = \frac{\varphi}{\dot{\eta}}$ and $\dot{\zeta} = \frac{\dot{\eta}\dot{\varphi} - \ddot{\eta}\varphi}{\dot{\eta}^2}$, so that

$$B(s,t) = \frac{1}{\pi}\mathrm{Re}\left[\dot{\eta}(s)\dot{\eta}(t)\left(\frac{\zeta(t) - \zeta(s)}{\eta(t) - \eta(s)}\right)\right],$$

and

$$\lim_{s \to t}\frac{\zeta(t) - \zeta(s)}{\eta(t) - \eta(s)} = \frac{\dot{\varphi}(t)}{\dot{\eta}(t)}.$$

Thus, the values of $B(s,t)$ on the diagonal are

$$B(t,t) = \frac{1}{\pi}\mathrm{Re}\left[\dot{\eta}(t)\dot{\eta}(t)\left(\frac{\frac{\dot{\eta}(t)\dot{\varphi}(t) - \ddot{\eta}(t)\varphi(t)}{\dot{\eta}(t)^2}}{\dot{\eta}(t)}\right)\right]$$

$$= \frac{1}{\pi}\left[\dot{\varphi}(t) - \varphi(t)\,\mathrm{Re}\left(\frac{\ddot{\eta}(t)}{\dot{\eta}(t)}\right)\right]. \tag{5.82}$$

Then, we define the integral operator with kernel B as

$$\mathbf{B}\varphi = \int_I B(s,t)dt \tag{5.83}$$

So, the uniquely solvable integral equation (5.78) become

$$\psi + \mathbf{v}^*\psi + \mathbf{J}\psi = -\mathbf{B}\varphi \tag{5.84}$$

and it has the integral form

$$\psi(s) + \int_I V^*(s,t)\psi(t)dt + \int_I J(s,t)\psi(t)dt = -\int_I B(s,t)dt. \tag{5.85}$$

5.3.6 Computing $f(z)$ and $f'(z)$

After solving for ψ from the integral equation (5.84), the boundary values of the function f' are computed from (5.38) and (5.55), i.e.,

$$-i\dot{\eta}(t)\,f^{-}{}'(\eta(t)) = \varphi(t) + i\psi(t). \tag{5.86}$$

In view of (5.86), the Cauchy integral formula implies that the function $f'(z)$ can be calculated for $z \in \Omega^{-}$ by

$$\begin{aligned} f^{-}{}'(z) &= \frac{1}{2\pi i}\int_{I}\frac{f'(\eta)}{\eta - z}\,d\eta = \frac{1}{2\pi i}\int_{I}\frac{\varphi(t)+i\psi(t)}{-i\dot{\eta}(t)}\,\frac{\dot{\eta}(t)}{\eta(t)-z}\,dt \\ &= \frac{1}{2\pi i}\int_{0}^{2\pi}\frac{\varphi_1(t)+i\psi_1(t)}{-i\dot{\eta}_1(t)}\,\frac{\dot{\eta}_1(t)}{\eta_1(t)-z}\,dt + \frac{1}{2\pi i}\int_{0}^{2\pi}\frac{\varphi_2(t)+i\psi_2(t)}{-i\dot{\eta}_2(t)}\,\frac{\dot{\eta}_2(t)}{\eta_2(t)-z}\,dt. \end{aligned} \tag{5.87}$$

i.e., the function $f^{-}{}'(z)$ is represented by a Cauchy type integral. To calculate the values of *f*, we apply the formula of the integral representation of an analytic function the *m*th derivative of which is representable by the Cauchy type integral (Gakhov, 1966). The kernel of this formula depends on the integration of the Cauchy kernel. In our case, the formula is

$$\begin{aligned} f(z) &= -\frac{1}{2\pi i}\int_{I} f^{-}{}'(\eta)\log\left(1-\frac{\eta(t)}{z}\right)\dot{\eta}(t)dt \\ &= -\frac{1}{2\pi i}\int_{0}^{2\lambda} f_1^{-}{}'(\eta_1)\log\left(1-\frac{\eta_1(t)}{z}\right)\dot{\eta}_1(t)dt - \frac{1}{2\pi i}\int_{0}^{2\pi} f_2^{-}{}'(\eta_2)\log\left(1-\frac{\eta_2(t)}{z}\right)\dot{\eta}_2(t)dt. \end{aligned} \tag{5.88}$$

Hence, the unique solution of the Neumann problem, *u*, can be evaluated from

$$u(z) = \mathrm{Re}[\,f(z)]. \tag{5.89}$$

5.4 Numerical Implementations of the Boundary Integral Equation

Since the function $A(t) = \dot{\eta}$ and $\eta(t)$ are 2π-periodic, the integrals in the integral equation (5.85) can be best discretized on an equidistant grid by the trapezoidal rule, i.e., the integral operators are discretized by the Nyström method with the trapezoidal rule as the quadrature rule.

Let *n* be a given integer and define the *n* equidistant collocation points t_j by

$$t_j = (j-1)\frac{2\pi}{n}, \qquad j = 1,2,\ldots,n. \tag{5.90}$$

Then, using the Nyström method with the trapezoidal rule to discretized the integral equation (5.85), we obtain the linear systems

$$\psi_n(t_i) + \frac{2\pi}{n}\sum_{j=1}^{n} V^*(t_i,t_j)\psi_n(t_j) + \frac{2\pi}{n}\sum_{j=1}^{n} J(t_i,t_j)\psi_n(t_j) = -\frac{2\pi}{n}\sum_{j=1}^{n} B(t_j,t_i). \tag{5.91}$$

with $i = 1,2,\ldots,n$ and ψ_n is an approximation to ψ.

Let $\mathbf{I}$ be the $n\times n$ matrix. Also let $\mathbf{I}_{11}$ and $\mathbf{I}_{22}$ be the $n\times n$ matrix and $\mathbf{I}_1$ be the $n\times 1$ vector whose elements are all unity. Defining the matrix $\mathbb{V} = [\mathbb{V}_{ji}], \mathbb{J} = [\mathbb{J}_{ji}], \mathbb{B} = [\mathbb{B}_{ji}]$ and vector $\mathbb{x} = [\mathbb{x}_j], \mathbb{y} = [\mathbb{y}_j]$ by

$$\mathbb{V}_{ji} = \frac{2\pi}{n}V^*(t_i,t_j) = \frac{2\pi}{n}\begin{cases} \dfrac{1}{\pi}\operatorname{Im}\left(\dfrac{\dot{\eta}(t_j)}{\eta(t_j)-\eta(t_k)}\right), & t_i \neq t_j \\ \dfrac{1}{2\pi}\operatorname{Im}\left(\dfrac{\ddot{\eta}(t_i)}{\dot{\eta}(t_i)}\right), & t_i = t_j \end{cases} \tag{5.92}$$

$$\mathbb{J}_{ji} = \frac{2\pi}{n}J(t_i,t_j) = \frac{2\pi}{n}\left(\frac{1}{2\pi}\right)\mathbf{I}_{11} + \frac{2\pi}{n}\left(\frac{1}{2\pi}\right)\mathbf{I}_{22} \tag{5.93}$$

$$\mathbb{B}_{ji} = -\frac{2\pi}{n} B(t_j, t_i)$$

$$= -\frac{2\pi}{n} \begin{cases} \dfrac{1}{\pi} \operatorname{Re}\left(\dfrac{\dot{\eta}(t_i)\varphi(t_j) - \dot{\eta}(t_j)\varphi(t_i)}{\eta(t_j) - \eta(t_i)} \right), & t_j \neq t_k \\ \dfrac{1}{\pi} \left[\dot{\varphi}(t_j) - \varphi(t_j) \operatorname{Re}\left(\dfrac{\ddot{\eta}(t_j)}{\dot{\eta}(t_j)} \right) \right], & t_j = t_k \end{cases} \tag{5.94}$$

$$\mathbb{x}_j = \psi_n(t_j) \tag{5.95}$$

and

$$\mathbb{y}_j = \mathbb{B}_{ji} \mathbf{I}_1 . \tag{5.96}$$

Hence, the application of Nyström method to the uniquely solvable integral equation (5.85) leads to the following n by n linear system

$$(\mathbf{I} + \mathbf{V} + \mathbf{J})\mathbf{x} = \mathbf{y}. \tag{5.97}$$

By solving the linear system (5.97) using Gaussian elimination, we obtain $\psi_n(t_j)$ for $j = 1,2,\ldots,n$ Then the approximate solution $\psi_n(t)$ can be calculated for all $t \in [0,2\pi]$ using the Nyström interpolating formula, i.e., the approximation $\psi_n(t)$ of the integral equation (5.83) is given by

$$\psi_n(t) = -\frac{2\pi}{n} \sum_{i=1}^{n} B(t,t_i) \frac{2\pi}{n} - \sum_{i=1}^{n} V^*(t_i,t)\psi_n(t_i) - \frac{2\pi}{n} \sum_{i=1}^{n} J(t_i,t)\psi_n(t_i). \tag{5.98}$$

By obtaining ψ, (5.87) and (5.88) implies that the function $f^{-}{'}(z)$ and $f(z)$ can be calculated for $z \in \Omega^{-}$.

5.5 Examples

Examples 5.1

In our first example, we consider a doubly connected region, Ω_1 as shown in Figure 5.3. The boundaries of this region are parameterized by the functions

$\Gamma_1 = \eta_1(t) = 0.5e^{-it}$,

$\Gamma_2 = \eta_2(t) = 1 + 0.25e^{-it}$.

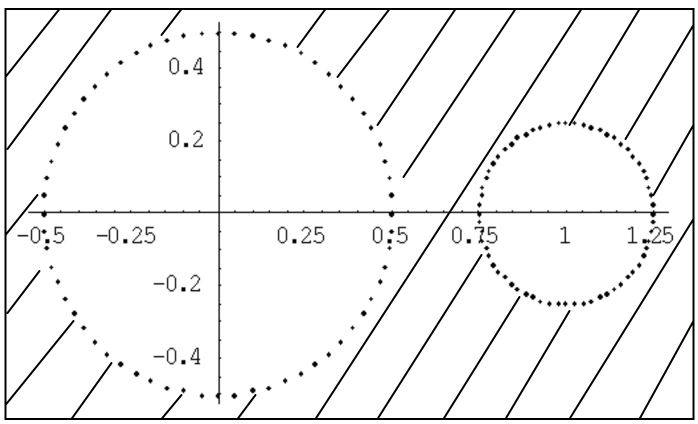

Figure 5.3: The test region Ω_1 for Example 5.1.

We choose the function

$$f(z) = \frac{1}{z},$$

which is analytic in Ω_1. Then the function

$$u(z) = \mathrm{Re}(f(z)) = \frac{x}{x^2 + y^2},$$

solves the Neumann problem uniquely in this region with the boundary condition

$$\frac{\partial u(\eta)}{\partial \mathbf{n}} = \gamma(t) = \frac{1}{|\dot{\eta}(t)|}\mathrm{Re}[-i\dot{\eta}(t) f^{-}{}'(\eta(t))].$$

We describe the error by infinity-norm error $\|f'(z)-f_n'(z)\|_\infty$, $\|u(z)-u_n(z)\|_\infty$ where $f_n'(z)$ and $u_n(z)$ are the numerical approximations of $f'(z)$ and $u(z)$ respectively. This example has been solved with the test points $z=\rho e^{\frac{i\pi}{4}}$, $\rho = 0.6, 1.2, 1.8$. The results are shown in Tables 5.1 and 5.2.

Table 5.1: The error $\|f'(z)-f_n'(z)\|_\infty$ for Example 5.1.

ρ	n = 16	n = 32	n = 64	n = 128
0.6	0.0054561	0.00815023	0.000023774	2.03469e-10
0.8	0.000847448	4.59178 e-07	1.35003e-13	8.95605e-16
1.2	5.6913e-07	4.73066e-13	1.11026e-16	1.47313e-16
1.8	1.8605e-09	9.14561e-17	5.94638e-17	1.06928e-17

Table 5.2: The error $\|u(z)-u_n(z)\|_\infty$ for Example 5.1.

ρ	n = 16	n = 32	n = 64	n = 128
0.6	0.00385804	0.000104632	1.55175 e-07	6.69242e-13
0.8	0.00002819	7.87121e-09	1.44329e-15	1.11022e-16
1.2	2.83839e-08	1.21044e-14	0	4.44089e-16
1.8	6.25139e-11	0	5.5512e-17	5.5512e-17

The test point when ρ = 1.8 has higher degree of accuracy than the other test points because it situated far away from the boundary.

Example 5.2

In this example, we consider a doubly connected region, Ω_2 as shown in Figure 5.4. The boundaries of this region are parameterized by the functions

$\Gamma_1 = \eta_1(t) = 0.5e^{-it}$,
$\Gamma_2 = \eta_2(t) = 1 + 0.25e^{-it}$

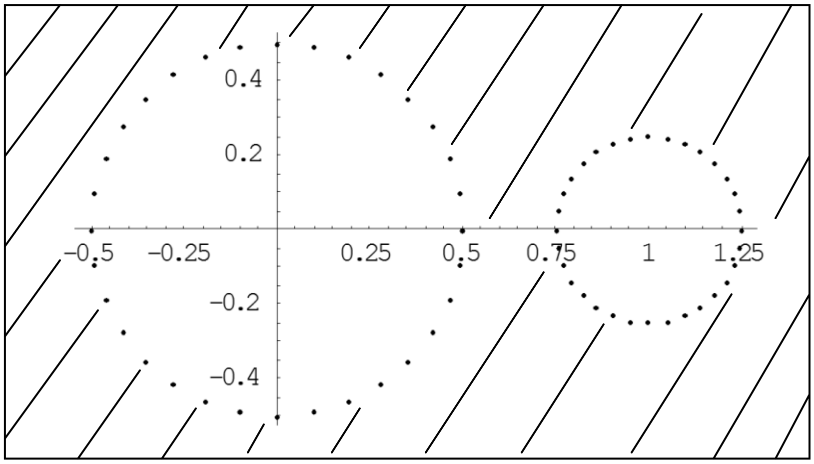

Figure 5.4: The test region Ω_2 for Example 5.2.

We choose the function

$$f(z) = \frac{1}{z^2}$$

which is analytic in Ω_2. Then the function

$$u(z) = \mathrm{Re}(f(z)) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

solves the Neumann problem uniquely in this region with the boundary condition

$$\frac{\partial u(\eta)}{\partial \mathbf{n}} = \gamma(t) = \frac{1}{|\dot{\eta}(t)|}\mathrm{Re}[-i\dot{\eta}(t) f^{-}'(\eta(t))].$$

We describe the error by infinity-norm error $\| f'(z) - f_n'(z)\|_\infty$, $\|u(z) - u_n(z)\|_\infty$ where $f_n'(z)$ and $u_n(z)$ are the numerical approximations of $f'(z)$ and $u(z)$ respectively. We choose three test points in this example. The results are shown in Tables 5.3 and 5.4.

Table 5.3: The error $\|f'(z) - f_n'(z)\|_\infty$ for Example 5.2.

z	n = 16	n = 32	n = 64	n = 128
0.4 + 0.6i	0.0151803	4.34525e-05	3.53997e-10	1.83103e-15
0.4 + i	7.49893e-06	3.46779e-11	1.11022e-16	4.44089e-16
0.75 + 0.2i	0.078725	0.00156144	5.70282e-07	7.6034e-14
2	3.34694e-8	3.56424e-16	2.45275e-16	3.2536e-16

Table 5.4: The error $\|u(z) - u_n(z)\|_\infty$ for Example 5.2.

z	n = 16	n = 32	n = 64	n = 128
0.4 + 0.6i	0.000243573	3.82728e-07	1.72273e-12	5.55112e-16
0.4 + i	3.78735e-07	1.02396e-12	1.11022e-16	1.11022e-16
0.75 + 0.2i	0.00179845	5.25571e-06	1.89133e-09	4.44089e-16
2	3.77351e-9	0	1.11022e-16	1.66533e-16

The result of this example shows that the test point at z = 0.4 + i which situated furthest away from the boundary has higher degree of accuracy than the other test points which is near to the boundary. When comparing the error between $\|f'(z) - f_n'(z)\|_\infty$ and $\|u(z) - u_n(z)\|_\infty$, the degree of accuracy between the computed $f_n'(z)$ is less than the computed $u_n(z)$ when the test point is near to the boundary.

Example 5.3

Let Ω_2 be a doubly connected region as shown in Figure 5.5. The boundaries of this region are parameterized by the functions

$\Gamma_1 = \eta_1(t) = (3 + \cos 3t)e^{-it}$,

$\Gamma_2 = \eta_2(t) = -6 + e^{-it}$.

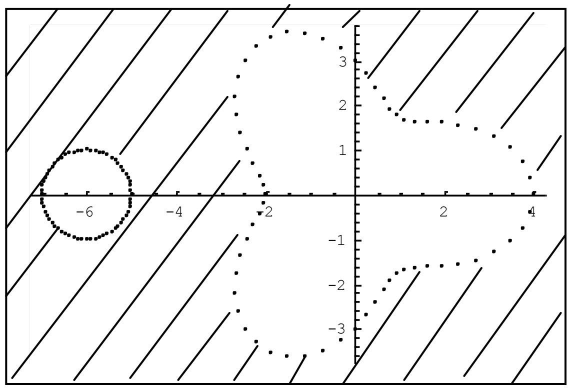

Figure 5.5: The test region Ω_3 for Example 5.3.

Consider the function *f* as in Example 5.1, i.e. $f(z)=\dfrac{1}{z}$. We choose five test points and the results are shown in Table 5.5 and 5.6.

Table 5.5: The error $\|f'(z)-f_n'(z)\|_\infty$ for Example 5.3.

z	n = 32	n = 64	n = 128	n = 256	n = 512
5	0.00010917	2.96287e-07	1.21875e-13	2.78679e-17	1.4008e-17
7*i*	1.61364e-08	2.12095e-14	1.04468e-17	4.35206e-18	1.10453e-17
1 – 2*i*	0.00708929	0.00023558	2.77209e-07	3.8408e-13	1.08389e-16
-5 + *i*	1.55593e-07	2.54067e-12	6.93889e-18	4.16695e-17	8.67362e-18
-2 + 3.7*i*	0.00997382	0.00267642	0.00011688	2.41082e-07	1.02812e-12
-8	5.0534e-09	2.35923e-16	8.67602e-18	1.04086e-17	1.26674e-17

Table 5.6: The error $\|u(z)-u_n(z)\|_\infty$ for Example 5.3.

z	n = 32	n = 64	n = 128	n = 256	n = 512
5	0.0000242572	3.29544e-08	1.21875e-13	8.32667e-17	1.11022e-16
7*i*	6.04787e-09	4.87903e-15	1.21295e-17	2.20229e-17	1.517044e-18
1 – 2*i*	0.000041245	4.31642e-06	3.48971e-09	2.35922e-15	5.55112e-17
-5 + *i*	8.94459e-08	4.14724e-13	8.32667 e-17	5.55112e-17	1.66533e-16
-2 + 3.7*i*	0.00112618	0.000166195	4.27878e-06	2.82063e-09	8.86791e-15
-8	1.26703e-10	1.38778e-17	1.38778e-17	2.77556e-17	5.55112e-17

From the results above, when the test points are near to the boundary, we get less degree of accuracy, and the degree of accuracy of the computed $f_n'(z)$ is less than the computed $u_n(z)$.

5.6 Conclusion

The reduction of the exterior Neumann problem on multiply connected region into the exterior Riemann-Hilbert problem is derived which leads to a non-uniquely solvable boundary integral equation. Additional constraint to the solution of the problem is required so that the problem is uniquely solvable. The related integral equation is modified to obtain uniquely solvable integral equation. Furthermore, we also presented the numerical implementations of the derived boundary integral equation in this chapter. The integral equation was solved by Nyström method with the trapezoidal rule as the quadrature rule. The exterior multiply connected Neumann problems were then solved numerically in three test regions using the proposed method. The numerical examples illustrated that the proposed method yields approximation of high accuracy.

CHAPTER 6

CONCLUSIONS AND SUGGESTIONS

6.1 Conclusions

A Neumann problem is a boundary value problem of determining a harmonic function $u(x,y)$ interior or exterior to a region with prescribed values of its normal derivative du/dn on the boundary. This research has presented several new boundary integral equations for the solution of Laplace's equation on both bounded and unbounded multiply connected regions, with the Neumann boundary condition. The method is based on uniquely solvable Fredholm integral equations of the second kind with the generalized Neumann kernel. The idea of formulation of these integral equations is firstly to reduce the Neumann problem on each region into the equivalent RH problem from which an integral equation is constructed. We have solved these integral equations numerically using Nyström method with the trapezoidal rule. Once we got the solutions of the integral equations, the solutions of the Neumann problems are within our reach. Several results of the numerical examples show the efficiency of our approach.

6.1.1 Name of articles/ manuscripts/ books published

- M.M.S. Nasser, A.H.M. Murid, M. Ismail, E. M. A. Alejaily, Boundary Integral Equations with the Generalized Neumann Kernel for Laplace's Equation in Multiply Connected Regions, Journal in Applied Mathematics and Computation, Vol. 217 (Jan 2011), 4710 – 4727. (IF: 1.125)
- E.M.A. Alejaily, Azlina Jumadi, Ali H.M. Murid, Hamisan Rahmat, "Computing the Solution of the Neumann Problem Using Integral Equation and Runge-Kutta Method", Proceedings of the Regional Annual Fundamental Science Symposium (RAFSS 2010), Ibnu Sina Institute, UTM, 51 – 61.
- Laey-Nee Hu, Ali H.M. Murid and Mohd Nor Mohamad, An Integral Equation Method for Conformal Mapping of Multiply Connected Regions onto an Annulus with Circular Slits via the Neumann Kernel, CD Proceedings of the 5th Asian Mathematical Conference (AMC) 2009.
- Eijaily M.A. Alejaily, Ali H.M. Murid & Mohamed M.S. Nasser, A Boundary Integral Equation for the Neumann Problem in Bounded Multiply Connected Region, Proceedings of the Simposium Kebangsaan Sains Matematik ke-17 (2009), 201-207.
- Azlina Jumadi, Munira Ismail & Ali H.M. Murid, An Integral Equation method for solving Exterior Neumann Problems on Smooth Regions, Proceedings of the Simposium Kebangsaan Sains Matematik ke-17 (2009), 208-215.
- Ejaily Milad Ahmed Alejaily, A Boundary Integral Equation for the Neumann Problem in Bounded Multiply Connected Region, M.Sc. Dissertation (2009) UTM.
- Azlina Jumadi, An Integral Equation Method for Solving Exterior Neumann Problems on Smooth Regions, M.Sc. Dissertation (2009) UTM.
- Chye Mei Sian, A Boundary Integral Equation for the Exterior Neumann Problem on Multiply Connected Region, M.Sc. Dissertation (2010) UTM.

6.1.2 Title of Paper Presentations (international/ local)

- International Conference on Computational and Function Theory, 8-12 June 2009, Bilkent University, Ankara, Turkey. Title of paper: A Boundary Integral Equation with the Generalized Neumann Kernel for Laplace's Equation in Multiply Connected Regions. Authors: Mohamed M.S. Nasser and Ali H.M. Murid (presenter).

- 5th Asian Mathematical Conference (AMC 2009), 22-26 Jun 2009, PWTC Kuala Lumpur. Title of paper: A Boundary Integral Equation for Conformal Mapping of Multiply Connected Regions onto the Circular Slit Regions Via The Neumann Kernel. Authors: Laey-Nee Hu (presenter), Ali H.M. Murid and Mohd Nor Mohamad.
- 2nd International Conference and Workshops on Basic and Applied Sciences and Regional Annual Fundamental Science Seminar (ICORAFFS 2009), 3-4 June 2009, The Zone Regency Hotel, Johor Bahru. Title of paper: *An Integral Equation Method for Solving Neumann Problems in Simply Connected Regions with Smooth Boundaries*. Authors: Ali H.M. Murid, Ummu Tasnim Husin (Presenter) and Hamisan Rahmat.
- Regional Annual Fundamental Science Symposium 2010 (RAFSS2010), 8-9 June 2010, Grand Seasons Hotel, Kuala Lumpur. Title: *Computing the Solution of the Neumann Problem Using Integral Equation and Runge-Kutta Method*. Authors: E. M. A. Alejaily, Azlina Jumadi, Ali H. M. Murid (presenter) and Hamisan Rahmat.

- Simposium Kebangsaan Sains Matematik Ke-17, 15-17 Dis 2009, Hotel Mahkota Melaka; Title of paper: A Boundary Integral Equation for the Neumann Problem in Bounded Multiply Connected Region,Penulis: E. M. A. Alejaily, A. H. M. Murid (presenter) and M.M.S. Nasser.
- Simposium Kebangsaan Sains Matematik Ke-17, 15-17 Dis 2009, Hotel Mahkota Melaka; Title of paper: An Integral Equation method for solving Exterior Neumann Problems on Smooth Regions,Penulis: Azlina Jumadi (presenter), Munira Ismail & Ali H.M. Murid.

6.1.3 Human Capital Development

- Four M.Sc. students (completed)
- One Research Officer

6.2 Suggestions for Future Research

This research develops a method to solve the Neumann problem within certain conditions and assumptions. For further research, we suggest the following

- We mentioned in the previous chapter that we lost some accuracy of values of $u(z)$ as we get closer to the boundaries. This lost, we think, can be reduced if we study the possibility of improving the formula (5.89) along the same lines as we have done for evaluating $f'(z)$.
- In this research we did not calculate the solution of the Neumann problem on the boundaries $u(\eta)$. This can be achieved if we calculate $f(\eta)$ from $f^{-\prime}(\eta)$. For that we suggest to treat the formula (5.86) as a differential equation and apply, for example, the Runge-Kutta method to solve it.
- The domains of the problems in this research are multiply connected regions with smooth boundaries. We propose extending our approach for multiply connected regions with nonsmooth boundaries.
- The general theories were developed for solving the interior and exterior Neumann problems on multiply connected region. Other potential applications need to be explored. Probably some extensions or modifications of the theories are required to obtain integral equations related to mixed boundary value problem with Dirichlet-Neumann boundary conditions.

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