Computation of the Riemann Map using Integral Equation Method and Cauchy's Integral Formula

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Abstract

The Riemann map is a conformal mapping that maps a simply connected region to a unit disk. Such a map has applications in fluid mechanics, electrostatics, and image processing. We present a numerical procedure for the computation of the Riemann map based on two stages. First we compute the boundary values of the Riemann map for the region we wish to map by solving an integral equation. Then we compute the Riemann map in the interior of the region using the well-known Cauchy's integral formula. Due to periodicity, trapezoidal rule is the most appealing procedure for these computations. We also provide some results of our numerical experiments using epitrochoid ("apple") as a test region.

Keywords: Riemann map, Integral equation, Cauchy's integral formula.

1. Introduction

An important and familiar tool of science and engineering since the development of complex analysis is conformal mapping. Conformal mapping uses functions of complex variables to transform complicated boundaries to simpler, more manageable configurations. A conformal mapping has the special property that angles between curves are preserved in magnitude as well as in direction (see Figure 1). Thus any set of orthogonal curves in the z-plane would therefore appear as another set of orthogonal curves in the w-plane.

An important fact about conformal mapping which accounts for much of its applications is that it the Laplace's equation is invariant under conformal mapping (Henrici, 1974, Section 5.6). This forms the basis of a method of solving numerous two-dimensional boundary-value problems such as the Dirichlet problem and the Neumann problem.

Figure 1: Conformality.

Figure 2: Dirichlet problem.

The exact solution of the Dirichlet problem (see Figure 2) for the unit disk is known:

$$\psi(w) = \int_0^1 \psi_0(t) \frac{1 - |w|^2}{|e^{2\pi i t} - w|^2} \, dt.$$

If $w = f(z)$ is a conformal mapping that transforms the region D to a unit disk E, Then the solution to the original problem is then found to be

$$\phi(z) = \frac{1}{2\pi i} \int_\Gamma \phi_0(t) \frac{1 - |f(z)|^2}{|f(t) - f(z)|^2} \frac{f'(t)}{f(t)} \, dt.$$

In various applied problems, by means of conformal maps, problems for certain "physical regions" are transplanted into problems on some standardized "model regions" where they can be solved easily. By transplanting back we obtain the solutions of the original problems in the physical regions. This process is used, for example, for solving boundary value problems problems about fluid flow, electrostatics, heat, mechanics, and aerodynamics. For these and other physical problems that use conformal mapping techniques, see, for example, the books by Henrici (1974), Churchill and Brown (1984), and Schinzinger and Laura (1991). Recently conformal maps have been proposed as a model for image representation by the human brain as well as face representation (Baricco et al., 1995). Since confromal maps are one-to-one, no information is lost, and their high non-linearity is expected to provide an easy way of distinguishing small differences between arbitrary regions and a fixed domain. Furthermore, conformal maps preserve some essential features of visual information.

Many methods for conformal mapping involve computing the boundary values of the mapping function by solving an integral equation for the boundary correspondence function. This is considered sufficient since in principle the interior values of the mapping function can be obtained by the well-known Cauchy's integral formula. Thus the question of computing the mapping of interior region is often neglected in many papers including Razali, Nashed, and Murid (1997, 1998, 2000). In this paper we describe a numerical method for computing the conformal mapping of the interior of simply connected regions onto a unit disk based on the integral equation developed in Murid, Nashed, and Razali (1997) and the Cauchy's integral formula. The numerical procedure has two parts. First we compute the boundary values of the Riemann map for the region we wish to map by solving the integral equation. Then we compute the Riemann map in the interior of the region using the Cauchy's integral formula. Due to periodicity, trapezoidal rule is the most appealing procedure for these computations. We also provide some results of our numerical experiments using epitrochoid ("apple") as a test region.

In the second section of this paper we present the integral equation formulation, and in the third we give the numerical methods used to solve the integral equation and computing the Cauchy's integral formula to evaluate the mapping in the interior of the region. Results of numerical experiments are presented in the last section.

2. Integral equation and Cauchy's integral formula

For a simply connected region, a classical theorem which asserts the existence and uniqueness of a conformal map onto a unit disk, is the Riemann mapping theorem. Such a conformal map is known as the Riemann mapping function.

Theorem 1 (Riemann Mapping Theorem) *Let Ω be a simply connected region which is not the whole complex plane and let $a \in \Omega$. Then there exists* *a unique one-to-one analytic function $R : \Omega \to U = \{w : |w| < 1\}$ satisfying the conditions*

$$R(a) = 0, \qquad R'(a) > 0.$$

The mapping function R can be computed on the smooth boundary Γ of Ω by means of the formula

$$R(z) = \frac{1}{i} T(z) \frac{R'(z)}{|R'(z)|}, \qquad z \in \Gamma, \tag{1}$$

where $T(z) = z'(t)/|z'(t)|$ is the complex unit tangent vector to Γ at $z(t)$.

Several methods have been proposed in the literature for the numerical evaluation of the Riemann mapping function. Generally , these methods fall into two types: expansion and integral equation methods. Common expansion methods are the Bergman and the Szegö kernels method, and the Ritz variational methods. The integral equation methods are more preferable and effective for numerical conformal mapping.

The Szegö kernel $S(z, a)$ and the Bergman kernel $B(z, a)$ of a simply connected region in the complex plane are well-known reproducing kernels and their relationships with the Riemann mapping function are classical, i.e.,

$$R(z) = \frac{1}{i} T(z) \frac{S(z, a)^2}{|S(z, a)|^2}, \qquad z \in \Gamma. \tag{2}$$

$$R(z) = \frac{1}{i} T(z) \frac{B(z, a)}{|B(z, a)|}, \qquad z \in \Gamma. \tag{3}$$

Thus the Riemann map is completely determined by the Szegö kernel and the Bergman kernel functions. The fact that there is an efficient numerical method, based on the Kerzman-Stein-Trummer integral equation, for computing the Szegö kernel has been known since 1986 (Kerzman and Trummer, 1986).

Theorem 2 *Let Γ be of class C^2 Jordan curve. The Szegö kernel $S(z, a)$ is the unique continuous solution to the Kerzman-Stein-Trummer integral equation*

$$S(z, a) + \int_\Gamma A(z, w) S(w, a) \, |dw| = \overline{H(a, z)}, \qquad z \in \Gamma, a \in \Omega, \tag{4}$$

where

$$A(z, w) = \begin{cases} \overline{H(w, z)} - H(z, w), & \text{if } w, z \in \Gamma, w \neq z \\ 0, & \text{if } w = z \in \Gamma, \end{cases} \tag{5}$$

and

$$H(w, z) = \frac{1}{2\pi i} \frac{T(z)}{z - w}, \qquad w \in \overline{\Omega}, z \in \Gamma, w \neq z. \tag{6}$$

The following integral equation for the Bergman kernel discovered by Razali, Nashed, and Murid (1997) can also be used effectively for numerical conformal mapping:

Theorem 3 *Let* Γ *be an analytic Jordan curve. The function* $\tilde{B}(z,a) = T(z)B(z,a)$ *is the unique continuous solution to the integral equation*

$$\tilde{B}(z,a) + \int_\Gamma N(z,w)\tilde{B}(w,a)\,|dw| = -\frac{1}{\pi}\frac{\overline{T(z)}}{(\overline{z}-\overline{a})^2}, \qquad z \in \Gamma, \tag{7}$$

where

$$N(z,w) = \begin{cases} \dfrac{1}{\pi}\mathrm{Im}\left[\dfrac{T(z)}{z-w}\right], & \text{if } w, z \in \Gamma, w \neq z \\[2ex] \dfrac{1}{2\pi}\dfrac{\mathrm{Im}\left[z''(t)\overline{z'(t)}\right]}{|z'(t)|^3}, & \text{if } w = z \in \Gamma. \end{cases} \tag{8}$$

Once the boundary values of the mapping function R are known, the values of the mapping function may be calculated by quadrature at arbitrary interior points of their domains of defination through the following theorem.

Theorem 4 (Cauchy's Integral Formula) *Let* f *be analytic everywhere within and on a simple closed contour* C*, taken in the positive sense. If* z *is any point inside* C*, then*

$$w = f(z) = \frac{1}{2\pi i}\int_C \frac{f(\tau)}{\tau - z}\,d\tau \tag{9}$$

If we introduce $\tau = z(t)$, $0 \leq t \leq \beta$, and $f(z) = R(z)$ represents the Riemann mapping function which maps the interior of C onto the unit disk U, then equation (9) becomes

$$w = R(z) = \frac{1}{2\pi i}\int_C \frac{R(z(t))z'(t)}{z(t) - z}\,dt \tag{10}$$

Since the image point $R(z(t))$ describes the unit circle, we can write

$$R(z(t)) = e^{i\theta(t)}.$$

The function $\theta(t)$ is called the **boundary correspondence function** for the map R. Thus we can rewrite (10) as

$$w = R(z) = \frac{1}{2\pi i}\int_0^\beta \frac{e^{i\theta(t)}z'(t)}{z(t) - z}\,dt \tag{11}$$

3. Numerical implementation

Using a parametric representation $z(t)$ of Γ, $0 \le t \le \beta$, the integral equation (7) becomes

$$\phi(t) + \int_0^\beta v(t,s)\phi(s)\,ds = \psi(t), \tag{12}$$

where for $0 \le s, t \le \beta$,

$$\phi(t) = |z'(t)|\tilde{B}(z(t),a), \tag{13}$$

$$\psi(t) = -\frac{1}{\pi}\frac{\overline{z'(t)}}{(\overline{z(t)} - \overline{a})^2}, \tag{14}$$

$$v(t,s) = \begin{cases} \dfrac{1}{\pi}\mathrm{Im}\left[\dfrac{z'(t)}{z(t)-z(s)}\right], & \text{if } t \ne s, \\ \dfrac{1}{2\pi}\mathrm{Im}\left[\dfrac{z''(t)}{z'(t)}\right], & \text{if } t = s. \end{cases} \tag{15}$$

The new kernel $v(t,s)$ is also known as the parametric Neumann kernel (Henrici, 1986, p. 394).

Since the functions ϕ, ψ, and v in (12) are β-periodic, an appealing procedure for solving (12) numerically is using the Nyström's method (Atkinson, 1976) with the trapezoidal rule which is equivalent to the Fourier method (Berrut and Trummer, 1987). The trapezoidal rule is the most accurate method for integrating periodic functions (Davis and Rabinowitz , 1984, pp. 134-142). Choosing n equidistant collocation points $t_i = (i-1)\beta/n$ and the trapezoidal rule for Nyström's method to discretize (12), we obtain

$$\phi(t_i) + \frac{\beta}{n}\sum_{j=1}^{n} v(t_i,t_j)\phi(t_j) = \psi(t_i), \qquad 1 \le i \le n. \tag{16}$$

Defining the matrix Q by $Q_{ij} = \beta v(t_i,t_j)/n$ and $x_i = \phi(t_i)$, $y_i = \psi(t_i)$, equation (16) can be rewritten as an n by n system

$$(I+Q)\mathbf{x} = \mathbf{y}. \tag{17}$$

Since (12) has a unique solution, then for a wide class of quadrature formula the system (17) also has a unique solution, as long as n is sufficiently large (Atkinson, 1976).

Once the solution $x_i = \phi(t_i)$ has been computed, the boundary value $R(z(t_i))$ is calculated by

$$R(z(t_i)) = -i\frac{x_i}{|x_i|}.$$

Note that discretization of (12) provides us with a very good "natural" interpolation formula (Atkinson, 1976):

$$\phi(t) = \psi(t) - \frac{\beta}{n}\sum_{j=1}^{n} v(t,t_j)\phi(t_j).$$

If the computation of the boundary correspondence function is required, let $\theta(t)$ be the boundary correspondence function to a representation $z = z(t)$, $0 \le t \le \beta$, of Γ. Then

$$R(z(t)) = e^{i\theta(t)} \tag{18}$$

where R is the Riemann mapping function. Differentiating (18) yields

$$R'(z(t))z'(t) = i\theta'(t)e^{i\theta(t)}. \tag{19}$$

Therefore

$$\theta(t) = \arg\left(-iR'(z(t))z'(t)\right),$$

and on using (1) and (3) leads to

$$\begin{aligned}\theta(t) &= \arg\left(-iB(z(t),a)z'(t)\right)\\ &= \arg\left(-iB(z(t),a)|z'(t)|T(z(t))\right)\\ &= \arg\left(-i\tilde{B}(z(t),a)|z'(t)|\right).\end{aligned}$$

Finally using (13), we can compute the boundary correspondence function (without integration) by the formula

$$\theta(t) = \arg\left(-i\phi(t)\right). \tag{20}$$

In principle, the image of an interior point z_0 under the Riemann map may be computed via the Cauchy integral formula in the form of

$$R(z_0) = \frac{1}{2\pi i}\int_0^\beta \frac{R(z(t))z'(t)}{z(t)-z_0}\,dt. \tag{21}$$

Since the integrand is β-periodic, the trapezoidal rule is most favourable to evaluate (21) numerically. Choosing n equidistant collocation points $t_i = (i-1)\beta/n$ and the trapezoidal rule to discretize (21), we obtain

$$RMC(z_0) = \frac{\beta}{2\pi i n}\sum_{j=1}^{n}\frac{R(z(t_i))z'(t_i)}{z(t_i)-z_0} = -\frac{\beta}{2\pi n}\sum_{j=1}^{n}\frac{x_i z'(t_i)}{|x_i|(z(t_i)-z_0)},$$

where $RMC(z_0)$ is an approximation of $R(z_0)$.

4. Numerical results

For our numerical experiments, we have used an epitrochoid (”apple”) as a test region whose exact boundary correspondence is known. The apple has the complex parametric representation

$$\partial\Omega : z(t) = e^{it} + \frac{\alpha}{2}e^{2it}, \qquad 0 \le t \le 2\pi, \quad 0 \le \alpha < 1$$

The boundary correspondence of the Riemann map is given by

$$\theta(t) = t, \quad 0 \le t \le 2\pi.$$

The Cartesian equation of a radial line joining the origin to $z(t)$ is given by

$$y = \frac{\mathrm{Im}\,[z(t)]}{\mathrm{Re}\,[z(t)]}x, \quad 0 \le x \le \mathrm{Re}\,[z(t)].$$

We then choose n equidistant collocation points $t_k = 2\pi(k-1)/n$, $1 \le k \le n$, and m equidistant collocation points $x_j = \mathrm{Re}\,[z(t)](j-1)/m$, $1 \le j \le m$. The image of each collocation point $z = x + iy = z(j,k)$ on the radial line under the Riemann map is then computed via the Cauchy integral formula in the form of

$$RM(j,k) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{it} z'(t)}{z(t) - z}\, dt.$$

The value of this integral is calculated numerically using the MATHEMATICA software (Wolfram, 1991). The image points are then connected by a smooth curve using the command PlotJoined of MATHEMATICA.

The circular grids inside the apple have the form

$$z(\tau) = re^{i\tau}.$$

For $0 \le r \le 1 - \alpha/2$, the circular grids are complete circles. While for $1 - \alpha/2 < r < 1 + \alpha/2$, the circular grids are not complete circles but circular arcs, each intercepting the apple at two points. To determine these two ponits, we solve

$$re^{i\tau} = e^{it} + \frac{\alpha}{2}e^{2it}.$$

for τ. Comparing the real and imaginary parts, gives

$$r\cos\tau = \cos t + \frac{\alpha}{2}\cos 2t,$$
$$r\sin\tau = \sin t + \frac{\alpha}{2}\sin 2t.$$

Squaring these equations and adding, we get

$$\cos t = \frac{1}{\alpha}\left(r^2 - 1 - \frac{\alpha^2}{4}\right).$$

Substituting this result into

$$r\cos\tau = \cos t + \frac{\alpha}{2}\cos 2t = \cos t + \frac{\alpha}{2}(2\cos^2 t - 1),$$

leads to

$$\cos\tau = \frac{1}{\alpha r}\left(r^2 - 1 - \frac{\alpha^2}{4}\right)\left(r^2 - \frac{\alpha^2}{4}\right) - \frac{\alpha}{2r}$$

Thus the two solutions are

$$\tau = \pm \arccos \left[\frac{1}{\alpha r} \left(r^2 - 1 - \frac{\alpha^2}{4} \right) \left(r^2 - \frac{\alpha^2}{4} \right) - \frac{\alpha}{2r} \right].$$

We then choose N equidistant collocation points $r_k = (1 + \alpha/2)(k-1)/n$, $1 \leq k \leq N$, and M equidistant collocation points $\tau_j = 2\pi(j-1)/M$, $1 \leq j \leq M$. The image of each collocation point $z = re^{i\tau} = z(j,k)$ on the circular grids under the Riemann map is then computed via the Cauchy integral formula as described earlier. In our experiments, we have chosen $a = 0.9$, $0 \leq r \leq 0.5$ with stepsize 0.05 for complete circles, and $0.55 \leq r \leq 1.4$ with stepsize 0.05 for circular arcs (see Figure 3 for the case $n = 128$).

Figure 3: Polar grids of an apple.

The entire computer programming was done using the MATHEMATICA package (Wolfram, 1991) in single precision (16 digit machine precision). We list the sup-norm error $\|\theta(t) - \theta_n(t)\|_\infty$, where $\theta(t)$ is the exact boundary correspondence function and $\theta_n(t)$ is the approximation obtained at the collocation points. We also list the sup-norm error $\|RMC(j,k) - RM(j,k)\|_\infty$, $1 \leq j \leq m$, $1 \leq k \leq n$, where $RMC(j,k)$ is the computed Riemann map at selected interior points based on integral equation while $RMC(j,k)$ is the computed Riemann map at those points based on the exact boundary correspondence function. In all our experiments we have chosen $m = 10$. The graphical result for $n = 128$ is plotted in Figure 4. Table show the results for the sup-norm error.

The numerical example illustrate that the present method based on the integral equation and the Cauchy integral formula can be used to produce approximations of acceptable accuracy for numerical conformal mapping of its interior region, provided the interior points are not very close to the boundary. As the interior point z_0 tends to the boundary, the kernel of the

Cauchy's integral formula, which contain the factor $1/(z(t) - z_0)$, tends to an unbounded limit, making it difficult to compute the mapping accurately.

Figure 4: Riemann map of an apple.

Table 1: Error norm.

n	$\|\theta(t) - \theta_n(t)\|_\infty$	$\|RMC(j,k) - RM(j,k)\|_\infty$		
		Radial lines	Complete circles	Circular arcs
64	8.0(−07)	5.2(−03)	9.6(−08)	4.3(−03), ($r \leq 1.1$)
128	2.0(−12)	2.9(−05)	1.7(−12)	2.0(−05), ($r \leq 1.1$)
				7.8(−03), ($r \leq 1.2$)
256	8.9(−16)	9.1(−10)	8.8(−13)	4.5(−10), ($r \leq 1.1$)
				6.5(−07), ($r \leq 1.2$)
				3.4(−04), ($r \leq 1.3$)

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