8

AN INTEGRAL EQUATION FOR CONFORMAL MAPPING OF TRIPLY CONNECTED REGIONS ONTO A DISK WITH CIRCULAR SLITS

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This paper presents an integral equation method for computing the conformal mapping function $f(z)$ of triply connected regions onto a disk of unit radius with two concentric circular slits of radii $\mu_1 < 1$ and $\mu_2 < 1$. By using the boundary relationship satisfied by the mapping function, a related system of nonlinear integral equations via Neumann kernel is constructed involving the unknown μ_1 and μ_2. Together with some normalizing conditions, a unique solution to the system is then computed using an optimization method. Numerical implementation on some test regions will also be presented.

8.1 OVERVIEW

Integral equation method for conformal mapping of multiply connected regions is presently still a subject of interest. Nehari [14, p.335] described the five types of slit region as important canonical regions for conformal mapping of multiply connected regions. They are the discs with concentric circular slits, an annulus with concentric circular slits, the circular slit region, the radial slit region and the parallel slit region. However, exact mapping functions are not known except for some special regions. Several methods have been proposed in the literature for the numerical approximation for conformal mapping of multiply connected regions [1,3,4,5, 7,11,12,13,15,16,17]. One of the methods is the integral equation method.

A derivation of boundary integral equation satisfied by a function on a doubly connected region with Neumann kernel onto a disk with a circular slit has been presented in Murid and Hu [10]. The theoretical development is based on the boundary integral equation for conformal mapping of doubly connected regions derived by Murid and Razali [12] which was limited to doubly connected regions. Murid & Mohamed [11] and Mohamed [8] have also discussed numerical conformal mapping of doubly connected regions onto an annulus via the Kerzman-Stein and the Neumann kernel. Recently, conformal mapping of multiply connected regions onto an annulus with circular slits is also discussed in Murid and Hu [9]. But Murid & Razali [9], Murid & Mohamed [8] and Murid & Hu [9,10] have not yet performed any numerical experiments on conformal mapping of triply connected regions onto a disk with circular slits.

In this paper we describe an integral equation method for computing the conformal mapping of triply connected regions onto a disk with concentric circular slits. The integral equation is satisfied by $f'(z)$, μ_1 and μ_2. For numerical experiment, we discretized the integral equation and imposed some normalizing conditions. The system obtained is solved using Lavenberg-Marquadt algorithm. Then, the boundary values of $f(z)$ is completely determined from the boundary values of $f'(z)$ through a boundary relationship. We presents a numerical result as well as comparisons with the results of Reichel [16] and Kokkinos *et. al.* [7].

8.2 BOUNDARY INTEGRAL EQUATION FOR CONFORMAL MAPPING OF TRIPLY CONNECTED REGIONS WITH NEUMANN KERNEL

Let Γ_0, Γ_1 and Γ_2 be three smooth Jordan curves in the z-plane such that Γ_1 and Γ_2 lies in the interior of Γ_0. Let $w = f(z)$ be the analytic function which maps Ω conformally onto a disk with circular slits of radii $\mu_1 < 1$ and $\mu_2 < 1$ (see Figure 8.1). The function f could be made unique by prescribing that

$$f(a) = 0, \; f'(a) > 0 \text{ or } f(z^*) = 1,$$

$a \in \Omega$ and $z^* \in \Gamma_0$ are fixed points. The boundary values of f can be represented in the form

$$f(z_0(t)) = e^{i\theta_0(t)}, \; \Gamma_0 : z = z_0(t), \; 0 \le t \le \beta_0,$$
$$f(z_1(t)) = \mu_1 e^{i\theta_1(t)}, \; \Gamma_1 : z = z_1(t), \; 0 \le t \le \beta_1,$$
$$f(z_2(t)) = \mu_2 e^{i\theta_2(t)}, \; \Gamma_2 : z = z_2(t), 0 \le t \le \beta_2,$$

where $\theta_0(t)$, $\theta_1(t)$ and $\theta_2(t)$ are the boundary correspondence functions of Γ_0, Γ_1 and Γ_2 respectively.

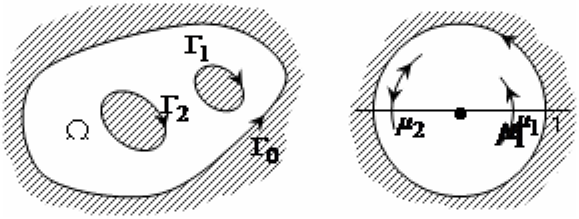

Figure 8.1. Mapping of a triply connected region.

Denote the unit tangent to Γ at $z(t)$ by $T(z(t)) = z'(t)/|z'(t)|$. Then it can be shown that

$$f(z_0(t)) = \frac{1}{i} T(z_0(t)) \frac{\theta_0'(t)}{|\theta_0'(t)|} \frac{f'(z_0(t))}{|f'(z_0(t))|} = \frac{1}{i} T(z_0(t)) \frac{f'(z_0(t))}{|f'(z_0(t))|}, \quad z_0 \in \Gamma_0,$$

$$f(z_1(t)) = \frac{1}{i} T(z_1(t)) \frac{\theta_1'(t)}{|\theta_1'(t)|} \frac{f'(z_1(t))}{|f'(z_1(t))|} = \pm \frac{\mu_1}{i} T(z_1(t)) \frac{f'(z_1(t))}{|f'(z_1(t))|}, \quad z_1 \in \Gamma_1,$$

$$f(z_2(t)) = \frac{1}{i} T(z_2(t)) \frac{\theta_2'(t)}{|\theta_2'(t)|} \frac{f'(z_2(t))}{|f'(z_2(t))|} = \pm \frac{\mu_2}{i} T(z_2(t)) \frac{f'(z_2(t))}{|f'(z_2(t))|}, \quad z_2 \in \Gamma_2$$

. (8.6)

Note that $\theta_1'(t)$ and $\theta_2'(t)$ may be positive or negative since the circular slits are traversed twice (see Figure 8.1). Thus, $\theta_1'(t)/|\theta_1'(t)| = \pm 1$ and $\theta_2'(t)/|\theta_2'(t)| = \pm 1$. If we square both sides of boundary relationship (8.4), (8.5) and (8.6), the results can be unified as

$$f(z)^2 = -|f(z)|^2 \, T(z)^2 \frac{f'(z)^2}{|f'(z)|^2}, \quad z \in \Gamma.$$

Recently, Murid and Hu [10] have shown that the mapping function f of multiply connected regions satisfies the integral equation

$$g(z,a) + PV\int_{\Gamma} N^*(z,w) g(w,a) \, |dw| = |f(z)|^2 \, h(a,z), \quad z \in \Gamma,$$

where

$$g(z,a) = f'(a) T(z) f'(z),$$

$$h(a,z) = -\frac{\overline{T(z)}}{(\bar{a} - \bar{z})^2},$$

$$N^*(z,w) = \frac{1}{2\pi i} \left[\frac{T(z)}{(z-w)} - \frac{|f(z)|^2 \, \overline{T(z)}}{|f(w)|^2 \, (\bar{z} - \bar{w})} \right].$$

Since Ω is a triply connected region, the single integral equation in (8.8) can be separated into a system of equations

$$g(z_0,a)+\int_{\Gamma_0} N(z_0,w)\, g(w,a)\,|dw|-\int_{-\Gamma_1} P_0(z_0,w)\, g(w,a)\,|dw| -\int_{-\Gamma_2} Q_0(z_0,w)\, g(w,a)\,|dw| = h(a,z_0),$$

$z_0 \in \Gamma_0$, (8.9)

$$g(z_1,a)+\int_{\Gamma_0} P_1(z_1,w)\, g(w,a)\,|dw|-\int_{-\Gamma_1} N(z_1,w)\, g(w,a)\,|dw| -\int_{-\Gamma_2} Q_1(z_1,w)\, g(w,a)\,|dw| = \mu_1^2 h(a,z_1),$$

$z_1 \in \Gamma_1$, (8.10)

$$g(z_2,a)+\int_{\Gamma_0} P_2(z_2,w)\, g(w,a)\,|dw|-\int_{-\Gamma_1} Q_2(z_2,w)\, g(w,a)\,|dw| -\int_{-\Gamma_2} N(z_2,w)\, g(w,a)\,|dw| = \mu_2^2 h(a,z_2),$$

$z_2 \in \Gamma_2$, (8.11)

where

$$P_0(z,w)=\frac{1}{2\pi i}\left[\frac{T(z)}{(z-w)}-\frac{\overline{T(z)}}{\mu_1^2(\bar z-\overline{w})}\right],$$

$$Q_0(z,w)=\frac{1}{2\pi i}\left[\frac{T(z)}{(z-w)}-\frac{\overline{T(z)}}{\mu_2^2(\bar z-\overline{w})}\right],$$

$$P_1(z,w)=\frac{1}{2\pi i}\left[\frac{T(z)}{(z-w)}-\frac{\mu_1^2\overline{T(z)}}{(\bar z-\overline{w})}\right],$$

$$Q_1(z,w)=\frac{1}{2\pi i}\left[\frac{T(z)}{(z-w)}-\frac{\mu_1^2\overline{T(z)}}{\mu_2^2(\bar z-\overline{w})}\right],$$

$$P_2(z,w)=\frac{1}{2\pi i}\left[\frac{T(z)}{(z-w)}-\frac{\mu_2^2\overline{T(z)}}{(\bar z-\overline{w})}\right],$$

$$Q_2(z,w)=\frac{1}{2\pi i}\left[\frac{T(z)}{(z-w)}-\frac{\mu_2^2\overline{T(z)}}{\mu_1^2(\bar z-\overline{w})}\right],$$

$$N(z,w)=\begin{cases}\dfrac{1}{2\pi\mathrm{i}}\left[\dfrac{T(z)}{z-w}-\dfrac{\overline{T(z)}}{\bar{z}-\bar{w}}\right], & \text{if } w,z\in\Gamma, w\neq z,\\ \dfrac{1}{2\pi}\dfrac{\mathrm{Im}\left[z''(t)\overline{z'(t)}\right]}{\left|z'(t)\right|^3}, & \text{if } w=z\in\Gamma.\end{cases}$$

The kernel *N* is also known as the Neumann kernel. There are five unknown quantities in the integral equations (8.9), (8.10) and (8.11), namely, $g(z_0,a)$, $g(z_1,a)$, $g(z_2,a)$, μ_1 and μ_2. Naturally it is also required that the unknown mapping function $f(z)$ be single-valued in the problem domain [5], that is,

$$\int_{-\Gamma_1} f'(z)dz = 0,$$
$$\int_{-\Gamma_2} f'(z)dz = 0\,,$$

which implies

$$\int_{-\Gamma_1} g(w,a)\,|\,dw\,|= 0,$$
$$\int_{-\Gamma_2} g(w,a)\,|\,dw\,|= 0.$$

Several conditions can be obtained to help achieve uniqueness. We first consider applying the condition $f(z_0(0))=1$. From (8.4), this implies $g(z_0(0),a)/\,|\,g(z_0(0),a)\,|=\mathrm{i}$, which means

$$\mathrm{Re}[g(z_0(0),a)]=0,\ \ \mathrm{Im}[g(z_0(0),a)/\,|\,g(z_0(0),a)\,|=1.$$

If the region is symmetric with respect to the axes, we can also impose the conditions

$$\mathrm{Re}[g(z_1(0),a)]=0\,,\quad \mathrm{Re}[g(z_2(0),a)]=0\,.$$

Thus the system of integral equations comprising of (8.9), (8.10), (8.11) with conditions (8.14), (8.15), (8.16) and (8.17) should lead to a unique solution.

8.1 NUMERICAL IMPLEMENTATION

Using the parametric representations $z_0(t)$ of Γ_0 for $t: 0 \le t \le \beta_0$, $z_1(t)$ of $-\Gamma_1$ for $t: 0 \le t \le \beta_1$ and $z_2(t)$ of $-\Gamma_2$ for $t: 0 \le t \le \beta_2$, equations (8.9), (8.10), (8.11), (8.14) and (8.15) become

$$g(z_0(t),a)+\int_0^{\beta_0} N(z_0(t),z_0(s))\, g(z_0(s),a)\,|z_0'(s)|\,ds-\int_0^{\beta_1} P_0(z_0(t),z_1(s))\, g(z_1(s),a)\,|z_1'(s)|\,ds$$

$$-\int_0^{\beta_2} Q_0(z_0(t),z_2(s))\, g(z_2(s),a)\,|z_2'(s)|\,ds = h(a,z_0(t)), \quad z_0(t)\in\Gamma_0$$

, (8.18)

$$g(z_1(t),a)+\int_0^{\beta_0} P_1(z_1(t),z_0(s))\, g(z_0(s),a)\,|z_0'(s)|\,ds-\int_0^{\beta_1} N(z_1(t),z_1(s))\, g(z_1(s),a)\,|z_1'(s)|\,ds$$

$$-\int_0^{\beta_2} Q_1(z_1(t),z_2(s))\, g(z_2(s),a)\,|z_2'(s)|\,ds = \mu_1^2 h(a,z_1(t)), \quad z_1(t)\in\Gamma_1$$

(8.19)

$$g(z_2(t),a)+\int_0^{\beta_0} P_2(z_2(t),z_0(s))\, g(z_0(s),a)\,|z_0'(s)|\,ds$$

$$-\int_0^{\beta_1} Q_2(z_2(t),z_1(s))\, g(z_1(s),a)\,|z_1'(s)|\,ds$$

$$-\int_0^{\beta_2} N(z_2(t),z_2(s))\, g(z_2(s),a)\,|z_2'(s)|\,ds = \mu_2^2 h(a,z_2(t)), \quad z_2(t)\in\Gamma_2$$

$$\int_0^{\beta_1} g(z_1(s),a)\,|z_1'(s)|\,ds = 0,$$

$$\int_0^{\beta_2} g(z_2(s),a)\,|z_2'(s)|\,ds = 0.$$

Multiplying (8.18), (8.19) and (8.20) respectively by $|z_0'(t)|$, $|z_1'(t)|$ and $|z_2'(t)|$ gives

$$|z_0'(t)|\, g(z_0(t),a) + \int_0^{\beta_0} |z_0'(t)|\, N(z_0(t), z_0(s))\, g(z_0(s),a)\, |z_0'(s)|\, ds$$

$$-\int_0^{\beta_1} |z_0'(t)|\, P_0(z_0(t), z_1(s))\, g(z_1(s),a)\, |z_1'(s)|\, ds - \int_0^{\beta_2} |z_0'(t)|\, Q_0(z_0(t), z_2(s))\, g(z_2(s),a)\, |z_2'(s)|\, ds$$

$$= |z_0'(t)|\, h(a, z_0(t)), \quad z_0(t) \in \Gamma_0,$$

$$|z_1'(t)|\, g(z_1(t),a) + \int_0^{\beta_0} |z_1'(t)|\, P_1(z_1(t), z_0(s))\, g(z_0(s),a)\, |z_0'(s)|\, ds$$

$$-\int_0^{\beta_1} |z_1'(t)|\, N(z_1(t), z_1(s))\, g(z_1(s),a)\, |z_1'(s)|\, ds - \int_0^{\beta_2} |z_1'(t)|\, Q_1(z_1(t), z_2(s))\, g(z_2(s),a)\, |z_2'(s)|\, ds$$

$$= \mu_1^2\, |z_1'(t)|\, h(a, z_1(t)), \quad z_1(t) \in \Gamma_1,$$

$$|z_2'(t)|\, g(z_2(t),a)$$

$$+ \int_0^{\beta_0} |z_2'(t)|\, P_2(z_2(t), z_0(s))\, g(z_0(s),a)\, |z_0'(s)|\, ds$$

$$- \int_0^{\beta_1} |z_2'(t)|\, Q_2(z_2(t), z_1(s))\, g(z_1(s),a)\, |z_1'(s)|\, ds$$

$$- \int_0^{\beta_2} |z_2'(t)|\, N(z_2(t), z_2(s))\, g(z_2(s),a)\, |z_2'(s)|\, ds$$

$$= \mu_2^2\, |z_2'(t)|\, h(a, z_2(t)), \quad z_2(t) \in \Gamma_2.$$

We next define

$$\phi_0(t) = |z_0'(t)|\, g(z_0(t),a),$$
$$\phi_1(t) = |z_1'(t)|\, g(z_1(t),a),$$
$$\phi_2(t) = |z_2'(t)|\, g(z_2(t),a),$$
$$\gamma_0(t) = |z_0'(t)|\, h(a, z_0(t)),$$
$$\gamma_1(t) = |z_1'(t)|\, h(a, z_1(t)),$$
$$\gamma_2(t) = |z_2'(t)|\, h(a, z_2(t)),$$
$$K_{00}(t_0, s_0) = |z_0'(t)|\, N(z_0(t), z_0(s)),$$
$$K_{01}(t_0, s_1) = |z_0'(t)|\, P_0(z_0(t), z_1(s)),$$
$$K_{02}(t_0, s_2) = |z_0'(t)|\, Q_0(z_0(t), z_2(s)),$$
$$K_{10}(t_1, s_0) = |z_1'(t)|\, P_1(z_1(t), z_0(s)),$$
$$K_{11}(t_1, s_1) = |z_1'(t)|\, N(z_1(t), z_1(s)),$$
$$K_{12}(t_1, s_2) = |z_1'(t)|\, Q_1(z_1(t), z_2(s)),$$
$$K_{20}(t_2, s_0) = |z_2'(t)|\, P_2(z_2(t), z_0(s)),$$

$$K_{21}(t_2,s_1) = |z_2'(t)|\, Q_2(z_2(t), z_1(s)),$$
$$K_{22}(t_2,s_2) = |z_2'(t)|\, N(z_2(t), z_2(s)).$$

Thus, equations (8.23), (8.24), (8.25), (8.21) and (8.22) can be briefly written as

$$\phi_0(t) + \int_0^{\beta_0} K_{00}(t_0,s_0)\phi_0(s)ds - \int_0^{\beta_1} K_{01}(t_0,s_1)\phi_1(s)ds - \int_0^{\beta_2} K_{02}(t_0,s_2)\phi_2(s)ds = \gamma_0(t),$$

$$\phi_1(t) + \int_0^{\beta_0} K_{10}(t_1,s_0)\phi_0(s)ds - \int_0^{\beta_1} K_{11}(t_1,s_1)\phi_1(s)ds - \int_0^{\beta_2} K_{12}(t_1,s_2)\phi_2(s)ds = \gamma_1(t),$$

$$\phi_2(t) + \int_0^{\beta_0} K_{20}(t_2,s_0)\phi_0(s)ds - \int_0^{\beta_1} K_{21}(t_2,s_1)\phi_1(s)ds - \int_0^{\beta_2} K_{22}(t_2,s_2)\phi_2(s)ds = \gamma_2(t)$$

$$\int_0^{\beta_1} \phi_1(s)ds = 0,$$

$$\int_0^{\beta_2} \phi_2(s)ds = 0.$$

We choose $\beta_0 = \beta_1 = \beta_2 = 2\pi$ and n equidistant collocation points $t_i = (i-1)\beta_0/n$, $1 \le i \le n$ on Γ_0, m equidistant collocation points $t_{\tilde{i}} = (\tilde{i}-1)\beta_1/m$, $1 \le \tilde{i} \le m$ on Γ_1 and l equidistant collocation points $t_{\hat{i}} = (\hat{i}-1)\beta_2/l$, $1 \le \hat{i} \le l$ on Γ_2. Applying the Nyström's method with trapezoidal rule to discretize equations (8.26) to (8.30), we obtain

$$\phi_0(t_i) + \frac{\beta_0}{n}\sum_{j=1}^{n} K_{00}(t_i,t_j)\phi_0(t_j) - \frac{\beta_1}{m}\sum_{\tilde{j}=1}^{m} K_{01}(t_i,t_{\tilde{j}})\phi_1(t_{\tilde{j}}) - \frac{\beta_2}{l}\sum_{\hat{j}=1}^{l} K_{02}(t_i,t_{\hat{j}})\phi_2(t_{\hat{j}}) = \gamma_0(t_i)$$

$$\phi_1(t_{\tilde{i}}) + \frac{\beta_0}{n}\sum_{j=1}^{n} K_{10}(t_{\tilde{i}},t_j)\phi_0(t_j) - \frac{\beta_1}{m}\sum_{\tilde{j}=1}^{m} K_{11}(t_{\tilde{i}},t_{\tilde{j}})\phi_1(t_{\tilde{j}}) - \frac{\beta_2}{l}\sum_{\hat{j}=1}^{l} K_{12}(t_{\tilde{i}},t_{\hat{j}})\phi_2(t_{\hat{j}})$$

$$\phi_2(t_{\hat{i}}) + \frac{\beta_0}{n}\sum_{j=1}^{n} K_{20}(t_{\hat{i}},t_j)\phi_0(t_j) - \frac{\beta_1}{m}\sum_{\tilde{j}=1}^{m} K_{21}(t_{\hat{i}},t_{\tilde{j}})\phi_1(t_{\tilde{j}}) - \frac{\beta_2}{l}\sum_{\hat{j}=1}^{l} K_{22}(t_{\hat{i}},t_{\hat{j}})\phi_2(t_{\hat{j}}) = \gamma_2(t_{\hat{i}})$$

$$\sum_{\tilde{j}=1}^{m} \phi_1(t_{\tilde{j}}) = 0 \, ,$$
$$\sum_{\hat{j}=1}^{l} \phi_2(t_{\hat{j}}) = 0 \, .$$

Equations (8.31) to (8.35) lead to a system of $(n+m+l+2)$ non-linear complex equations in n unknowns $\phi_0(t_i)$, m unknowns $\phi_1(t_{\tilde{i}})$, l unknowns $\phi_2(t_{\hat{i}})$, as well as the unknown parameters μ_1 and μ_2. By defining the matrices

$$x_{0i} = \phi_0(t_i) \, , \qquad x_{1\tilde{i}} = \phi_1(t_{\tilde{i}}) \, , \quad x_{2\hat{i}} = \phi_2(t_{\hat{i}}) \, ,$$
$$\gamma_{0i} = \gamma_0(t_i) \, , \quad \gamma_{1\tilde{i}} = \gamma_1(t_{\tilde{i}}) \, , \quad \gamma_{2\hat{i}} = \gamma_2(t_{\hat{i}}) \, ,$$
$$B_{ij} = \frac{\beta_0}{n} K_{00}(t_i, t_j) \, , \qquad C_{i\tilde{j}} = \frac{\beta_1}{m} K_{01}(t_i, t_{\tilde{j}}) \, ,$$
$$D_{i\hat{j}} = \frac{\beta_2}{l} K_{02}(t_i, t_{\hat{j}}) \, , \quad E_{\tilde{i}j} = \frac{\beta_0}{n} K_{10}(t_{\tilde{i}}, t_j) \, ,$$
$$F_{\tilde{i}\tilde{j}} = \frac{\beta_1}{m} K_{11}(t_{\tilde{i}}, t_{\tilde{j}}) \, , \qquad G_{\tilde{i}\hat{j}} = \frac{\beta_2}{l} K_{12}(t_{\tilde{i}}, t_{\hat{j}}) \, ,$$
$$H_{\hat{i}j} = \frac{\beta_0}{n} K_{20}(t_{\hat{i}}, t_j) \, , \quad J_{\hat{i}\tilde{j}} = \frac{\beta_1}{m} K_{21}(t_{\hat{i}}, t_{\tilde{j}}) \, ,$$
$$L_{\hat{i}\hat{j}} = \frac{\beta_2}{l} K_{22}(t_{\hat{i}}, t_{\hat{j}}) \, .$$

The system of equations (8.31), (8.32) and (8.33) can be written as $n+m+l$ by $n+m+l$ system of non-linear equations

$$[I_{nn} + B_{nn}]x_{0n} - C_{nm}x_{1m} - D_{nl}x_{2l} = \gamma_{0n} \, ,$$
$$E_{mn}x_{0n} + [I_{mm} - F_{mm}]x_{1m} - G_{ml}x_{2l} = \gamma_{1m} \, ,$$
$$H_{ln}x_{0n} - J_{lm}x_{1m} + [I_{ll} - L_{ll}]x_{2l} = \gamma_{2l} \, ,$$

The result in matrix form for the system of equations (8.36), (8.37) and (8.38) is

$$\begin{pmatrix} I_{nn}+B_{nn} & \cdots & -C_{nm} & \cdots & -D_{nl} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ E_{mn} & \cdots & I_{mm}-F_{mm} & \cdots & -G_{ml} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ H_{l\,n} & \cdots & -J_{lm} & \cdots & I_{ll}-L_{ll} \end{pmatrix} \begin{pmatrix} x_{0n} \\ \vdots \\ x_{1m} \\ \vdots \\ x_{2l} \end{pmatrix} = \begin{pmatrix} \gamma_{0n} \\ \vdots \\ \gamma_{1m} \\ \vdots \\ \gamma_{2l} \end{pmatrix}.$$

Defining

$$\boldsymbol{A} = \begin{pmatrix} I_{nn}+B_{nn} & \cdots & -C_{nm} & \cdots & -D_{nl} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ E_{mn} & \cdots & I_{mm}-F_{mm} & \cdots & -G_{ml} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ H_{l\,n} & \cdots & -J_{lm} & \cdots & I_{ll}-L_{ll} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_{0n} \\ \vdots \\ x_{1m} \\ \vdots \\ x_{2l} \end{pmatrix} \text{ and } \mathbf{y}$$

$$= \begin{pmatrix} \gamma_{0n} \\ \vdots \\ \gamma_{1m} \\ \vdots \\ \gamma_{2l} \end{pmatrix},$$

where the $(n + m + l) \times (n + m + l)$ system can be written briefly as $\boldsymbol{A}\mathbf{x} = \mathbf{y}$. Separating $\boldsymbol{A}$, $\boldsymbol{x}$ and $\mathbf{y}$ in terms of the real and imaginary parts, the system can be written as

$$\text{Re } A \text{ Re } \boldsymbol{x} - \text{Im } A \text{ Im } \boldsymbol{x} + \text{i } (\text{Im } A \text{ Re } \boldsymbol{x} + \text{Re } A \text{ Im } \boldsymbol{x}) = \text{Re } \mathbf{y} + \text{i Im } \mathbf{y}.$$

The single $(n + m + l) \times (n + m + l)$ complex system (8.40) above is equivalent to the $2(n + m + l) \times 2(n + m + l)$ system matrix involving the real (Re) and imaginary (Im) of the unknown functions, that is,

$$\begin{pmatrix} \operatorname{Re} A & \cdots & -\operatorname{Im} A \\ \vdots & \ddots & \vdots \\ \operatorname{Im} A & \cdots & \operatorname{Re} A \end{pmatrix} \begin{pmatrix} \operatorname{Re} x \\ \vdots \\ \operatorname{Im} x \end{pmatrix} = \begin{pmatrix} \operatorname{Re} y \\ \vdots \\ \operatorname{Im} y \end{pmatrix}.$$

Note that the matrix in (8.41) contains the unknown parameters μ_1 and μ_2. Since $\phi = \operatorname{Re}\phi + i\operatorname{Im}\phi$, equations (8.34), (8.35), (8.16) and (8.17) become

$$\sum_{\tilde{j}=1}^{m} (\operatorname{Re} x_{1\tilde{j}} + i \operatorname{Im} x_{1\tilde{j}}) = 0$$

$$\sum_{\hat{j}=1}^{l} (\operatorname{Re} x_{2\hat{j}} + i \operatorname{Im} x_{2\hat{j}}) = 0,$$

$$\operatorname{Re} x_{01} = 0,$$

$$\operatorname{Im}\left[x_{01} / \sqrt{(\operatorname{Re} x_{01})^2 + (\operatorname{Im} x_{01})^2} \right] = 1,$$

$$\operatorname{Re} x_{11} = 0, \quad \operatorname{Re} x_{21} = 0.$$

The system of equations (8.41) to (8.46) is an over-determined system of nonlinear equations involving $2(n + m + l) + 6$ equations in $2(n + m + l) + 2$ unknowns. We use a modification of the Gauss-Newton called the Lavenberg-Marquardt algorithm [2] to solve this non-linear least square problem. The Lavenberg-Marquardt algorithm is an iterative procedure with starting value denoted as $\mathbf{x}^0$. This initial approximation, which, if at all possible, should be well-informed guess and generate a sequence of approximations $\mathbf{x}^1$, $\mathbf{x}^2$, $\mathbf{x}^3$, … base on the formula

$$\mathbf{x}^{k+1} = \mathbf{x}^k - H(\mathbf{x}^k)\mathbf{f}(\mathbf{x}^k), \qquad \lambda^k \geq 0,$$

where $H(\mathbf{x}^k) = ((J_{\mathbf{f}}(\mathbf{x}^k))^T J_{\mathbf{f}}(\mathbf{x}^k) + \lambda^k I)^{-1} (J_{\mathbf{f}}(\mathbf{x}^k))^T$ and $J_{\mathbf{f}}(\mathbf{x})$ denotes the Jacobian of $\mathbf{f}$ at $\mathbf{x}$. Here, $\mathbf{x}$ stands for the $(2n + 2m + 2l + 2)$ vector (Re x_{01}, Re x_{02}, …, Re x_{0n}, Re x_{11}, Re x_{12}, …, Re x_{1m}, Re x_{21}, Re x_{22}, …, Re x_{2l}, Im x_{01}, Im x_{02}, …, Im x_{0n}, Im x_{11}, Im x_{12}, …, Im x_{1m}, Im x_{21}, Im x_{22}, …, Im x_{2l}, μ_1, μ_2), and $\mathbf{f} = (f_1$, f_2, …,

$f_{2n+2m+2l+6}$).

The strategy for getting the initial estimation is to provide rough estimates of the slit radius, $\mu_1 \approx 0.8$ and $\mu_2 \approx 0.7$ for the test region. Then the non-linear system of equations (8.39), (8.34) and (8.35) reduces to over-determined linear system. Writing the over-determined system as $\mathbf{Bx} = \mathbf{y}$, we use the least-squares solutions of $\mathbf{Bx} = \mathbf{y}$ which are precisely the solutions of $\mathbf{B}^T\mathbf{Bx} = \mathbf{B}^T\mathbf{y}$ [6]. The solutions are then taken as initial estimation. In our experiments, we have chosen the number of collocation points on Γ_0, Γ_1 and Γ_2 being equal, i.e., $N = n = m = l$.

The system of equations (8.41) with (8.42), (8.43), (8.44), (8.45) and (8.46) are then solved for the unknown function

$$\phi_0(t) = |z_0'(t)|\; f'(a)T(z_0(t))\, f'(z_0(t)) \,,$$
$$\phi_1(t) = |z_1'(t)|\; f'(a)T(z_1(t))\, f'(z_1(t)) \,,$$
$$\phi_2(t) = |z_2'(t)|\; f'(a)T(z_2(t))\, f'(z_2(t)) \,,$$

μ_1 and μ_2. Finally, the boundary correspondence functions $\theta_0(t)$, $\theta_1(t)$ and $\theta_2(t)$ are then computed approximately by the formula

$$\theta_0(t) = \text{Arg}\, f(z_0(t)) \approx \text{Arg}\,(-\mathrm{i}\phi_0(t)) \,,$$
$$\theta_1(t) = \text{Arg}\, f(z_1(t)) \approx \text{Arg}\,(\pm\mathrm{i}\phi_1(t)) \,,$$
$$\theta_2(t) = \text{Arg}\, f(z_2(t)) \approx \text{Arg}\,(\pm\mathrm{i}\phi_2(t)) \,.$$

8.4 NUMERICAL RESULTS

For numerical experiment, we have used an ellipse and two circle as a test regions (see Figure 8.2). Let

$$\Gamma_0 : \{z_0(t) = 2\cos t + \mathrm{i}\sin t\} \,,$$
$$\Gamma_1 : \{z_1(t) = 0.5(\cos t + \mathrm{i}\sin t)\} \,,$$

$$\Gamma_2 : \{z_2(t) = 1.2 + 0.3(\cos t + \mathrm{i}\sin t)\},\ t : 0 \le t \le 2\pi .$$

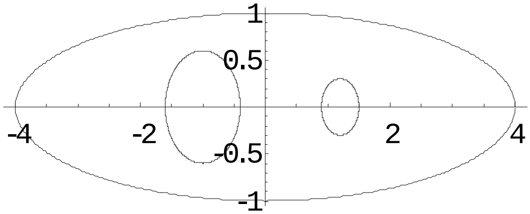

Figure 8.2. Ellipse/two circle

We have adopted the example from Reichel [16] and Kokkinos *et al*. [7] for comparison of μ_1 and μ_2 (see Table 8.1 and 8.2). Since the conditions of the problems are somewhat different, μ_0 = 1 in ours and μ_0 = 2.5 in Reichel's and Kokkinos *et al.*, our radius should be multiplied by 2.5. Values of μ_1 and μ_2 in Reichel are denoted here by $\mu_{1,R}$ and $\mu_{2,R}$ respectively. While the values of μ_1 and μ_2 in Kokkinos *et al*. are denoted here by $\mu_{1,K}$ and $\mu_{2,K}$ respectively. All the computations were done using MATHEMATICA package Wolfram [18] in single precision (16 digit machine precision).

Table 8.1. Radii comparison with Reichel [16].

N	$\|\mu_1 \times 2.5 - \mu_{1,R}\|_\infty$	$\|\mu_2 \times 2.5 - \mu_{2,R}\|_\infty$
64	1.8 (-02)	6.0 (-04)

Table 8.2. Radii comparison with Kokkinos *et al*. [7].

N	$\|\mu_1 \times 2.5 - \mu_{1,K}\|_\infty$	$\|\mu_2 \times 2.5 - \mu_{2,K}\|_\infty$
64	1.8 (-02)	5.9 (-04)

8.5 CONCLUSION

In this paper we have constructed a system of integral equations for numerical conformal mapping from a triply connected regions onto a disk with concentric circular slits of radii μ_1 and μ_2. The system involved the Neumann kernel and unknown parameters μ_1 and μ_2. Due to the presence of μ_1 and μ_2 in the kernel, the discretized integral equation leads to a system of nonlinear equations which is solved using optimization method. A mapping of the test region was computed numerically using the proposed method. The advantage of our method is that it calculates the boundary correspondence functions and the unknown parameters μ_1 and μ_2 simultaneously.

8.6 REFERENCES

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