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REPRESENTATION OF THE CROSS-SECTIONAL STRESS DISTRIBUTION USING FOURIER'S SERIES

A. Y. Mohd Yassin and D. A. Nethercot
Lecturer, Faculty of Civil Engineering, Universiti Teknologi Malaysia, 81310, Johor Bahru, Malaysia
airil.mohd-yassin@imperial.ac.uk
Head and Professor, Civil and Environmental Engineering Department, Imperial College London
d.nethercot@imperial.ac.uk

ABSTRACT: A generic procedure is presented for the determination of the moment capacity of a complex beam's cross-section. The use of a general expression - possible due to the single-rule representation of the stress distribution using Fourier's series - allows for a compact computer coding and easier input data. The series converges to the idealized values obtained using a bilinear material curve as the number of terms in the series increases.

Keywords – moment capacity, stress distribution, Fourier's series, composite beam

1. INTRODUCTION

Since a beam is primarily a flexural member, the most important design requirement is usually the provision of adequate moment capacity. This is actually the internal moment produced by the flexural stress during the bending of the beam. The degree of difficulty in determining the moment capacity increases as the beam becomes more complex; this can be due to geometric and material configurations of the beam's cross-section. Such complexities make the determination of the moment capacity a task best handled by a computer program. In developing a generic computer program, it is better to use general mathematical expressions which can cover a wide range of possibilities with minimal end-user intervention. In the determination of the moment capacity of a beam, the use of general expressions requires that the stress distribution be expressed as a single-rule function – this is the focus of this paper.

1.1 Cross-sectional Stress Development in a Beam

The moment capacity of a beam depends on the stress it can develop at the most critical cross-section. The distribution of the stress on the other hand, depends on the straining of the cross-section; usually assumed to obey the stress-strain curve of the material. For steel, a bilinear curve is usually adopted whilst for concrete, a rigid plastic behaviour is assumed (the material is either fully stressed or not stressed at all). Figure 1 presents a series of stress profiles for a beam based on the assumption of a bilinear elastic-plastic material. Figure 1a is the stress profile during the elastic stage where none of the fibers has yielded. Once yielding occurs the beam is said to be in the elasto-plastic stage where it contains a mixture of yielded and unyielded portions. This stage is shown in Figure 1b. If the loading further increases, the stress will continue to develop in the unyielded portion of the beam until the cross-section becomes fully yielded, leading to a stage called the plastic stage, as shown in Figure 1c. This, however, is a rather simplified concept as this condition is impossible to achieve due to the singularity at the neutral axis. Finally, there is a possibility for some of the extreme fibers to develop extra stresses due to strain hardening, but this stage is normally and conservatively ignored in design.

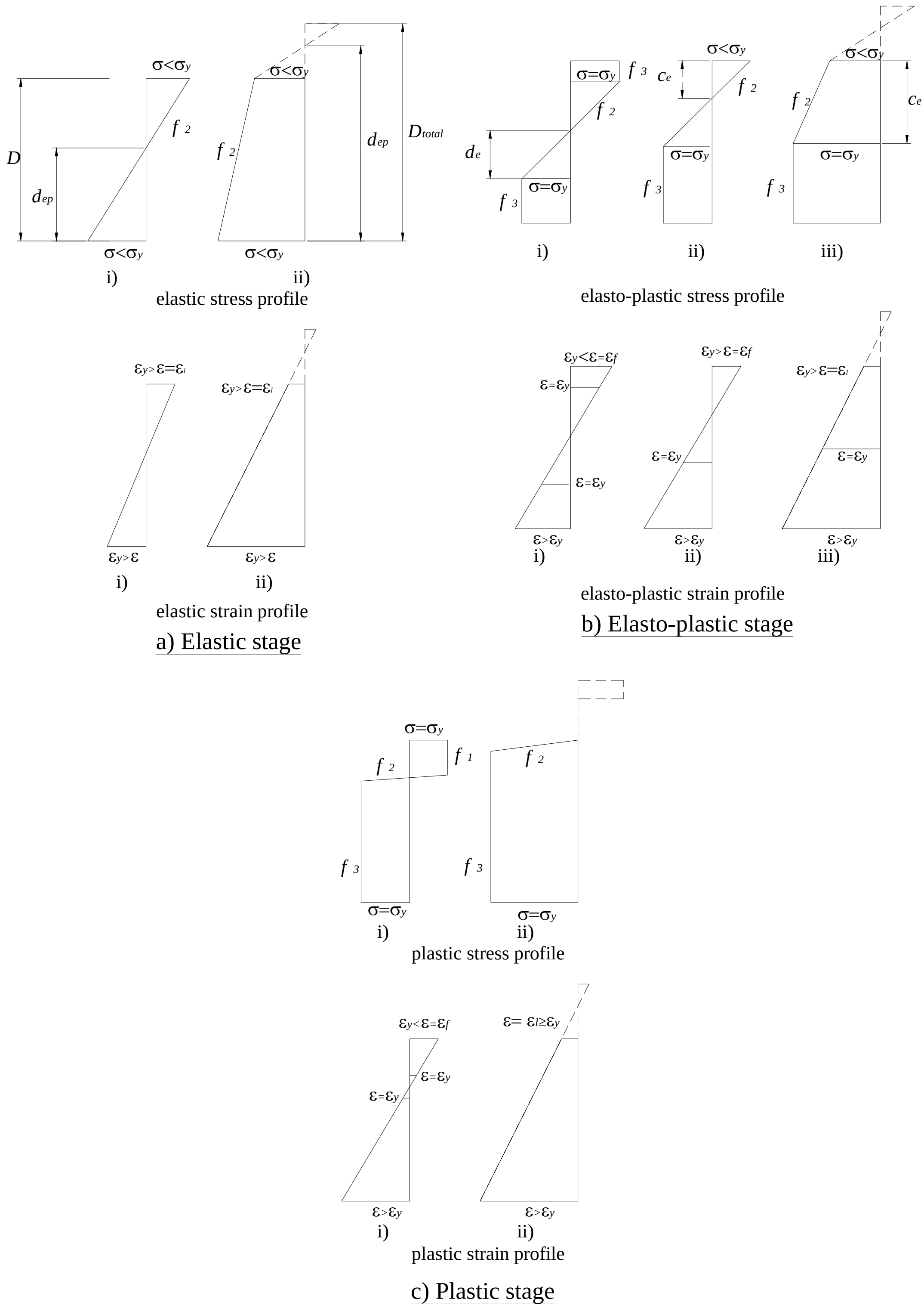


Figure 1. Various stress stages

1.2 Calculation of Moment Capacity

The moment capacity of a beam can be calculated from:

$$M = \int_A \sigma(y) y dA \tag{1}$$

where $\sigma(y)$ is the cross-sectional stress distribution or the shape of the stress profile of the beam, expressed in the *y*-direction. Although Equation (1) is general, it has not been widely used in its integral form. For the elastic stage, the integration can be readily solved to give the elastic moment capacity, M_e as:

$$M_e = \frac{\sigma_y I}{c} \tag{2}$$

where σ_y is the yield stress, *I* is the second moment of area of the beam, and *c* is the distance of the centroid of the beam to the top surface of the beam. The form of equation (2) is possible because the stress profile at the limit of the elastic stage can be expressed as a linear function as follows:

$$\sigma(y) = -\frac{\sigma_y y}{c} \tag{3}$$

In calculating the moment capacity for the plastic stage, termed the plastic moment capacity, M_p, the rectangular stress block method is usually adopted. In this method, the location of the plastic neutral axis or PNA is sought first by satisfying the cross-sectional equilibrium. Once this is done, the resultants of the compressive and tensile portions of the cross-section are calculated as the product of the material yield stress (or design strength) and the relevant area. The resultant force is assumed to act at the centroid of the stress block. The plastic moment capacity is then obtained as the summation of the first moment of these resultants about any reference point given as:

$$M_p = \sum_{i=1}^{n} R_i z_i \tag{4}$$

where *R* and *z* refer to resultant and lever arm, respectively and *n* refers to the total number of elements in a cross-section. Since the elasto-plastic condition contains both yielded and unyielded portions, the elasto-plastic moment capacity, M_{ep} can be obtained from the combination of Equations (2) and (4) as:

$$M_{ep} = \sum_{i=1}^{n_p} R_i z_i + \frac{\sigma_y I_e}{c_e} \tag{5}$$

where n_p refers to the total number of yielded elements, I_e is the second moment of area of the elastic portion and c_e is the distance of the centroid to the highest level of the elastic portion, as shown in Figure 1b.

1.3 Premature Failures and the Limited Development of Stress

The attainment of the plastic condition is the optimum case for the design of a beam as it provides the maximum moment capacity. However, this requires an excessive amount of straining or rotation of the cross-section and is possible only if premature failures are prevented. For a steel beam acting alone, the most likely type of premature failure is local buckling. For a composite beam, in addition to local buckling, there are two other possible failures, crushing of the concrete prior to the full yielding of the steel section and fracture of the shear connection due to insufficient ductility. For a reinforced concrete beam provided with tension materials i.e. fibers or steel plates, in addition to the above failures, another possibility is fracture failure of the rebar. These premature failures dictate the stress distribution within a cross-section. If premature failure occurs before the yielding of the extreme fiber, the moment capacity of the beam must be calculated using Equation (2) by replacing σ_y with the maximum stress. If premature failure is prevented from occurring at least up to the attainment of the full yielding of the cross-section, Equation (4) is used. If premature failure occurs during the elasto-plastic stage of the beam, Equation (5) must be used instead.

2. THE NEED FOR A SINGLE-RULE FUNCTION AND THE USE OF A GENERAL EXPRESSION

The need for separate treatments for calculating the moment capacity of a beam at various stress stages occurs - despite the availability of the general expression of Equation (1) - because the stress profile of the elasto-plastic and the plastic stages can no longer be expressed as a single-rule function. As shown in Figure 1a, while the elastic stress profile can be expressed as a single linear function, f_2, other stress profiles consist of multi-rule functions. For elasto-plastic conditions, there are three possible conditions, as shown in Figure 1b, depending on the degree of symmetricality of the beam about the major axis. A low degree of symmetricality can be caused, for example, by a composite configuration. For the plastic stage, the stress distribution can be envisaged as having f_2 very slightly inclined over the major axis which is actually closer to the actual condition than is assumed in the rectangular stress block method.

The calculation of the moment capacity is obviously more difficult for the elasto-plastic condition. The difficulty increases as the cross-section of the beam becomes more complex, both geometrically and materially. Currently, the authors are developing a new type of composite beam known as Precast Cold-formed Composite Beam or PCFC beam which consists of a closed cold-formed steel section of arbitrary shape surrounded by concrete. The cross-section of the beam is shown in Figure 2 and details are given in Yassin and Nethercot (2006). For such sections the use of the general expression of Equation (1) becomes attractive as it can eliminate the need for:

i) the determination of the centroid of the yielded element (as this is implicitly done by the integration)
ii) separate treatments for various stress stages

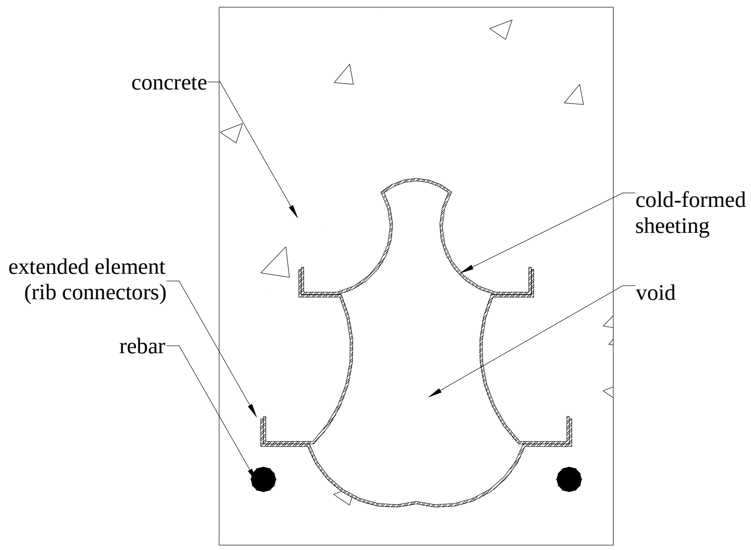


Figure 2. Cross-section of PCFC beam

Also, the use of a general expression is important especially in producing a generic computer program. The use of a general expression not only allows for a more compact coding but also reduces the size of the input data required. However, the use of Equation (1) requires the stress profile to be expressed as a single-rule function, achieved herein by the use of Fourier's series. To note, the function must also be able to represent all the possible profiles of the various stress stages.

2.1 Representation of the Stress Profile using Fourier's Series

The multi-rule functions of the stress profile of a cross-section can be stated as:

$$\sigma(y) = \begin{cases} f_1 & if \quad d_{ep} + d_e < y < D \\ f_2 & if \quad d_{ep} - d_e < y < d_{ep} + d_e \\ f_3 & if \quad 0 < y < d_{ep} - d_e \end{cases}$$

where, D and d_{ep} are the height of the beam and the distance of the neutral axis from the bottom surface of the beam, respectively, as shown in Figure 1. d_e is either the compressive or the tensile depth of the unyielded portion measured from the neutral axis, given as:

$d_e = -\frac{\varepsilon_y}{\varepsilon_l}\left(D - d_{ep}\right)$	(6)

where ε_y is the yielding strain and ε_l is the limiting strain or the premature failure strain, defined as the maximum compressive strain of the considered section at the occurrence of the premature failure, given as:

$\varepsilon_l = \frac{\varepsilon_f\left(D - d_{ep}\right)}{D_{total} - d_{ep}}$	(7)

where D_{total} is the total height of the beam, for example, the height of the composite cross-section and ε_f is the premature failure strain, also shown in Figure 1. Based on Equation (7), if D=D_{total}, then $\varepsilon_l = \varepsilon_f$ which refers to either one of the following (or both):

i) the premature failure is the failure of the considered section itself, or/and
ii) the extreme fiber of the considered section coincides with the extreme fiber of the failed material

It can also be deduced from Equation (7) that ε_l can have a negative value. If this is the case, it means that the considered section is completely subjected to tensile stress. In considering the various shapes of the stress profile, the following limits must also be imposed:

i) if $d_{ep} + d_e > D$, the summation must be taken as equal to D
ii) if $d_{ep} - d_e < 0$, the residual must be taken as equal to zero

The multi-rule functions, f_1, f_2 and f_3 are given as:

$f_1 = -\sigma_y$	(8)

$f_2 = -\frac{\sigma_y}{d_e}\left(y - d_{ep}\right)$	(9)

$f_3 = -f_1$	(10)

The nth Fourier's sine term coefficient can be determined as:

$$b_n = \frac{2}{D}\left[\int_{d_{ep}+d_e}^{D} f_1 \sin\left(\frac{n\pi y}{D}\right)dy + \int_{d_{ep}-d_e}^{d_{ep}+d_e} f_2 \sin\left(\frac{n\pi y}{D}\right)dy + \int_{0}^{d_{ep}-d_e} f_3 \sin\left(\frac{n\pi y}{D}\right)dy\right] \tag{11}$$

The sine half-range expansion of the Fourier's series of the stress profile can thus be given as:

$$\sigma_{Fourier} = \sum_{1}^{n} b_n \sin\left(\frac{n\pi y}{D}\right) \tag{12}$$

Equation (12) is the approximation of the stress profile which converges to the idealized shape as the number of terms increases. By inserting Equation (12) into Equation (1), the moment capacity of a beam can be calculated for various stress stages as:

$$M = \int_A \left[\sum_{1}^{n} b_n \sin\left(\frac{n\pi y}{D}\right)\right] y dA \tag{13}$$

The use of Equations (12) and (13) in representing the stress profile of various stress stages and to calculate the moment capacity of a beam is demonstrated in the following example.

3. APPLICATIONS

A program has been written using Matlab7 to implement the above procedure and has been used to perform the calculations for the following examples.

3.1 Example 1 (Convergence of the Fourier's Series)

In this first example, it will be shown that, 1), for a given elasto-plastic stress distribution, the Fourier's series representation of the stress distribution converges to the idealized shape as the number of terms increases, and 2) the series is able to represent automatically all possible stress distributions. To achieve purpose 1), an elasto-plastic stress distribution having the following properties is used:

Yield stress, σ_y = 275 N/mm^2, Yield strain, ε_y = 1400 microstrain,
Limiting strain, ε_l = 2200 microstrain, Beam's depth, D = 400 mm,
Location of neutral axis, $d_{ep} = D/2$.

The above stress distribution is shown by the solid line in Figure 3.

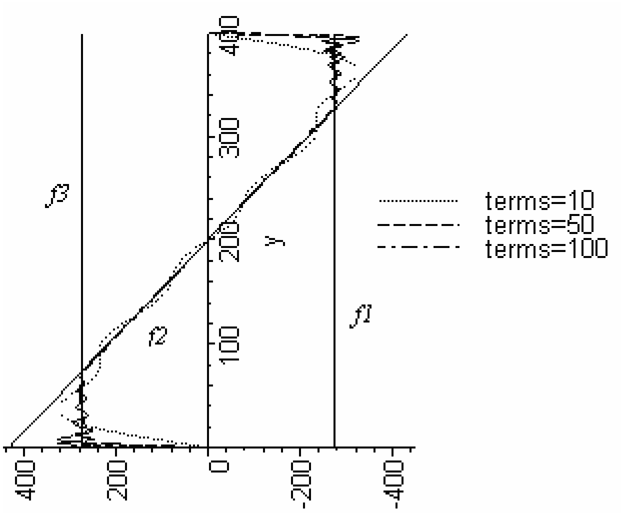
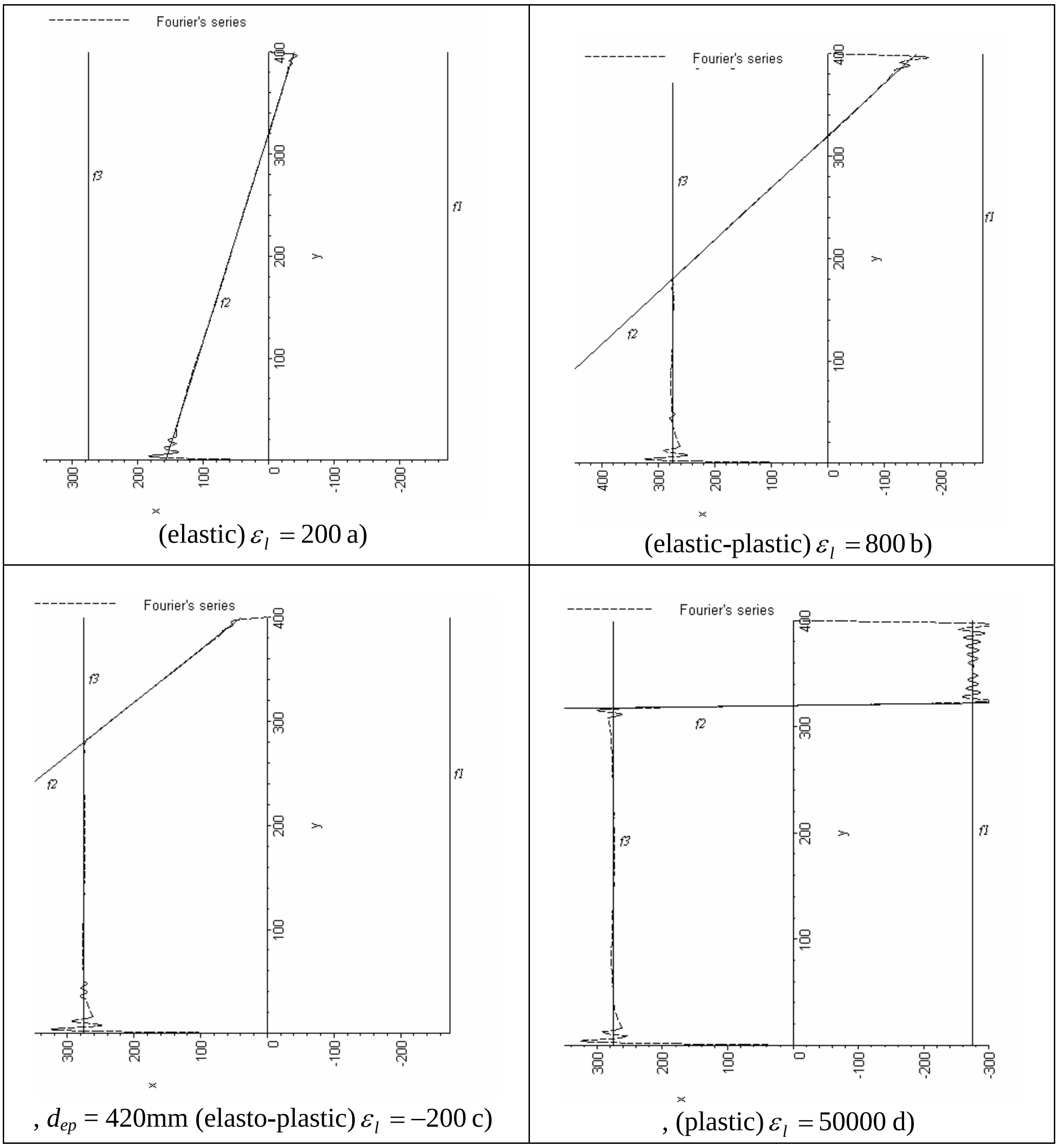


Figure 3. Convergence of the Fourier's series

Based on Figure 3, it can be seen that, the stress distribution has been successfully represented as a single-rule function by using Fourier's series. It can also be seen that the series converges to the idealized shape (solid line) as the number of terms increases.

To achieve purpose ii), the limiting strain, ε_l is varied; possibly caused by different types of premature failure and composite configuration as allowed by Equation (7). To allow for all possible states of stress, a higher location of neutral axis is chosen (d_{ep} = 320mm, except for Figure (4c)); eccentric location of neutral axis may be caused, for example, by composite configuration. To note, for a complex cross-section, determination of d_{ep} requires iteration but the process is not detailed herein. There are 100 terms in the series. Figure 4 exhibits the generality of the Fourier's series since it is able to dictate automatically the appropriate state of the stress based on the input data.



(elastic) $\varepsilon_l = 200$ a)

(elastic-plastic) $\varepsilon_l = 800$ b)

, d_{ep} = 420mm (elasto-plastic) $\varepsilon_l = -200$ c)

, (plastic) $\varepsilon_l = 50000$ d)

Figure 4. Generality of Fourier's series representation of the stress distribution

3.2 Example 2 (Calculation of the Cross-sectional Properties of a Beam using Integration Method)

Yassin and Nethercot (2006) have proposed a generic procedure for determining the cross-sectional properties of a complex composite beam. The novel feature of the procedure is the use of functions to describe the shape of the different elements in a cross-section; this permits determination of the cross-sectional properties through appropriate integrations. Firstly, the basic concept of the procedure is briefly demonstrated herein through the determination of the area of the beam shown in Figure 5. By understanding this, it is possible to see the synchronization of this paper with the proposed procedure. The area of the beam is determined first symbolically and then numerically. Once these are established, the plastic moment capacity of the beam is then calculated.

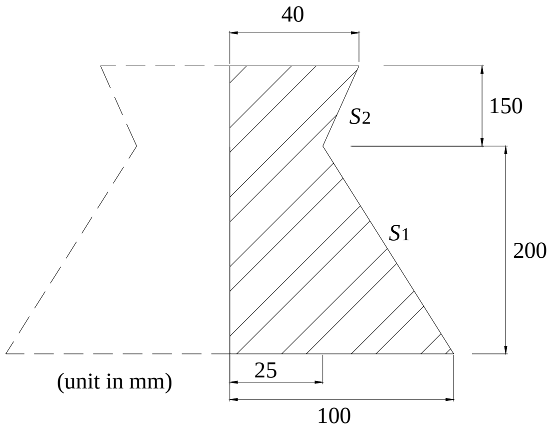


Figure 5. A complex beam's cross-section

Since the beam is symmetrical about the minor axis, only half the beam is considered. Let the outer shape of the beam be represented by functions S_1 and S_2. These functions are derived in the y-axis direction by taking the mid-bottom point of the beam as the origin. The area of the beam is now determined symbolically as:

$$\frac{A}{2} = \int_0^{h_1} S_1 dy + \int_{h_1}^{h_1+h_2} S_2 dy \qquad (14)$$

Now determine the area numerically. Based on the dimensions shown, it can be derived that S_1 = 100–0.375y and S_2 = 0.1y + 5. The area is thus determined as:

$$\frac{A}{2} = \left[100y - \frac{0.375y^2}{2} \Big|_0^{200} \right] + \left[\frac{0.1y^2}{2} + 5y \Big|_{200}^{350} \right] = 17375mm^2$$

The above procedure, although it might appear cumbersome for hand calculation and typical beam configurations, can be very efficient for computer aided calculation and for complex composite beam configurations. It allows for compact computer coding and easier input data even for complex shapes and the end user is required to provide appropriate functions to describe the complex shape, as detailed in Yassin and Nethercot (2006). By inserting Equation (14) into (13), the moment capacity of the beam shown in Figure 5 can be determined as:

$$\frac{M}{2} = \int_0^{h_1} S_1 \left[\sum_1^n b_n \sin\left(\frac{n\pi y}{D}\right) \right] y dy + \int_{h_1}^{h_1+h_2} S_2 \left[\sum_1^n b_n \sin\left(\frac{n\pi y}{D}\right) \right] y dy \qquad (15)$$

The values of the plastic moment capacity of the beam, determined using Equation (15) for various numbers of terms and two different limiting strains are given in Table 1 and plotted in Figure 6.

Table 1. Plastic moment capacities

Number of terms	Plastic moment capacity ($1x10^8$ Nmm)		
	Present formulation		Rectangular stress block
	$\varepsilon_l = 25000$	$\varepsilon_l = 50000$	
10	1.4662	1.4604	2.0267
30	1.8525	1.8532	
50	1.9201	1.9227	
100	1.9743	1.9741	
150	1.9921	1.9921	
200	2.0011	2.0010	

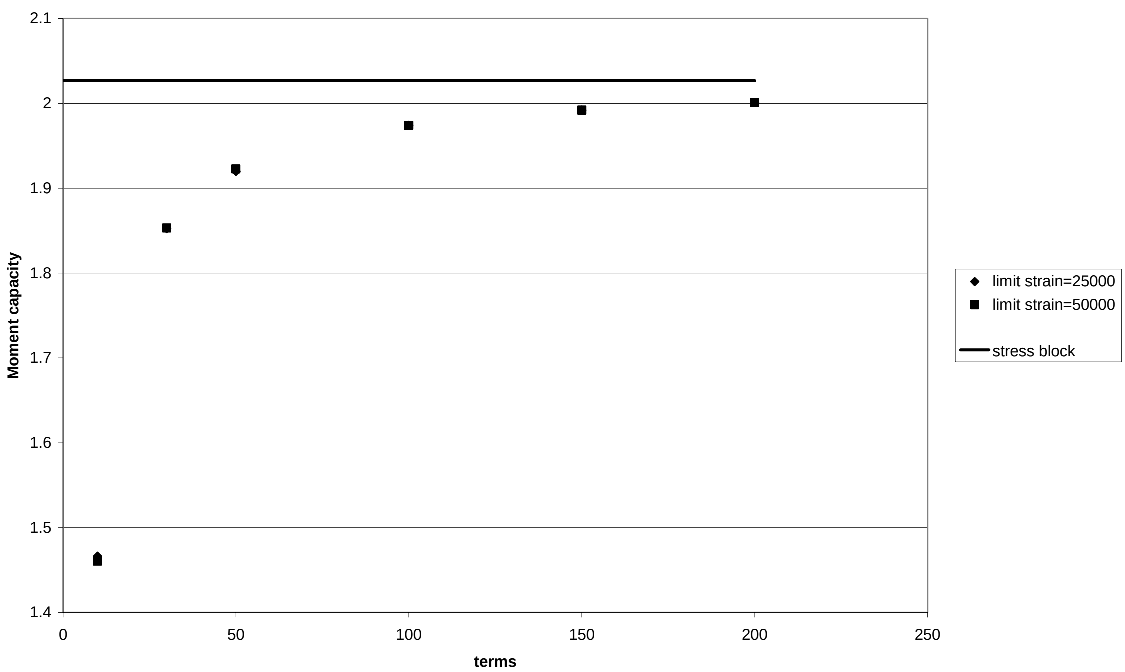


Figure 6. Convergence of the plastic moment capacities

Based on Figure 6, it can be seen that the values of the plastic moment capacities converge to the idealized value of the rectangular stress block method. Although it may not be obvious from Figure 6, it can be observed from Table 1 that the convergence rate of the moment capacity calculated based on $\varepsilon_l = 50000$ is slightly higher than that calculated using $\varepsilon_l = 25000$. This is expected since function f_2 of the former is flatter, which is closer to the idealized condition. The causes for the different values between the present formulation and the rectangular stress block method are identified as follows:

i) the noise borne with the series approximation (can be reduced by increasing the number of terms)

ii) ripples in the vicinity of the discontinuity (at the intersection between original functions and at the upper and bottom surface of the beam), a phenomenon known as *Gibbs phenomenon*.

While item i) will theoretically vanish as the number of terms approaches infinity, item ii) will always remain, although the magnitude reduces as the number of terms increases. The fact that the rectangular stress method always gives an upper bound result, as shown in Figure 6, means that these items not major concerns.

4. CONCLUSION

This paper has presented a generic procedure for determining the moment capacity of a complex cross-section. The novel feature of the procedure is the single-rule representation of the stress distribution using Fourier's series which allows the moment capacity of the beam to be determined by a general expression without the need for the separate treatment of the various stress stages.

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6. REFERENCES

Matlab7,Version 7.0.0 19920 9 (R14), 2004, Copyright The MathWorks, Inc.

Yassin, A. Y. M. and Nethercot, D. A., 2006, *Cross-sectional Properties of Complex Composite Beams*, accepted for publication in "Engineering Structures".