STRAIGHT LINE AND CIRCULAR ARC METHODS FOR DEVELOPING G^1 AND G^2 INVOLUTE CURVES

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Abstract.   Parametric polynomial curves such as Bezier, Ball, B-splines, Non-uniform B-splines (NURBS) are used for free form curve design. In this paper, we classify these curves as conventional curves. The flexibility of these curves deems suitable for use in the interactive design of curves. On the contrary, these curves cannot be used for highways, railways and robot trajectory designs as the signed curvature of these curves are difficult to control. As a result, the designer has to integrate a time consuming fair process. There are unconventional curves with easy control of the curvature namely, Euler and equiangular spirals. Unfortunately, the formulation of these spirals involves Fresnal integral and exponential functions respectively, which results in extra overhead and implementation. This paper introduces two type of curves which are generated from an evolute-involute process. The first type of involute curve(s) is generated using straight line(s) as the evolute(s) and named IFSL. The second type of involute curve(s) is generated based on circular arc(s) and a straight line and named IFCA.

Keywords: Computer Aided Geometric Design (CAGD), involute curves, geometric continuity of degree 1 (G^1) and 2 (G^2), circular arcs, spirals

Abstrak.   Lengkung polinomial berparameter seperti Bezier, Ball, splin-B dan Splin-B tak seragam digunakan dalam mereka bentuk lengkung bebas. Dalam kajian ini, kami telah mengkategori lengkung di atas sebagai lengkung konvensional. Fleksibiliti lengkung-lengkung tersebut membolehkannya diguna dalam mereka bentuk lengkung secara interaktif. Namun demikian, lengkung konvensional ini tidak boleh diguna untuk mereka bentuk lebuh raya, landasan keretapi dan trajektori robot kerana kelengkungan bertanda bagi lengkung-lengkung tersebut sukar dikawal. Oleh yang demikian, pereka bentuk harus menerapkan proses saksama yang memakan masa. Terdapat juga lengkung tak konvensional yang mudah dikawal dari segi kelengkungan, misalnya lingkaran Euler dan sama sudut equiangular. Walau bagaimanapun, mereka bentuk lengkung tak konvensional melibatkan kamiran Fresnal dan fungsi eksponen masing-masingnya yang mengakibatkan overhed dan implementasi. Kajian ini memperkenalkan dua jenis lengkung yang dijana melalui proses evolut-involut. Lengkung involut jenis pertama dijana menggunakan garis lurus selaku evolut dan dinamakan IFSL. Lengkung involut jenis kedua dijana berdasarkan segmen bulatan dan garis lurus, dan dinamakan IFCA.

Kata kunci: Reka Bentuk Geometri Dibantu Komputer (RGBK), lengkung involut, keselanjaran geometri berdarjah 1 (G^1) dan 2 (G^2), segmen bulatan, lingkaran

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1.0 INTRODUCTION

The curvature is used as a measure of curve fairness for technical design specifically for aesthetic design, such as car bodies. A curve is said to be fair if it has relatively few curvature monotonous segments [1]. Fair curves are not just important for Computer Aided Design (CAD) and Computer Aided Geometric Design (CAGD) applications, but they play essential roles in highways and robot trajectory designs as well. Inspection of curvature has been used as a passive shape interrogation tool in the field of aesthetic design.

A significant disadvantage of using the conventional curves (Bezier, Ball, B-splines and non-uniform B-splines) is that their curvature is a complicated function of its parameter. Hence, it is not easy to use them in the design of curvature controlled curves. In general, a spiral is said to be a plane curve traced by a point which winds about a fixed pole from which it continually recedes [2]. In CAGD, a spiral is a curve with monotone curvature of constant sign [3]. Two types of spirals which have been widely used in curve design are Euler and equiangular spirals.

The Euler spiral (also called Cornu's spiral or clothoid) was first studied by Euler in 1781 in connection with an investigation of an elastic spring [4]. The Euler spiral has been used for highways and railways route design [5]. This is because the curvature for clothoid is linearly related to its arc length. The standard clothoid [6] with scaling factor S is:

$$K(\theta) = S \begin{pmatrix} C(\theta) \\ S(\theta) \end{pmatrix}, \qquad 0 \le \theta \tag{1}$$

where scaled versions of Fresnal integrals $C(\theta)$ and $S(\theta)$ are:

$$C(\theta) = \int_0^\theta \frac{\cos u}{\sqrt{u}} du, \qquad S(\theta) = \int_0^\theta \frac{\sin u}{\sqrt{u}} du, \qquad 0 \le \theta \tag{2}$$

The equiangular spiral (also called logarithmic spiral) was first discovered by Descartes, followed by its properties of self reproduction by Bernoulli [4]. The radius of curvature of logarithmic spiral is proportional to its arc length. Thus, the task of constructing a smooth planar spline curve with the given end curvature is straightforward using a logarithmic spiral. A logarithmic spiral $L(\theta)$ in plane [7] can be defined by:

$$L(\theta) = r \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \qquad r = ae^{b\theta}, \qquad 0 \le \theta \tag{3}$$

where e is the exponential function, r is the distance from the origin, θ is the angle from the x-axis and, a and b are arbitrary constants.

Both Euler and equiangular spirals have their own drawbacks, for instance, clothoid involves integration which is numerically expensive (Simpson rule is widely used for numerical integration). Logarithmic spirals involve exponential, sine and cosine functions and it cannot be represented by a Bezier curve of finite degrees [7]. Thus, researchers look into methods of approximating the spirals by conventional curves [7-9]. A notable research on developing spirals by using cubic Bezier can be seen in [10, 11].

2.0   PROBLEM STATEMENT

The idea of evolutes originated from Huygens [4] in 1673, in conjunction with his studies on light. However, it is said that the concept had existed around 200 BC where it first appeared in the fifth book of Apollonius's conic sections [4]. Conversely, the idea of involute of a circle was discussed and utilized by Huygens [4] in 1693 in connection with his study of clocks without pendulums for service on ships of the sea. Nowadays, involutes of circle are being used in the design of gear teeth [2].

Involutes can be drawn easily by mechanical means: if a string is attached to a point on a curve, lying along the tangent to the curve at that point and is 'wrapped up' on to the curve, the locus of any point of the string is an involute of the given curve [12]. The original curve is called an evolute. A curve can have any number of involutes, thus a curve is an evolute of each of its involutes and an involute of its evolute. The normal to a curve is tangent to its evolute and the tangent to a curve is normal to its involutes. The evolute of a curve is also the envelope of the normal to the curve [13].

The involute of a convex curve in the plane is a spiral in the plane, e.g., the involute of a circle is a spiral (also called anti-clothoid) and the involute of an equiangular spiral is another equiangular spiral.

The idea of using evolute-involute process for curve design was first introduced by Kuroda and Mukai [14] whereby circular arcs have been used as the evolutes to develop G^2 involute curves. Recently, Goodman *et al.* [15] have proposed the generation of an involute spiral that matches G^2 Hermite data by means of a polynomial and a rational Pythagorean hodograph as the evolutes.

In this paper, we propose a different method to design G^1 arc splines by using straight lines as the evolutes. The resultant arc splines are named as Involutes from straight lines (IFSL). The design of G^2 spirals using circular arcs and a straight line as the evolutes is elaborated in Section 5. The resultant spirals are named as Involutes from Circular Arcs (IFCA). The proposed spiral has sufficient degree of freedom which makes it suitable for free form curve design.
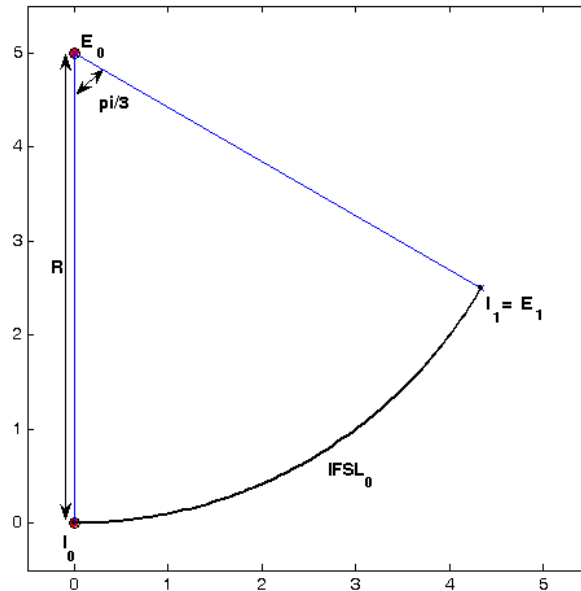
3.0   A SEGMENT OF IFCA

Let a straight line defined from $E_0(a_0, b_0)$ to $E_1(a_1, b_1)$ be the evolute. For simplicity, let $E_0 = (0, R_0)$, where R_0 is the distance from the origin. The length of the stated line is denoted by L_0 and the angle formed between the y-axis and the evolute line by θ_0. By using a trigonometric function, one may produce an involute segment, $IFSL_0$, denoted by $A(\theta_0)$ in the form of a parametric equation with θ_0 as the parameter. The end points are denoted as $I_0(x_0, y_0) = (0, 0)$ and $I_1(x_1, y_1) = E_1$. Notationwise:

$$E_0(a_0, b_0) = [0, R_0], \qquad E_1(a_1, b_1) = I_1(x_1, y_1), \tag{4}$$

$$I_0(x_0, y_0) = [0, 0], \quad I_1(x_1, y_1) = \left[R_0 \sin \theta_0, R_0 (1 - \cos \theta_0) \right], \quad 0 \leq \theta_0 \leq 2\pi \tag{5}$$

$$A_0(\theta) = \left[R_0 \sin \theta_0, R_0 (1 - \cos \theta_0) \right], \qquad 0 \leq \theta \leq \theta_0 \tag{6}$$

From (6), one may generate an involute which is in the form of a circular arc with radius $R_0 = L_0$. Figure 1 shows an example of generating a IFSL segment with $R_0 = 5$ and $\theta_0 = \pi/3$. One may choose a desired θ_0 to design a convex $IFSL_0$ curve, for instance, if $\theta_0 = 2\pi$, the resulting $IFSL_0$ curve is a circle.

Figure 1   A segment of IFSL

4.0   A GENERAL ALGORITHM TO DEVELOP IFSL SPLINE

To generate $IFSL_i$ spline with $(n + 1)$ number of segments where $i = 0, ..., n$, and $0 \leq \theta_i \leq 2\pi$, we need to calculate the evolutes of a straight line. For now, consider $R_0 = \sum_{i=0}^{n} L_i$ (see *Remark 1*). The interpolating point of the evolutes can be defined recursively as:

$$E_{i+1}(a_{i+1}, b_{i+1}) = \left[a_i + L_i \, \sin\left(\sum_{i=0}^{n} \theta_i \right), b_i - \left(L_i \, \cos\left(\sum_{i=0}^{n} \theta_i \right) \right) \right], \qquad (7)$$

Next, the end points of involutes are determined by letting $R_{i+1} = R_i - L_i$ as:

$$I_{i+1}(x_{i+1}, y_{i+1}) = \left[a_i + R_i \, \sin\left(\sum_{i=0}^{n} \theta_i \right), b_i - \left(R_i \, \cos\left(\sum_{i=0}^{n} \theta_i \right) \right) \right] \qquad (8)$$

Finally, for $i = 1, ..., n$, $IFSL_i$ spline is defined as:

$$A_i(\theta) = \left[a_i + R_i \, \sin\theta, b_i - R_i \, \cos\theta \right], \qquad \text{where } \sum_{i=0}^{n-1} \theta_i \leq \theta \leq \sum_{i=0}^{n} \theta_i \qquad (9)$$

The signed curvature of $IFSL_i$ spline, $\kappa(\theta)$ is calculated as:

$$\kappa_i(\theta) = \frac{A_i'(\theta) \times A_i''(\theta)}{\left\| A_i'(\theta) \right\|^3} = \frac{1}{R_i} \qquad (10)$$

where the symbol $\times$ represents the two-dimensional cross product .The norm or length of $A_i'(\theta)$ is denoted as $\left\| A_i'(\theta) \right\|$. Equation (10) proofs that IFSL spline consists of circular arcs (the curvature of a circular arc is constant).

*Remark 1*:   If the major radius, $R_0 = \sum_{i=0}^{n} L_i$, then the generated IFSL segments are joined with geometric continuity of degree 1, GC^1, and $E_{n+1} = I_{n+1}$. If $R_0 > \sum_{i=0}^{n} L_i$ (or $R_0 < \sum_{i=0}^{n} L_i$), then the generated segments may not be G^1 (a cusp may occur) and $E_{n+1} \neq I_{n+1}$. As a result, it is advisable to fix $R_0 = \sum_{i=0}^{n} L_i$ for free form curve design in accordance with the evolute-involute process.

We developed a package using MATLAB®7.0 to design IFSL spline interactively. By default, θ_i is set to be 10 degree and L_i is set with a uniform length: $L_i = R_{0/n+1}$. Figure 2 illustrates four examples of developing IFSL spline with 20 segments when $R_0 = 30$.

Figure 2(a) shows an example of IFSL spline generated using default values θ_i of and L_i. Each IFSL segment is G^1, it shares the same interpolating point and tangent vectors at the joint. Figure 2(b) (and Figure 2(c)) shows the resulting IFSL spline

(a)

(b)

(c)

(d)
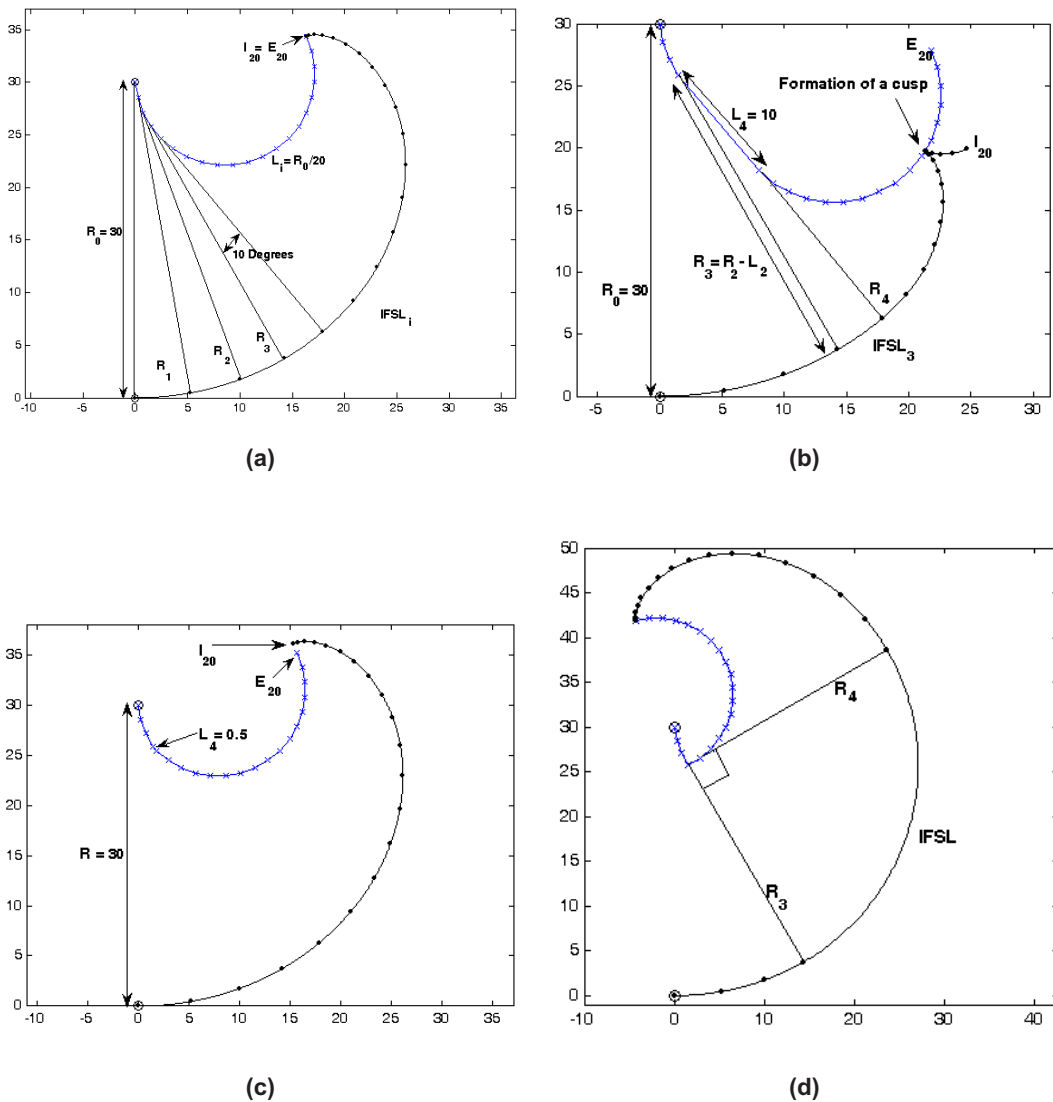
Figure 2   IFCA spline consists of circular arcs

when $R_0 > \sum_{i=0}^{n} L_i$ (and $R_0 < \sum_{i=0}^{n} L_i$). Figure 2(d) shows an example of curve design using $\theta_3 = \pi/2$.

As a result, the designer may implement the proposed algorithm to design a family of convex splines by varying the values of θ_i and L_i, where $0 \le \theta_i \le 2\pi$ and

$$R_0 = \sum_{i=0}^{n} L_i.$$

5.0   A SEGMENT OF IFCA

In this section, the construction of a IFCA spiral segment (denoted as $B(\theta)$ in parametric form) is shown. Figure 3 illustrates a IFCA$_0$ spiral.
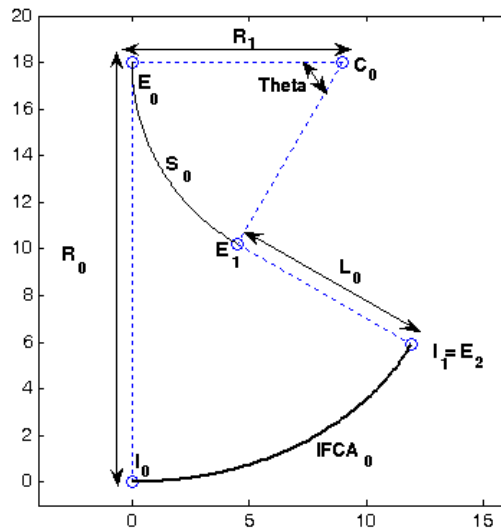

Figure 3   IFCA$_0$ spiral segment

Let a circular arc be defined from $E_0(a_0, b_0)$ to $E_1(a_1, b_1)$ and a straight line from $E_1(a_1, b_1)$ to $E_2(a_2, b_2)$ be the evolute, where $E_0 = (0, R_0)$. The length of the evolute line is denoted as L_0, the length of the circular arc segment is denoted as S_0 and the angle of the segment is denoted as θ_0. The properties of IFSL$_0$ spiral segment are defined as:

$$E_0\left(a_0, b_0\right) = [0, R_0], \qquad C_0\left(p_0, q_0\right) = (R_1, R_0), \qquad (11)$$

$$E_1\left(a_1, b_1\right) = [p_0 - p_0 \cos \theta_0, q_0 - p_0 \sin \theta_0], \qquad (12)$$

$$I_1(x_1, y_1) = [a_1 + L_0 \sin \theta_0, b_1 - L_0 \cos \theta_0], \tag{13}$$

where $L_0 = R_0 - S_0$, $S_0 = R_1\theta_0$ and $0 \le \theta_0 \le 2\pi$. Hence, IFSL_0 is defined as:

$$B_0(\theta) = \left[(p_0 - R_1 \cos \theta) + (R_0 - R_1\theta)\sin \theta, (q_0 - R_1 \sin \theta) - (R_0 - R_1\theta)\cos \theta_0 \right], \tag{14}$$

where $0 \le \theta \le \theta_0$. Figure 3 shows an example of IFCA_0 spiral segment generated with $R_0 = 18$, $R_1 = 9$ and $\theta_0 = \pi/3$.

The signed curvature of IFSL_0 spiral segment is calculated as:

$$\kappa_0(\phi) = \frac{1}{R_0 - R_1 \cdot \phi} \tag{15}$$

where $0 \le \phi \le \theta_0$.

6.0 A GENERAL ALGORITHM TO GENERATE SPIRAL SEGMENTS

This section elaborates the algorithm to generate IFSL_i spline with $(n+1)$ segments

where $i = 1, ..., n$, and $0 \le \theta_i \le 2\pi$. A necessary condition is $R_0 = \left(\sum_{i=0}^{n} S_i \right) + L_n$,

where $S_i = R_{i+1}\theta_i$ and $L_i = L_{i-1} - S_i$. The centre point of the circular arc is are calculated as:

$$C_i(p_i, q_i) = \left[p_{i-1} - (R_i - R_{i+1})\cos\left(\sum_{i=0}^{n-1}\theta_i \right), q_{i-1} - (R_i - R_{i+1})\sin\left(\sum_{i}^{n-1}\theta_i \right) \right] \tag{16}$$

where R_i denotes the radius of circular arc (evolute). The interpolating point of the evolutes is calculated as:

$$E_i(a_{i+1}, b_{i+1}) = \left[p_i - R_{i+1}\cos\left(\sum_{i=0}^{n}\theta_i \right), q_i - R_{i+1}\sin\left(\sum_{i=0}^{n}\theta_i \right) \right], \tag{17}$$

The interpolating point of the IFSL_i spiral segment is calculated as:

$$I_{i+1}(x_{i+1}, y_{i+1}) = \left[a_{i+1} + L_i\cos\left(\sum_{i=0}^{n}\theta_i \right), b_{i+1} - L_i\sin\left(\sum_{i=0}^{n}\theta_i \right) \right], \tag{18}$$

Finally, the general equation for IFSL_i spline is:

$$B_i(\theta) = \begin{pmatrix} (p_i - R_{i+1}\cos\theta) + (L_{i-1} - R_{i+1}\cdot\phi)\sin\theta \\ (q_i - R_{i+1}\sin\theta) - (L_{i-1} - R_{i+1}\cdot\phi)\cos\theta \end{pmatrix}^T, \tag{19}$$

where $\sum_{i=0}^{n-1}\theta_i \le \theta \le \sum_{i=0}^{n}\theta_i$ and $0 \le \phi \le \theta_i$.

Upon algebraic simplification, the parametric equation of $\kappa(\theta)$, can be written in a simple form, where $0 \le \phi \le \theta_i$:

$$\kappa_i(\phi) = \frac{1}{L_{i-1} - R_{i+1}\cdot\phi} \tag{20}$$

Figure 4 illustrates an example of IFCA spline generated with 5 IFCA spiral segments with:

$$\{\theta_0, \theta_1, \theta_2, \theta_3, \theta_4\} = \{\pi/6, \pi/5, \pi/5, \pi/4, \pi/3\}, \quad \{R_1, R_2, R_3, R_4, R_5\} = \{8, 13, 5, 8, 17\}$$

and $R_0 = 60$. The curvature plot ($\kappa(\theta)$ against parameter θ_i) illustrated in Figure 5 indicates that IFCA segments consist of spirals with monotonic increase of positive signed curvature. To note, IFCA spiral segments are joined with GC^2 as these pieces share signed curvature values at I_{i+1}. The designer may fix the end curvatures by defining $\kappa_0(0) = \dfrac{1}{R_0}$ and $\kappa_n(\phi) = \dfrac{1}{L_{n-1} - R_{n+1}\cdot\phi}$, where $0 \le \phi \le \theta_n$.
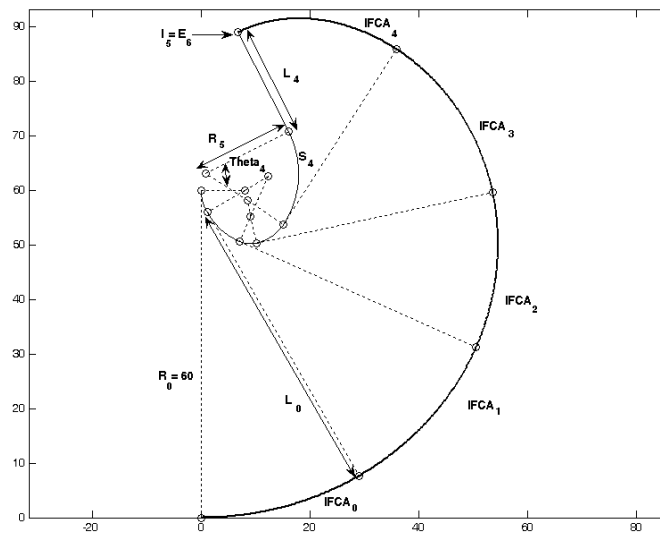
Figure 4 Five segments of IFSL spirals joined with G^2
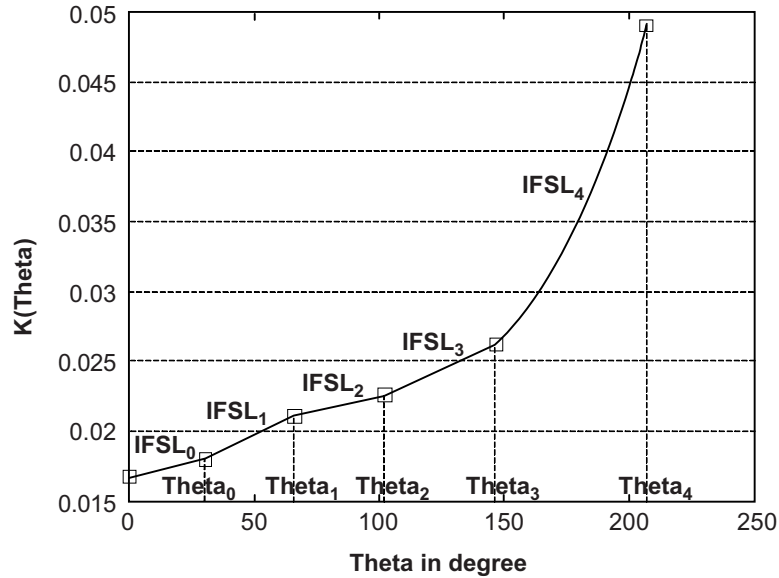
Figure 5   Curvature plot of IFSL spline

7.0   CONCLUSION

A general algorithm to develop G^1 IFSL and G^2 IFCA has been shown in this paper. The IFSL and IFCA splines consist of circular arcs and spirals respectively. These curves can be used for free form curve design, e.g., railways and highways design and robot trajectories where the signed curvature of proposed curves can be controlled to a certain extent. Future work would include identifying the necessary and sufficient conditions that can be matched for a given G^2 Hermite data.

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