CAPABILITY AND HOMOLOGICAL FUNCTORS OF INFINITE TWO-GENERATOR GROUPS OF NILPOTENCY CLASS TWO

NOR MUHAINIAH BINTI MOHD ALI

UNIVERSITI TEKNOLOGI MALAYSIA

CAPABILITY AND HOMOLOGICAL FUNCTORS OF INFINITE TWO-GENERATOR GROUPS OF NILPOTENCY CLASS TWO

NOR MUHAINIAH BINTI MOHD ALI

A thesis submitted in fulfilment of the
requirements for the award of the degree of
Doctor of Philosophy (Mathematics)

Faculty of Science
Universiti Teknologi Malaysia

JANUARY 2009

ABSTRACT

A group is called capable if it is a central factor group. Baer characterized finitely generated abelian groups which are capable as those groups which have two or more factors of maximal order in their direct decomposition. The capability of groups have been determined for infinite metacyclic groups and for 2-generator p-group of nilpotency class two (p prime). The remaining case for capability of 2-generator group of nilpotency class two is the infinite case where the groups have been classified by Sarmin in 2002. Let R be the class of infinite 2-generator groups of nilpotency class two. Using their classification and non-abelian tensor squares, the capability of groups in R are determined. Brown and Loday in 1984 and 1987 introduced the nonabelian tensor square of a group to be a special case of the nonabelian tensor product which has its origin in algebraic K-theory as well as in homotopy theory. The homological functors have been determined for infinite metacyclic groups and non-abelian 2-generator p-groups of nilpotency class two. Therefore, the homological functors including the exterior square, the symmetric square and the Schur multiplier of groups in R will also be determined in this research.

ABSTRAK

Suatu kumpulan dipanggil kumpulan berupaya jika ia adalah kumpulan faktor berpusat. Baer telah mencirikan bahawa suatu kumpulan abelan berpenjana terhingga yang merupakan suatu kumpulan berupaya mempunyai dua atau lebih faktor peringkat maksima dalam penghuraian terus kumpulan tersebut. Keberupayaan bagi suatu kumpulan telah ditentukan bagi kumpulan kitaran meta tak terhingga dan bagi kumpulan-p berpenjana 2 dengan kelas nilpoten dua (p nombor perdana). Kes yang tinggal bagi keberupayaan suatu kumpulan bagi kumpulan berpenjana 2 dengan kelas nilpoten dua adalah kes tak terhingga di mana kumpulan tersebut telah diklasifikasikan oleh Sarmin dalam tahun 2002. Katalah R adalah kelas bagi kumpulan tak terhingga berpenjana 2 dengan kelas nilpoten dua. Dengan menggunakan klasifikasi dan kuasa dua tensor tak abelan bagi kumpulan dalam R, keberupayaan bagi kumpulan R akan ditentukan. Brown dan Loday dalam tahun 1984 dan 1987 telah memperkenalkan kuasa dua tensor tak abelan bagi suatu kumpulan sebagai kes istimewa bagi hasil darab tensor tak abelan yang berasal dari algebra teori-K begitu juga dalam teori homotopi. Fungtor homologi telah ditentukan bagi kumpulan kitaran meta tak terhingga dan bagi kumpulan-p berpenjana 2 yang tak abelan dengan kelas nilpoten dua. Fungtor homologi termasuk kuasa dua peluaran, kuasa dua simetrik dan pekali Schur bagi semua kumpulan dalam R juga akan ditentukan dalam penyelidikan ini.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

$a\|b$	-	a divides b
$\|G\|, \|x\|$	-	Order of the group G, the order of the element x
$\mathbb{Z}$	-	Integers, the infinite cyclic group
$\mathbb{Z}_n$	-	Cyclic group of order n
$\mathbb{Z}/n\mathbb{Z}$	-	Integers modulo n
$H \leq G$	-	H is a subgroup of G
$G \cong H$	-	G is isomorphic to H
$G \oplus H$	-	Direct sum of G and H
$G \times H$	-	Direct product of G and H
$G \rtimes H$	-	Semidirect product of G and H
$G \otimes H$	-	Tensor product of G and H
$G \otimes G$	-	Tensor square of G
$G \otimes_{\mathbb{Z}} G$	-	Abelian tensor square of G
$G \tilde{\otimes} G$	-	Symmetric square of G
$G \wedge G$	-	Exterior square of G
$M(G)$	-	Schur multiplier of G
$C_G(A)$	-	Centralizer of A in G
$Z(G)$	-	Center of the group G
$\langle x \rangle$	-	Group generated by the element x
$ker(\kappa)$	-	Kernel of the homomorphism κ
$N \lhd G$	-	N is a normal subgroup of G
gH, Hg	-	Left coset, and right coset of H with coset representative g
$\|G : H\|$	-	Index of the subgroup H in the group G

$\mathrm{Inn}(G)$	-	Inner automorphism group of G
$T(G)$	-	Torsion subgroup of G
${}^{g}h$	-	The conjugate of h by g
$[g,h]$	-	The commutator of g and h
G'	-	The commutator subgroup of G
$\subset$	-	Proper subset
$\subseteq$	-	Subset
$\in$	-	Element of
$\notin$	-	Not element of
$\cap$	-	Intersection
$\wedge$	-	Wedge product
$1_{\wedge}$	-	Identity of exterior square
$\neq$	-	Not equal to
$<$	-	Less than
$\leq$	-	Less than or equal to
$>$	-	Greater than
$\geq$	-	Greater than or equal to
X	-	Direct product

CHAPTER 1

INTRODUCTION

1.1 Introduction

Philip Hall in 1940 [1] has pointed the way towards the classification of prime power order. He said, "The question of what conditions a group G must fulfill in order that it may be the central quotient group of another group H,

$$G = H/Z(H)$$

is an interesting one. However, while it is easy to write down a number of necessary conditions, it is not so easy to be sure that they are sufficient."

A group G is said to be capable if there exists a group K such that $K/Z(K)$ is isomorphic to G. There are groups which are not capable for example a group G with $G/T(G)$ of rank bigger than 1. The question of which finite p-groups are capable plays an important role in their classification. Since the classification of p-groups are not settled yet, we can classify all finite p-groups by knowing which groups from that classification are capable groups.

The nonabelian tensor square is a special case of the nonabelian tensor product which has its origin in homotopy theory. Topologists are interested in explicit case for example explicit tensor squares. In this research, we focus on the nonabelian tensor square $G \otimes G$ of a group G. The nonabelian tensor square of a group G is generated by the symbols $g \otimes h$, where g, $h \in G$ subject to the relations

$$gg'\otimes h = ({}^{g}g'\otimes{}^{g}h)(g\otimes h) \text{ and } g\otimes hh' = (g\otimes h)({}^{h}g\otimes{}^{h}h')$$

for all g, g', h, $h' \in G$, where ${}^{h}g = hgh^{-1}$ denotes the conjugate of g by h. In 2000 and 2003, the homological functors have been determined for infinite metacyclic groups and for the non-abelian 2-generator p-groups of class 2, respectively. In this research, using the classification of the infinite non-abelian 2-generator groups of nilpotency class 2 given by Sarmin in [2], we will determine which groups are capable, and we will also determine some homological functors for all groups in the classification, including the symmetric squares, the exterior squares and the Schur multiplier.

1.2 Research Background

The classification of all two-generator p-groups of class two and determination of their nonabelian tensor squares have been done in [3] for the case p prime, $p\neq 2$, in [4] for the case $p = 2$, and in [2] for the infinite case. The nonabelian tensor squares, capability and homological functors for many groups have been determined by many authors. In [5], the nonabelian tensor squares for 2-Engel group have been determined. In [6], the nonabelian tensor squares have been determined for infinite metacyclic groups. The nonabelian tensor squares of the free 2-Engel groups have been determined by Blyth et all, [7] in 2004. Earlier work by Miller [8] and Dennis [9] is in context with Schur multipliers and algebraic K-theory, respectively are referred. Beuerle and Kappe in [6] determined the capability using nonabelian tensor square and homological functors of infinite metacyclic groups. Recent work of Bacon and Kappe in [10] focuses the capability and homological functors for p-group of nilpotency class two. Although, the classification of non-abelian 2-generator 2-groups was determined in [4], the capability of these groups was determined by Magidin in [11]. This research extends the applications on a wider basis which cover infinite two-generator groups of nilpotency class two.

1.3 Problem Statement

To determine the capability of all infinite two-generator groups of nilpotency class two and to compute their homological functors .

1.4 Research Objectives

The objectives of this research are:

1. To determine which of the groups in the classifications of infinite 2-generator groups of nilpotency class 2 are capable.
2. To compute various functors of infinite 2-generator groups of nilpotency class 2 including the symmetric squares, the exterior squares and the Schur multiplier.

1.5 Scope of the Study

This research will focus only on the classification of infinite two-generator groups of nilpotency class two.

1.6 Significance of Findings

The benefit of this research will be the development of new theorems with proofs. The result obtained can be used for further research in related areas. Articles or papers regarding this research have been and will be presented nationally and internationally, and will be sent to be published in national and international journals.

1.7 Research Methodology

The research starts from studying the classification of two-generator p-groups of class two and the infinite case. As a preliminary research and warm up, finite groups in the classification of finite 2-generator groups of nilpotency class two are studied in details and the applications of these finite groups in probability theory have been investigated. The results of this probability theory can be found in [12]. Then research articles on the capability and homological functors by several authors were studied. These include Beuerle and Kappe (2000), Bacon and Kappe (2003), and Magidin (2006). The determination of capable groups and various functors of all infinite two-generator groups of nilpotency class two will be given. Some appropriate theorems with the proofs will be developed. These results will be analyzed by Groups, Algorithms and Programming (GAP) software.

1.8 Groups, Algorithms and Programming (GAP)

GAP is a system for computational discrete algebra, with emphasis on computational group theory. GAP provides a programming language, a library of functions that implement algebraic algorithms written in the GAP language as well as libraries of algebraic objects such as for all non-isomorphic groups up to order 2000.

GAP was started in 1986 at Lehrstuhl D fur Mathematik, RWTH Aachen, Germany. After 1997, the development of GAP was coordinated from St Andrews. Nowadays, GAP is developed and maintained by GAP centers in Aachen and Braunschweig in Germany, Fort Collins in the USA and St Andrews in Scotland.

GAP is used in this research to verify the results found in determining the homological functors of the infinite two-generator groups of nilpotency class two of Type 2 and Type 4. Meanwhile for Type 1 and Type 3 groups, GAP is used to make a conjecture and futhermore to check the theoretical results obtained. Some examples and detail explainations of the use of GAP are presented in both Chapter 5 and Appendix A.

1.9 Thesis Organization

The first chapter serves as an introduction to the whole thesis. This chapter introduces the concepts of capability and homological functors of groups. Chapter 1 also includes research background, problem statement, research objectives and research methodology. The scope of the study and the significance of findings are also listed.

Chapter 2 includes the literature review of this research. Various works by different authors regarding the capability and homological functors are presented. This chapter also includes some basic results that will be used in the subsequent chapters. The proofs of some results are given while some are omitted.

Chapter 3 presents the results on the capability of infinite 2-generator groups of nilpotency class two. Some definitions and results on the capability by several authors are also presented in this chapter. These definitions and results are needed to prove our results in the main section of this chapter.

In Chapter 4, results on the homological functors of infinite 2-generator groups of nilpotency class two are presented. The results are separated in different theorems based on the types of groups and the condition of the parameters. The nonabelian tensor squares of infinite 2-generator groups of nilpotency class two had been given by [2] and are shown in the preliminary part of this chapter.

Chapter 5 provides GAP programmes that have been used to construct examples for the homological functors of infinite 2-generator groups of nilpotency class two (for each Types). GAP is used to check the theoretical results by the examples generated. The illustration of some examples and the detail explaination of the interpretation of the commands used are provided in this chapter.

The last chapter gives a summary of the whole research. It also gives some suggestions for future research in this area.