BOUNDARY INTEGRAL EQUATIONS APPROACH FOR NUMERICAL CONFORMAL MAPPING OF MULTIPLY CONNECTED REGIONS

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HU LAEY NEE

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To my beloved husband Kong King Hwa
and my three children, Wan Kieng, Wan Wee and Jing Yi
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ABSTRACT

Several integral equations involving the Kerzman-Stein and the Neumann kernels for conformal mapping of multiply connected regions onto an annulus with circular slits and onto a disk with circular slits are presented. The theoretical development is based on the boundary integral equation for conformal mapping of doubly connected region. The integral equations are constructed from a boundary relationship satisfied by a function analytic on a multiply connected region. The boundary integral equations involve the unknown parameter radii. For numerical experiments, discretizing each of the integral equations leads to a system of non-linear equations. Together with some normalizing conditions, a unique solution to the system is then computed by means of an optimization method. Once the boundary values of the mapping function are calculated, the Cauchy's integral formula has been used to determine the mapping function in the interior of the region. Typical examples for some test regions show that numerical results of high accuracy can be obtained for the conformal mapping problem when the boundaries are sufficiently smooth.

ABSTRAK

Beberapa persamaan kamiran melibatkan inti Kerzman-Stein dan Neumann untuk pemetaan konformal bagi rantau berkait berganda ke atas anulus dengan belahan membulat dan ke atas cakera dengan belahan membulat dipersembahkan. Pembangunan teori adalah berdasarkan persamaan kamiran sempadan bagi pemetaan konformal rantau berkait ganda dua. Persamaan-persamaan kamiran adalah dibangunkan dari hubungan sempadan yang ditepati oleh fungsi yang analisis dalam rantau berkait berganda. Persaman-persamaan kamiran sempadan ini melibatkan parameter jejari-jejari yang tidak diketahui. Untuk kajian berangka, setiap persamaan kamiran berkenaan telah didiskretkan menghasilkan suatu sistem persamaan tak linear. Bersama dengan beberapa syarat kenormalan, satu penyelesaian unik kepada sistem berkenaan dikira dengan kaedah pengoptimuman. Setelah nilai sempadan bagi fungsi pemetaan dikira, formula kamiran Cauchy digunakan untuk menentukan fungsi pemetaan terhadap rantau pedalaman. Contoh-contoh tipikal untuk beberapa rantau ujikaji telah menunjukkan keputusan berangka berketepatan tinggi boleh diperoleh untuk masalah pemetaan konformal dengan sempadan licin.

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$A(z, w)$	-	Kerzman-Stein kernel
$H(w, z)$	-	Cauchy kernel
$N(z, w)$	-	Neumann kernel
R	-	Riemann mapping function
$\mathbb{C}$	-	Set of complex numbers
$\Re$	-	Set of real numbers
$e, \exp$	-	Exponential($e \approx 2.718...$)
i	-	$\sqrt{-1}$
Im	-	Imaginary part
Re	-	Real
U	-	Unit disk
ϵ	-	Epsilon (small number $0 \leq \epsilon < 1$)
Γ	-	Curve (boundary of Ω)
Γ_0	-	Outer boundary of a doubly connected region
Γ_1	-	Inner boundary of a doubly connected region
π	-	Pi ($\pi \approx 3.142...$)
Ω	-	Connected region
$\in$	-	Component
$\sum$	-	Sum
$\int$	-	Integration

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CHAPTER 1

INTRODUCTION

1.1 Introduction

A conformal mapping, also called a conformal map, a conformal transformation, angle-preserving transformation, or biholomorphic map is a transformation $w = f(z)$ that preserves local angle. An analytic function is conformal at any point where it has nonzero derivatives. Conversely, any conformal mapping of a complex variable which has continuous partial derivatives is analytic.

Conformal mappings have been an important tool of science and engineering since the development of complex analysis. A conformal mapping uses functions of complex variables to transform a complicated boundary to a simpler, more manageable configuration. In various applied problems, by means of conformal maps, problems for certain *physical regions* are transplanted into problems on some standardized *model regions* where they can be solved easily. By transplanting back we obtain the solutions of the original problems in the physical regions. This process is used, for example, for solving problems about fluid flow, electrostatics, heat conduction, mechanics, aerodynamics and image processing. For these and other physical problems that use conformal mapping techniques, see, for example, the books by Henrici (1974), Churchill and Brown (1984), Schinzinger and Laura (1991) and Kythe (1998). For theoretical aspects of conformal mappings, see, e.g., Andersen *et al.* (1962), Hille (1962), Ahlfors (1979), Goluzin (1969), Nehari (1975), Henrici (1974), and Wen (1992).

A special class of conformal mappings that map any simply connected region onto a unit disk is called Riemann map. The Riemann mapping function is closely connected to the Szegö or the Bergman kernels. These kernels can be computed as a solution of second kind integral equations. Hence to solve the conformal mapping problem it is sufficient to compute the boundary values of either the Szegö or the Bergman kernel.

An integral equation of the second kind that expressed the Szegö kernel as the solution is first introduced by Kerzman and Trummer (1986) using operator-theoretic approach. Henrici (1986) gave a markedly different derivation of the Kerzman-Stein-Trummer integral equation based on a function-theoretic approach. The discovery of the Kerzman-Stein-Trummer integral equation, briefly KST integral equation, for computing the Szegö kernel later leads to the formulation of an integral equation for the Bergman kernel as given in Murid (1997) and Razali *et al.* (1997). Both integral equations can be used effectively for numerical conformal mapping of simply connected regions.

1.2 Background of The Problem

The practical limitation of conformal mapping has always been that only for certain special regions are exact conformal maps known and others have to be computed numerically.

Henrici (1986), Kythe (1998), Murid (1997), Schinzinger and Laura (1991), Trefethen (1986), Wegmann (2005) and Wen (1992) have surveyed some methods for numerical approximation of conformal mapping function such as expansion methods, iterative methods, osculation methods, integral equation method, Cauchy-Riemann equation methods and charge simulation methods. The integral equation methods mostly deal with computing the boundary correspondence function for solving numerical conformal mapping. This correspondence refer to a particular parametric representation of the boundary (Razali *et al.*, 1997; Henrici, 1986; Kerzman and Trummer, 1986).

Conformal mapping of multiply connected regions suffer form severe limitations compared to the simply connected region. There is no exact multiply equivalent of the Riemann mapping theorem that holds in multiply connected case. This implies that there is no guarantee that any two multiply connected regions of the same connectivity are conformally equivalent to each other.

Nehari (1975, p. 335), Bergman (1970) and Cohn (1967) described the five types of slit region as important canonical regions for conformal mapping of multiply connected regions, namely

(i) the disk with concentric circular slits (Figure 1.1a),

(ii) an annulus with concentric circular slits (Figure 1.1b),

(iii) the circular slit region (Figure 1.1c),

(iv) the radial slit region (Figure 1.1d), and

(v) the parallel slit region (Figure 1.1e).

The former two are bounded slit regions and the latter three are unbounded slit regions. It is known that any multiply connected region can be mapped conformally onto these canonical regions. In general the radii of the circular slits are unknown and have to be determined in the course of the numerical evaluation. However, exact mapping functions are not known except for some special regions.

By using a boundary relationship satisfied by a function analytic in a doubly connected region, Murid and Razali (1999) extended the construction to a

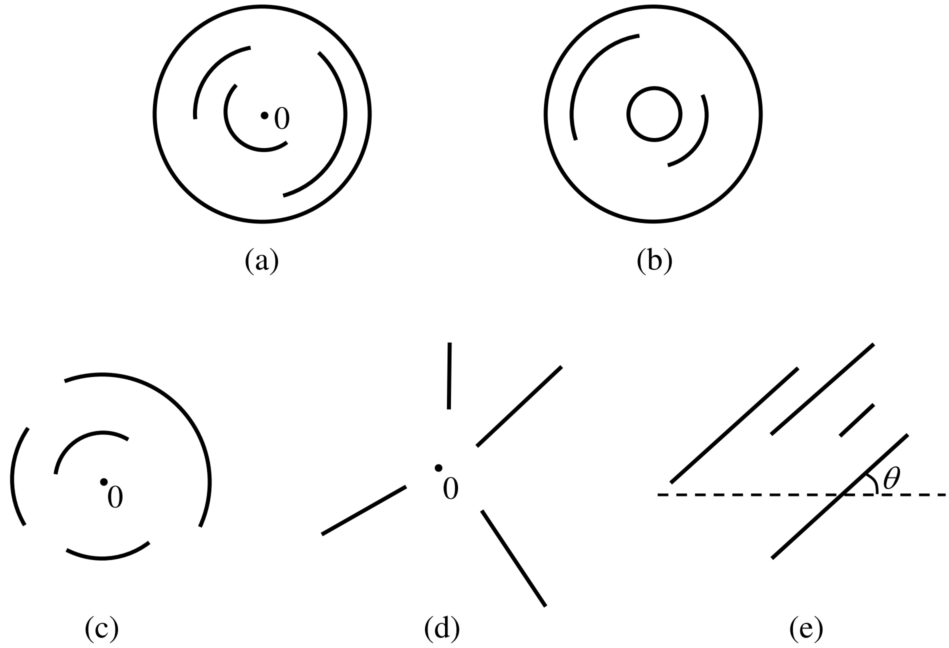

Figure 1.1: Canonical regions.

doubly connected region and obtained a boundary integral equation for conformal mapping of doubly connected regions. Special realizations of this boundary integral equation are the integral equations for conformal mapping of doubly connected regions via the Kerzman-Stein and the Neumann kernels. However, the integral equations are not in the form of Fredholm integral equations and no numerical experiments are reported in Murid and Razali (1999).

Murid and Mohamed (2007), and Mohamed and Murid (2007b) have formulated an integral equation method based on the Kerzman-Stein and the Neumann kernel for conformal mapping of doubly connected regions onto an annulus. The theoretical development is based on the boundary integral equations for conformal mapping of doubly connected regions derived by Murid and Razali (1999). For numerical experiments, the integral equations are discretized which lead to systems of non-linear equations. The systems obtained are solved simultaneously using Newton's iterative method. The numerical implementations on some test regions are reported only for doubly connected regions onto an annulus.

But Murid and Razali (1999), Murid and Mohamed (2007), and Mohamed and Murid (2007b) have not yet formulated an integral equation method based on the Kerzman-Stein and the Neumann kernels for conformal mapping of multiply connected regions onto the disk with concentric circular slits and an annulus with concentric circular slits.

1.3 Problem Statement

The research problem is to formulate new integral equations for conformal mapping of multiply connected regions with smooth boundaries onto the disk with concentric circular slits and an annulus with concentric circular slits via the Neumann kernel and the Kerzman-Stein that are suitable for numerical purposes.

1.4 Scope of Study

This research focuses on the integral equation method for the numerical computation of the conformal mapping of multiply connected regions. The theoretical development of the integral equation is based on the approach given by Murid and Razali (1999) for doubly connected regions.

In this study, some new boundary integral equations will be derived for conformal mapping of multiply connected regions via the Kerzman-Stein and the Neumann kernels. These integral equations will be applied to multiply connected regions onto an annulus with concentric circular slits and the disk with concentric circular slits. For numerical experiments, these integral equations will be discretized that might leads to a system of equations. Some normalizing conditions might be needed to help achive unique solutions.

The research will also describe a numerical procedure based on Cauchy integral formula for computing the mapping of interior points. The research will present numerical examples to highlight the advantages of using the proposed method.

1.5 Research Objectives

The objectives of this research are:

1. To improve and extend the construction of integral equation related to a boundary relationship satisfied by a function analytic in a doubly connected region by Murid and Razali (1999) to multiply connected regions.

2. To derive new boundary integral equation for conformal mapping of multiply connected regions onto a disk with concentric circular slits via the Neumann kernel.

3. To derive new boundary integral equations for conformal mapping of multiply connected regions onto an annulus with circular slits via the Neumann kernel and the Kerzman-Stein kernel.

4. To use the integral equations to solve numerically the boundary values of the conformal mapping of multiply connected regions onto an annulus with concentric circular slits and the disk with concentric circular slits.

5. To use the Cauchy's integral formula to determine the interior values of mapping functions.

6. To make numerical comparison of the proposed method with exact solution or with some existing methods.

1.6 Thesis Outline

This thesis consists of six chapters. The introductory Chapter 1 details some discussion on the introduction, background of the problem, problem statement, objectives of research, scope of the study and chapter organization.

Chapter 2 gives an overview of methods for conformal mapping in particular of multiply connected regions as well as the conformal mapping of multiply connected regions. We discuss some theories of the Riemann mapping function. We also present some exact conformal mapping of doubly connected regions for certain special regions like annulus, frame of limacon, elliptic frame, frame of Cassini's oval and circular frame. Some numerical methods that have been proposed in the literature for conformal mapping of multiply connected regions are also presented in the Section 2.6 of Chapter 2. The boundary integral equation for conformal mapping of doubly regions derived by Murid and Razali (1999) is also presented.

In Chapter 3, we construct new boundary integral equation related to a boundary relationship satisfied by an analytic function on multiply connected regions. The theoretical development is based on the boundary integral equation for conformal mapping of doubly connected region derived by Murid and Razali (1999) who have constructed an integral equation for the mapping of doubly connected regions onto an annulus involving the Neumann kernel. By using the boundary relationship satisfied by the mapping function, a related system of integral equation is constructed, including the unknown parameter radii. We apply the new boundary integral equation for conformal mapping of multiply connected regions onto a disk with circular slits and onto an annulus with circular slits via the Neumann and the Kerzman-Stein kernels. Special cases of this result are the integral equations involving the Kerzman-Stein and the Neumann kernels related to conformal mapping of doubly connected regions onto an annulus obtained by Murid and Mohamed (2007), and Mohamed and Murid (2007b).

In Chapter 4, we apply the result of Chapter 3 to derive a new boundary integral equation related to conformal mapping $f(z)$ of multiply connected region onto an annulus with circular slits. We discretized the integral equation and imposed some normalizing conditions different from Murid and Mohamed (2007), and Mohamed and Murid (2007b) for the case doubly connected region via the Kerzman-Stein and the Neumann kernels. We also extend the construction of the boundary integral equation in Chapter 3 to a triply connected regions. The boundary values of $f(z)$ is completely determined from the boundary values of $f'(z)$ through a boundary relationship. Discretization of the integral equation leads to a system of non-linear equations. Together with some normalizing conditions, we show how a unique solution to the system can be computed by means of an optimization method. We report our numerical results and give comparisons with existing method for some test regions.

In Chapter 5, we apply the result of Chapter 3 to derive a new boundary integral equation related to conformal mapping $f(z)$ of multiply connected region onto a disk with circular slits. Discretization of the integral equation leads to a system of non-linear equations. Together with some normalizing conditions, we show how a unique solution to the system can be computed by means of an optimization method. Once the boundary values of the mapping function f are known, we use the Cauchy's integral formula to determine the interior values of the mapping function. Numerical experiments on some test regions are also reported.

Finally the concluding chapter, Chapter 6, contains a summary of all the main results and several recommendations. There are two appendices in this thesis. Appendix A presents the list of the papers that have been published, presented or submitted for publication or presentation during the author's candidature. Appendix B presents a sample of the computer program used for a test region.