AN INITIAL STUDY OF COUPLED FINITE ELEMENT LATTICE BOLTZMANN SIMULATION SCHEME

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ABSTRACT

A finite element lattice Boltzmann scheme was developed for the incompressible viscous fluid flow. The work was based on the two-dimensional nine-microvelocity lattice model (D2Q9). The simulations of simple fluid flow problems were carried out to validate the proposed approach. The results are compared well with the analytical solutions.

1. INTRODUCTION

Lattice Boltzmann (LB) simulation scheme was first introduced in 1980's by McNamara and Alder (1993). Since its introduction, this numerical scheme has been widely used in wide range of engineering application especially in fluid mechanics and heat transfer related fields. Many researchers have demonstrated that the LB scheme can also be applied in magnetohydrodynamics (Breyiannis and Valougeorgis, 2004), multiphase fluid flow (Halliday and Care, 1996), turbulence (Jonas *et al.*, 2006), flow in porous media (Chen and Doolen, 1998), microscale fluid flow (Lim et al., 2002) and recently compressible fluid flow (Mason, 2002).

Historically, LB scheme evolved from its precursor, the lattice gas automata method (Frisch et al., 1986). In LGA, computational space in discretized into square grids and Boolean digits were used to describe the existence of fluid particles. The fluid particles are allowed to travel on the links that connected the grids and collide simultaneously if two particles arrive at the same grid. Even with these simplifications to mimic the behavior of fluid flow at particle level, the LGA was successfully simulated fluid flow up to certain accuracy. However, the drawbacks of LGA were uncovered when the researchers tried to relate this scheme with macroscopic Navier-Stokes equation. They discovered that the LGA was suffered from lack of Gallilean invariants and high statistical noise. With these drawbacks, macroscopic equations cannot be correctly recovered and huge computational cost is required to simulate a relatively simple fluid flow phenomena.

Lattice Boltzmann scheme evolved as the results of the improvement of LGA method. A few modifications have been made such as the introduction of BGK collision (Bhatnagar et al., 1954) model and distribution function to replace Boolean representation. The square grids lattice structure is still the main characteristic in LB method. As in LGA, the time evolution in LB scheme consists of two main steps; propagation, where the distribution function shift to the new grid point and collision, the distribution function relax to equilibrium state according to Boltzmann H-theorem (Cercignani, 1988). The lattice structure determines the number of microvelocity in LB scheme. Researchers frequently use "DmQn" to describe the lattice model in their works. Here "m" represents the space dimension and "n" represents the number of microscopic velocity (Qian et al., 1992). Among the numbers of lattice models exist in the literature, D2Q9 is the most famous lattice model due its simplicity and effectiveness for complex geometry (Azwadi et al., 2006).

In this research, our works focused on further improvement of LB scheme in order to increase its accuracy and efficiency for the simulation of blood flow in capillary. In our approach, we combined conventional numerical method, finite element method with LB scheme. With this combination, Langragian method in the original LB scheme is transformed into Eulerian method. Even though the unconditional stable of the Langrigian original LB scheme is sacrificed, however, the introduction of finite element scheme is believed to overcome the restriction of uniform mesh which very crucial in obtaining higher accuracy in predicting blood flow behavior in capillary.

The rest of the paper is organized as follow. In the next section, the theory of two-dimensional of lattice Boltzmann scheme is discussed. Some simulation results of fluid flow with analytical solutions that were published elsewhere are repeated to demonstrate the capability of this scheme. The combination of finite element with lattice Boltzmann is shown section 3. Computational results of backward facing step flow using the proposed mathematical modeling are also shown in this section. Final section contains the concluding remarks.

2. LATTICE BOLTZMANN SCHEME

The governing equation of two-dimensional LB scheme is given by (Gladrow, 2000)

$$\partial f/\partial t + c(\partial f/\partial \mathbf{x}) = -(f - f^{eq})/\tau \quad (1)$$

where f is the distribution function with velocity c at position $\mathbf{x}$ and time t. τ is the time relaxation. The value of τ controls the amount of distribution function that relaxes to equilibrium state during collision process and f^{eq} is the Maxwell-Boltzmann equilibrium distribution function. Noted that the BGK collision function is used to describe the collision process in LB scheme.

In present research, D2Q9 lattice model is used. This equivalent to nine links connected to every grid points as shown in Fig. 1.

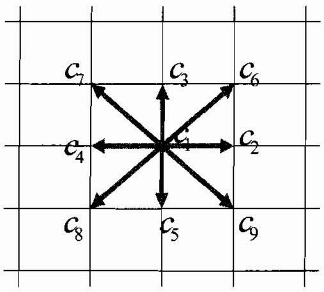

Fig. 1 D2Q9 lattice model.

Once the number microvelocity is decided (discretized in velocity space) for the distribution function, the governing equation can be rewritten as follow

$$\partial f_i/\partial t + c_i(\partial f_i/\partial \mathbf{x}) = -(f_i - f_i^{eq})/\tau \quad (2)$$

where i = 1 – 9. The discretized equilibrium distribution function is expressed as

$$f_i^{eq} = \rho w_i[1 + 3(c_i.u) + 4.5(c_i.u)^2 - 1.5u^2] \quad (3)$$

with w_i = 4/9 for i = 1, w_i = 1/9 for i = 2,3,4,5 and w_i = 1/36 for i = 6,7,8,9.

The local macroscopic variables such as density and velocity can be obtained through moment integration of distribution function as follow

$$\rho = \Sigma f_i \quad \rho u = \Sigma c_i f_i \quad (4)$$

The evolution of LB scheme consists of two steps. The initial value of ρ and u are specified at each grid points (x, y). Then the system evolves in the following steps

i) The value of f_i^{eq} on each grid point is calculated using Eq. 3 and the collision process can be computed according to left hand side of Eq. (1). The post collision value of f_i is henceforth obtained.

ii) After the collision, propagation takes place and the new value of f_i is obtained. The value of macroscopic at new time step can be calculated using Eq. (4). Then the collision and propagation processes are repeated.

2.1 Code Validation

The lattice Boltzmann scheme introduced in the proceeding section is used to simulate fluid flow in two-dimensional channel. The bounce-back boundary condition (Frisch et al., 1986) is applied on the top and bottom walls while periodic boundary condition at inlet and channel exit. The well-known analytical solution for this case, parabolic velocity profile, is compared with the results obtained from LB simulation and shown in Fig. 2.

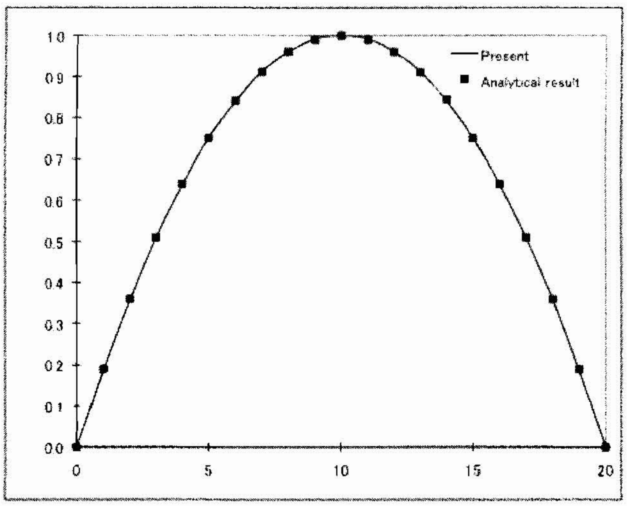

Fig. 2 Comparison of results between LB scheme and analytical solution.

From the comparison of results shown in Fig. 2, we can see that the LB scheme gives a very good agreement with the analytical solution for this case of fluid flow. This gives us confidence to apply LB scheme for a more complex fluid flow behavior such as blood flow in capillary.

3. THE FINITE ELEMENT LATTICE BOLTZMANN SCHEME

The choice of finite element was due to its flexibility on dealing with complex geometry and computational grids.

We start with the temporal discretisation of Eq. (1) as follow

$$\partial f/\partial t = (f^{n+1} - f^n)/\Delta t \tag{5}$$

where $f^n = f(x, t^n)$ denotes the distribution function's value at time $t = t^n$, Δt is the time increment, and $t^{n+1} = t^n + \Delta t$. In general we assume that f^n is already known and is used as an initial condition to advance the solution to time level t^{n+1}. Next, we introduce a relaxation parameter θ and write the solution f in the form

$$f = \theta f^{n+1} + (1 - \theta) f^n \tag{6}$$

The parameter θ is usually specified within the range $0 \leq \theta \leq 1$ and is used to control the accuracy and stability of the algorithm. In our case, $\theta = 0$ is selected to avoid complexity of the algorithm.

We now apply the Galerkin finite element method to Eq. (1). Suppose that the computational domain is discretized into a collection of finite elements, which are bilinear quadrilateral elements in the present study. The Galerkin approximation to the solution f in an element is given by

$$f_\Omega = N f_\Omega \tag{7}$$

where N is the shape function and Ω is the spatial domain.

By applying the Galerkin method, and substituting Eqs. (6) and (7) into Eq. (1) one obtains

$$Mf^{n+1} = Mf^n - Bf^n - (\Delta t/\tau)\, M(f^n + f^{n,eq}) + \text{boundary} \tag{8}$$

where matrices M and B are defined as

$$M = \int_\Omega N N^T d\Omega \quad \text{and} \quad B = \int_\Omega N(\partial N^T/\partial x)\partial\Omega \tag{9}$$

where N^T is the transpose matrix.

The macroscopic equation can be obtained via Chapmann-Enskog expansion of Eq. (1). Details of the derivation can be seen from Azwadi et al. (2007) and will not be shown here

$$\nabla \bullet u = 0 \tag{10}$$

$$\partial u/\partial t + u \nabla \bullet u = (\nabla p)/\rho + \tau \nabla^2 u/3 \tag{11}$$

By comparing Eq. (11) with the Navier-Stokes equation derived from Newton's second law, the time relaxation τ in microscopic equation can be related to viscosity υ in macroscopic equation as follow

$$\tau = 3\upsilon \tag{12}$$

3. SIMULATION RESULTS

Numerical simulation for expansion channel flow was carried out to test the validity of the proposed approach. Fig. 3 shows a sketch of the geometry of the flow problem used in the present study.

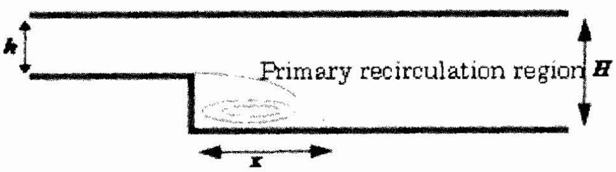

Fig. 3 Sketch of expansion channel flow.

The expansion ratio is defined by H/h where H is the channel height downstream of the step and h is the channel height of the inflow channel. The Reynolds number in this study is defined as

$$Re = UD/\upsilon \tag{13}$$

as in Armaly et al. (1983), where U is two-thirds of the maximum inlet velocity, which corresponds in the laminar case to the average inlet velocity, D is the hydraulic diameter of the inlet channel and is equivalent to twice its height D = 2h. At the outlet of the computational domain, the flow should be fully developed again. Hence the application of simple outflow conditions assuming zero gradients of all flow variables is typically sufficient.

Fig. 4 Streamline plot for expansion flow at Re = 100.

Fig. 4 shows streamline of steady state flow field for an expansion ratio 2.0 at Re = 100. As can be seen from the figure, the primary vortex was successfully simulated and this demonstrates the capability of the proposed numerical scheme.

CONCLUSION

This paper has demonstrated that the lattice Boltzmann simulation scheme can be an alternative approach in solving fluid flow problems. One of the major drawbacks in original lattice Boltzmann scheme, the restriction to uniform grid can be avoided by coupling lattice Boltzmann scheme with conventional numerical method. Our initial study on the finite element lattice Boltzmann simulation scheme have shown that this approach is a reliable and accurate approach in modeling fluid flow problem. The application to more complex fluid flow, blood flow in capillary will be our next future research.

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