Boundary Integral Equation for the Neumann Problem in Bounded Multiply Connected Region

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A dissertation submitted in partial fulfilment of the
requirements for the award of the degree of
Master of Science (Mathematics)

Faculty of Science

Universiti Teknologi Malaysia

OCTOBER 2009

ACKNOWLEDGEMENTS

Praise be to Allah, Lord of the Worlds, and prayers and peace be upon the Messenger of Allah. O Allah, to You belongs all praise for your guidance and your care.

First and foremost, I would like to express my profound thanks to my supervisor, Assoc. Prof. Dr. Ali Hassan bin Mohamed Murid, who has suggested the dissertation topic and directed the research. I thank him for his incisive guidance, supervision and encouragement to improve my work.

I would like also to thank Assist. Prof. Dr. Mohamed M. S. Nasser from Ibb University, Yemen for his valuable comments and generous support during his one month visit to IIS, UTM.

Finally, I would like to thank all those who have contributed to this work.

ABSTRACT

This research determines solutions of the Neumann problem in multiply connected regions by using the method of boundary integral equations. This method is widely used for solving boundary value problems. The method depends on reducing the boundary value problem in question to an integral equation on the boundary of the domain of the problem, and then solves this integral equation. Our approach in this research is to convert the Neumann problem into the Riemann-Hilbert problem and then derive an integral equation related to the Riemann-Hilbert problem. The derived integral equation is not uniquely solvable. The complete discussion on the solvability of the Neumann problem, the Riemann-Hilbert problem as well as the derived integral equation is presented. As an examination of the present method, some numerical examples for some different test regions are presented. These examples include comparison between the numerical results and the exact solutions.

ABSTRAK

Penyelidikan ini menyelesaikan masalah Neumann atas satah terkait berganda untuk menggunakan kaedah persamaan kamiran. Kaedah ini telah digunakan secara meluas dalam menyelesaikan masalah nilai sempadan. Kaedah ini bergantung kepada penurunan masalah nilai sempadan ke persamaan kamiran atas sempadan rantau dan seterusnya menyelesaikan persamaan kamiran tersebut. Pendekatan yang digunakan dalam penyelidikan ini adalah menukarkan masalah Neumann kepada masalah Riemann-Hilbert dan seterusnya membina persamaan kamiran yang berkaitan dengan masalah Riemann-Hilbert. Persamaan kamiran yang diperolehi tidak mempunyai penyelesaian unik. Perbincangan yang mendalam telah disampaikan berkaitan kebolehselesaikan masalah Neumann, masalah Riemann-Hilbert, dan persamaan kamiran yang diperolehi. Untuk mengaji kaedah yang dipersembahkan, beberapa contoh berangka melibatkan beberapa rantau terpilih disampaikan. Perbandingan berangka juga diberi antara keputusan berangka dengan penyelesaian tepat.

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CHAPTER 1

RESEARCH FRAMEWORK

1.1 Introduction

Partial differential equations play a vital role in natural sciences and technology. There are many phenomena in these fields that can be described as boundary value problems for partial differential equations. However, formulating and solving such problems is not easy especially when we talk about real modelling of those phenomena. Furthermore, it is also important to study existence and uniqueness of the solution of these problems. These issues were and still occupy the minds of mathematicians and engineers.

The last few decades have witnessed a great progress in computational mathematics and engineering. This opens the door to more researches in these fields. Numerical treatment is usually an important and necessary part to deal with boundary value problems. The nature of some problems imposes on the engineers and mathematicians some assumptions and limits in order to get solutions for those problems.

Neumann problem is classified as a boundary value problem associated with Laplace’s equation and Neumann boundary condition. Different types of Neumann problems occur naturally in some fields like electrostatics, fluid flow, heat flow and elasticity.

Boundary integral equation method is one of the common methods for solving Neumann problem. This method depends on reducing the boundary value problem in question to an integral equation on the boundary of the domain of the problem, and then solves this integral equation. This integral equation could be uniquely solvable or non- uniquely solvable, non-singular or singular. This depends on the original problem and the way of reduction. This method reduces our task to solve an integral equation only on the boundary of the region, thus reducing the dimension of the Neumann problem by one. Numerical treatment is usually needed to solve the resulting integral equation.

Our approach in this research is to reduce the Neumann problem to the Riemann-Hilbert problem in multiply connected region, and then derive an integral equation with the Neumann kernel related to the Riemann-Hilbert problem. This integral equation is the Fredholm integral equation of the second type.

1.2 Background of the Problem

The partial differential equation which is identified with the name of Pierre Simon Marquis de Laplace (1749-1827) is one of most important equations in mathematics which has wide applications to a number of topics relevant to mathematical physics and engineering [2]. The two-dimensional Laplace equation has the following form:

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1.1}$$

There are two important types of boundary value problems for Laplace's equation: the Dirichlet problem and the Neumann problem.

A function that solves Laplace's equation is called a harmonic function, or sometimes called potential function in physics and engineering. The problem that we are interested about is classified as a boundary value problem and it asks for a harmonic function on a defined region and satisfies a boundary condition related to the normal derivative of this function on the boundary of that region. Such a problem is called a Neumann problem (after the German mathematician Carl Gottfried Neumann (1832-1925)) and sometimes referred to as a Dirichlet problem of the second kind [12]. Mathematically, Neumann problem is to find a function u defined in a region Ω with boundary Γ which satisfies

$$\nabla^2 u = 0 \tag{1.2}$$

with boundary condition

$$\frac{\partial u}{\partial \mathbf{n}} = \gamma(t) \quad on \ \Gamma \tag{1.3}$$

and solvability condition

$$\int_\Gamma \gamma(t) \ ds = 0. \tag{1.4}$$

Here $\frac{\partial u}{\partial \mathbf{n}}$ denotes the directional derivative of u along the outward normal to the boundary Γ and condition (1.3) is called the Neumann condition. The last condition (1.4) is known as compatibility condition, which is necessary for the existence of a solution, where ds is the element of arc length on Γ. Even so, the solution of the Neumann problem is not unique; this is due to the presence of arbitrary constant in the solution.

Finding an exact analytical solution for the Neumann problem that described real physics situations is usually impossible. Many approximation and numerical methods have been developed to solve such type of problems. Boundary integral method is one of several methods that has been used to solve Neumann problem. Due to non-uniqueness of the Neumann problem, the boundary integral method leads to non-uniquely solvable integral equation. However, it is possible to overcome non-uniqueness by imposing additional condition(s) which is (are) consistent with the nature of the problem.

Through the previous research by Nasser [10], the interior and exterior Neumann problems are reduced to equivalent Dirichlet problems by using Cauchy-Riemann equation that are uniquely solvable. Then, boundary integral equations are derived for the Dirichlet problems.

1.3 Statement of the Problem

Recently Husin [21] and Murid et al. [1] have reduced the Neumann problem on a simply connected region to the Riemann-Hilbert problem. The Riemann-Hilbert problem is then formulated as a boundary integral equation which is uniquely solvable.

The main question here is how can we reduce the Neumann problem in multiply connected region to the Riemann-Hilbert problem and derive a uniquely solvable integral equation related to the Riemann-Hilbert problem?

1.4 Objectives of the Study

We can summarize the objectives of this research in the following:

- Define and study the solvability of the Neumann problem and Riemann-Hilbert problem in multiply connected region with smooth boundary.
- Reduce the Neumann problem in multiply connected region to the corresponding Riemann-Hilbert problem.

- Derive a boundary integral equation related to Riemann-Hilbert problem.
- Use appropriate numerical methods to solve this boundary integral equation and make comparisons with exact solution.

1.5 Scope of the Study

This research endeavours to construct an integral equation method for solving the Neumann problem in multiply connected region. This includes converting the Neumann problem into another boundary value problem which is the Riemann-Hilbert problem, and then solving it using the boundary integral equation method. Our approach will be based on a complex analysis framework. We will focus on bounded multiply connected region with smooth boundaries as a domain for the Neumann problem.

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