AN INTEGRAL EQUATION METHOD FOR SOLVING EXTERIOR NEUMANN PROBLEMS ON SMOOTH REGIONS

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To Mak, Abah, Jeli, Arul, Azhar, Ainul and abang.

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ABSTRACT

This work develops a boundary integral equation method for numerical solution of the exterior Neumann problem. An integral equation for solving the exterior Neumann problem in a simply connected region is derived in this dissertation based on the exterior Riemann-Hilbert problem. In the first step the exterior Neumann problem is reduced to an exterior Riemann-Hilbert problem for the derivative of an auxiliary function which is analytic in the region. Then, the exterior Riemann-Hilbert problem is transformed to a uniquely solvable Fredholm integral equation on the boundary of the region. Once this equation is solved, the auxiliary function and the solution of the exterior Neumann problem can be obtained. The efficiency of the method is illustrated by some numerical examples.

ABSTRAK

Kajian ini bertujuan untuk membina suatu kaedah persamaan kamiran sempadan untuk penyelesaian berangka bagi masalah Neumann luaran. Satu persamaan kamiran untuk menyelesaikan masalah Neumann luaran dalam rantau terkait mudah dibentuk dalam disertasi ini berdasarkan masalah Riemann-Hilbert luaran. Dalam langkah pertama, masalah Neumann luaran akan diturunkan kepada masalah Riemann-Hilbert luaran dalam rantau tersebut. Kemudian, masalah Riemann-Hilbert luaran akan dijelmakan kepada satu persamaan kamiran Fredholm yang mempunyai penyelesaian unik dalam rantau tersebut. Apabila persamaan ini diselesaikan jawapan untuk masalah Neumann luaran boleh dicari. Keberkesanan kaedah ini diilustrasikan dengan beberapa contoh berangka.

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CHAPTER 1

RESEARCH FRAMEWORK

1.1 Introduction

The problem of finding a function which is harmonic in a specified domain and which satisfies prescribed conditions on the boundary of that domain is abound in applied mathematics. If the values of the function are prescribed along the boundary, the problem is known as a boundary value problem of the first kind, or a Dirichlet problem. If the values of the normal derivative of the function are prescribed on the boundary, the boundary value problem is one of the second kind, or a Neumann problem (in honor of German mathematician Carl Gottfried Neumann). Modifications and combinations of those types of boundary condition also arise.

The Neumann problem is a class of fundamental boundary value problems for analytic functions, a living subject with a fascinating history and interesting applications. It is a boundary value problem for determining a harmonic function, $u(x, y)$ interior or exterior to a region with prescribed values of its normal derivative, $\frac{\partial u}{\partial n}$ on the boundary. Some examples are heat problems in an insulated plate, electrostatic potential in a cylinder and potential of flow around airfoil.

In its simplest form, the Neumann problem consists in finding a function $u(x, y)$ satisfying the following conditions:

i. u is continuous and differentiable in Ω and Γ.

ii. u is harmonic in Ω.

iii. $\Delta u(z) = 0$ for all z in Ω.

iv. If $\frac{\partial}{\partial \mathbf{n}}$ denotes differentiation in the direction of the exterior normal, then

$$\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\eta(t)} = \gamma(t), \qquad \eta(t) \in \Gamma. \tag{1.1}$$

which is known as a Neumann condition (see [1]).

The Neumann problem is often solved by conformal mapping for arbitrary simply connected region [2]. The basic technique is to transform a given boundary value problem in the xy plane into a simpler one in the uv plane where they can be solved easily. By transforming back to the original region, the desired answer is obtained.

An important fact about conformal mapping which accounts for much of its applications is that the Laplace's equation is invariant under conformal mapping. Although conformal mappings have been an important tool of science and engineering since the development of complex analysis, the practical use of the conformal maps has always been limited by the fact that exact conformal mappings are only known for special regions.

Other than conformal mapping, there are many techniques from various mathematics fields for solving the Neumann problem. Since 1900, lots of mathematicians started to investigate the types of Neumann problems and using different methods and approaches in order to formulate its solution such as finite difference method, finite element method, iterative method, collocation method and boundary integral equation method.

In this report we will only cover the boundary integral equation methods which can be used to solve the boundary value problem in its original region. The reformulation of the boundary value problem as an equivalent integral equation over the boundary reduces the dimensionality of the problem which makes the method an efficient tool for complicated engineering problems.

1.2 Background of the Problem

The boundary integral equation method is a classical method for solving the Neumann problem. The classical boundary integral equations for the Neumann problems are derived by representing the solutions of Neumann problems as the potential of a single layer; however, the integral equation for the interior Neumann problem is not uniquely solvable. Furthermore, extra calculations are required for determining the boundary values of the solutions of the Neumann problems from the solutions of the integral equation.

In this project, boundary integral equations will be derived for the exterior Neumann problem in simply connected regions Ω^-. The derived integral equations are Fredholm integral equations of the second kind with continuous kernels provided that the boundaries are sufficiently smooth.

1.3 Statement of the Problem

The research on boundary integral equations with the generalized Neumann kernel is still continuing. Through the previous research by Ummu Tasnim in [3], the interior Neumann problem is reduced to equivalent Riemann-Hilbert problem by using Cauchy-Riemann equations. Then, the boundary integral equation is derived for the Riemann-Hilbert problem based on an earlier work by Nasser [4]. In [4], the interior and exterior Neumann problems are reduced to equivalent Dirichlet problems by using Cauchy-Riemann equations. Then, the boundary integral equations are derived for the Dirichlet problem. This research continues the study on Neumann problems based on [3, 4, 5, and 6]. The aim of the study is to derive an integral equation for the exterior Neumann problem by reducing it to the exterior Riemann-Hilbert problem using Cauchy-Riemann equations. Furthermore, the analysis on the solvability for this integral equation will be determined as well.

1.4 Objectives of the Study

This study embarks on the following objectives:

i. To reduce the exterior Neumann problem to the exterior Riemann-Hilbert problem.
ii. To study the boundary integral equation for the exterior Riemann-Hilbert problem.
iii. To derive a Fredholm integral equation of the second kind for the Neumann problem based on the exterior Riemann-Hilbert problem in Ω^{-}.
iv. To determine the solvability of the formulated integral equation.
v. To provide a numerical technique for the boundary integral equation using software MATLAB.

1.5 Scope of the Study

This research will focus on the development of a numerical method for the exterior Neumann problem in a simply connected region Ω^-. Firstly, the exterior Neumann problem will be reduced to the exterior Riemann-Hilbert problem. Then, the boundary integral equation for the Neumann problem will be derived based on the exterior Riemann-Hilbert problem. Next, the solvability of the derived integral equation will be discussed. Then, a numerical method will be employed to solve the problem numerically through several examples.

1.6 Significance of the Study

The purpose of the study is to develop a new method for solving the exterior Neumann problem. The method is based on recent investigations on Ummu Tasnim's approach [3] for the interior Neumann problem and on the interplay of Riemann-Hilbert problems and Fredholm integral equations with generalized Neumann kernel [4, 6]. This approach will enrich the numerical procedure of solving exterior Neumann problem and enhance the numerical effectiveness of solving it.

1.7 Dissertation Organization

Chapter 1 contains the general introduction and background regarding of one type of Laplace's problem called the Neumann problem. In this chapter, we define our statement of the problem, objectives of the study, scope of the study and also significance of the study.

In Chapter 2 we give the historical background of the Neumann problem and the Riemann-Hilbert and a brief review of the boundary integral equation method for solving the Neumann problem. We also discuss some important facts that will be required later and also some auxiliary material related to Neumann problem and Riemann-Hilbert problem in Chapter 3.

In Chapter 4 we provide the theoretical formulation for the overall research. First, the reduction of the exterior Neumann problem into the exterior Riemann-Hilbert problem is explained. Then, the solvability of the derived Riemann-Hilbert problem is reviewed. Then the derived integral equation is used to solve numerically the exterior Neumann problem based on the derived Riemann-Hilbert problems.

In Chapter 5, we present the numerical implementations of the derived integral equation. A Nyström method will be used based on the trapezoidal rule. After that, some numerical examples on some test regions are reviewed. We then conclude this chapter with displaying some comparison result with the exact solution and some discussions on the result obtained.

Finally in Chapter 6, we conclude our project by summarizing every chapter and then stating some suggestions for the future research.