MATEMATIKA, 2007, Volume 23, Number 2, 75–81

Analysis of a Dengue Disease Transmission Model without Immunity

Yusof Yaacob
Department of Mathematics, Faculty of Science, Universiti Teknologi Malaysia
81310 UTM Skudai, Johor, Malaysia
e-mail: yby@mel.fs.utm.my

Abstract A dengue disease transmission model by Esteva & Vargas assumes that once a person recovers from the disease he or she will not be reinfected by the disease. However recovering from one of the four types of virus will not guarantee that a person is immuned to the other types. Hence it is reasonable to assume that the immune subpopulation is negligible. Consequently the model is reduced to a two-dimensional planar system. In this model, the endemic state is stable if the basic reproductive number of the disease is greater than one, and this result is similar to the result of the transmission model with immunity. For a relatively small series of outbreaks of the disease in population sufficiently large for the number of susceptible to remain effectively constant, the model is reduced to a population model for the group of infectives. Taking the incubation period into consideration, the model without immunity gives rise to a two-dimensional delay differential equations. The presence of the delay seems to destabilise the dynamics

Keywords Dengue disease, immunity, host-vector model, locally stable, globally stable, thresold parameter, delay.

1 Introduction

Dengue fever is transmitted to human by female aedes aegypti mosquitoes. Some known facts about aedes aegypti mosquitoes and the transmission process are as follows [4]:

(i) Mosquito biting occurs during day time (there is a tendency to be multiple biters).
(ii) Life time of mosquitoes is about 10 days.
(iii) Flying distance of mosquitoes is about 100 meters.
(iv) Mosquitoes lay eggs in a clean water environment (nesting sites are believed inside houses).
(v) Mosquitoes eggs hatch in 6-8 days, but could survive in much longer period and hatch after the first contact with water.
(vi) Survival rate from eggs to adults is very low.
(vii) Incubation period is about 14 days.
(viii) Virus lives in human body for about 7 days and then dies naturally.
(ix) Vertical transmission in mosquitoes is insignificant.

2 Host-Vector Model for Dengue Fever Transmission

Esteva & Vargas [2] developes a dengue fever transmission model by assuming that once a person recover from the disease, he or she will not be reinfected by the disease. The model also also assumes that the host population N_H is constant, that is the death rate and the birth rate equal to μ_H. The vector population is assumed to be governed by

$$\frac{d}{dt}N_V = A - \mu_V N_V$$

where μ_V is the mortality rate of the vector, and A is the recruitment rate. The mosquitoe model can be explained from the fact that only a small portion of the large supply of eggs survive to adult stage. Therefore A is independent from the adult population. The vector approaches to the equlibria A/μ_H as $t \to \infty$. The host population is subdivided into the susceptible S_H, the infective I_H and the recovered (and the immune) R_H. The vector population, due to a short life period, is subdivided into the susceptible S_V and the infective I_V. The host-vector model for the dengue transmission of Esteva & Vargas [2] is as follows:

$$\begin{aligned}
\frac{d}{dt}S_H &= \mu_H N_H - \frac{\beta_H b}{N_H} S_h I_V - \mu_H S_H \\
\frac{d}{dt}I_H &= \frac{\beta_H b}{N_H} S_H I_V - (\mu_H + \gamma_H) I_H \\
\frac{d}{dt}R_H &= \gamma_H I_H - \mu_H R_H \\
\frac{d}{dt}S_V &= A - \frac{\beta_V b}{N_H} S_V I_H - \mu_V S_V \\
\frac{d}{dt}I_V &= \frac{\beta_V b}{N_H} S_V I_H - \mu_V I_V
\end{aligned} \tag{1}$$

where

N_H = the host population
S_H = the number of susceptibles in the host population
I_H = the number of infectives in the host population
R_H = the number of immunes in the host population
N_V = the vector population
S_V = the number of susceptibles in the vector population
I_V = the number of infectives in the vector population
μ_H = the birth/death rate in the host population
μ_V = the death rate in the vector population
β_H = the transmission probabilty from vector to host
β_B = the transmission probabilty from host to vector
γ_H = the recovery rate in the host population
b = the biting rate of the vector

Equation (1) is reduced to three dimensional dynamic since

$$S_H + I_H + R_H = N_H \quad \text{and} \quad S_V + I_V = \frac{A}{\mu_V}.$$

The resulting dynamic, expressed in proportion

$$x = \frac{S_H}{N_H}, \quad y = \frac{I_H}{N_H} \quad \text{and} \quad z = \frac{I_V}{A/\mu_V}$$

is given by

$$\begin{aligned} \frac{dx}{dt} &= \mu_H(1-x) - \alpha x z \\ \frac{dy}{dt} &= \alpha x z - \beta y \\ \frac{dz}{dt} &= \gamma(1-z)y - \delta z \end{aligned} \tag{2}$$

where

$$\alpha = \frac{b\beta_H A}{\mu_V N_H}, \quad \beta = \gamma_H + \mu_H, \quad \gamma = b\beta_V \quad \text{and} \quad \delta = \mu_V.$$

The fixed points of the system are $F_1(1,0,0)$ and $F_2(x_0, y_0, z_0)$ where

$$x_0 = \frac{\mu_H + \beta\delta}{\gamma(\mu_H + \alpha)}, \quad y_0 = \frac{\mu_H(\alpha\gamma - \beta\delta)}{\beta\gamma(\mu_H + \alpha)} \quad \text{and} \quad z_0 = \frac{\mu_H(\alpha\gamma - \beta\delta)}{\alpha(\mu_H\gamma + \beta\delta)}.$$

The second fixed point exists only if the thresold parameter

$$R = \frac{\alpha\gamma}{\beta\delta} > 1.$$

The number $R_0 = \sqrt{R}$ is called the basic reproductive number. The following theorem is proved in [2].

Theorem 1 *If $R < 1$, the fixed point F_1 is globally stable. If $R > 1$, F_1 is unstable and the endemic fixed point F_2 is globally asymptotically stable.*

Global stability of F_1 for $R < 1$ is shown in [2] using Lyapunove function $\gamma y + \beta z$. For $R > 1$ the fixed point F_1 becomes locally unstable and F_2 becomes locally asymptotically stable. The global stability is shown using the property of stability of periodic orbits.

3 Dengue Fever Transmission Model without Immunity

Not enough information is known about the immunity after recovery. Four types of virus have been identified. Recovering from one of the four types of virus will not guarantee that a person will immune to the other types. Hence it is reasonable to assume that the immune

subpopulation is negligible. Assuming that the immune subpopulation is negligible, then the system of equations (1) is reduced to

$$\begin{aligned}
\frac{d}{dt}S_H &= \mu_H N_H - \frac{\beta_H b}{N_H} S_h I_V - \mu_H S_H \\
\frac{d}{dt}I_H &= \frac{\beta_H b}{N_H} S_H I_V - (\mu_H + \gamma_H) I_H \\
\frac{d}{dt}S_V &= A - \frac{\beta_V b}{N_H} S_V I_H - \mu_V S_V \\
\frac{d}{dt}I_V &= \frac{\beta_V b}{N_H} S_V I_H - \mu_V I_V
\end{aligned} \tag{3}$$

where

$$S_H + I_H = N_H \quad \text{and} \quad S_V + I_V = \frac{A}{\mu_A}.$$

Using the same notation for the propotion varibles y and z as before, the system becomes

$$\begin{aligned}
\frac{dy}{dt} &= \alpha(1-y)z - \beta y \\
\frac{dz}{dt} &= \gamma(1-z)y - \delta z
\end{aligned} \tag{4}$$

The linearisation of (4) at the fixed point $(0, 0)$ is given by

$$\begin{bmatrix} \dfrac{dy}{dt} \\ \dfrac{dz}{dt} \end{bmatrix} = \begin{bmatrix} -\beta & \alpha \\ \gamma & -\delta \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \tag{5}$$

and its charateristic equation is

$$\lambda^2 + (\beta + \delta)\lambda + (\beta\delta - \alpha\gamma) = 0.$$

The origin remains locally stable if $(\beta\delta - \alpha\gamma) > 0$. The same Lyapunove function $\gamma y + \beta z$ can be used to show that the origin is globally stable.

When $(\beta\delta - \alpha\gamma) < 0$ (this is equivalent to $R > 1$), the origin becomes unstable. In this case the second fixed point (y_0, z_0) comes into play, where

$$y_0 = \frac{\alpha\gamma - \beta\delta}{\gamma(\alpha + \beta)} \quad \text{and} \quad z_0 = \frac{\alpha\gamma - \beta\delta}{\alpha(\gamma + \delta)}.$$

The linearisation of (4) at the second fixed point (x_0, y_0) is given by

$$\begin{bmatrix} \dfrac{dy}{dt} \\ \dfrac{dz}{dt} \end{bmatrix} = \begin{bmatrix} \dfrac{-\gamma(\alpha+\beta)}{(\gamma+\delta)} & \dfrac{\alpha\beta(\gamma+\delta)}{\gamma(\alpha+\beta)} \\ \dfrac{\gamma\delta(\alpha+\beta)}{\alpha(\gamma+\delta)} & \dfrac{-\alpha(\gamma+\delta)}{(\alpha+\beta)} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \tag{6}$$

The characteristic equation for the second fixed point is

$$\lambda^2 + a_0\lambda + (\alpha\gamma - \beta\delta) = 0. \tag{7}$$

where

$$a_0 = \frac{\gamma(\alpha+\beta)^2 + \alpha(\delta+\gamma)^2}{(\alpha+\beta)(\gamma+\delta)}.$$

Both eigenvalues of (7) has negative real parts. Hence the second fixed point is locally stable. Global stability can be shown by using phase plane analysis. The following theorem is proved in [4].

Theorem 2 *Let*

$$\Omega = \left\{(x,y) : \frac{\beta y}{\alpha(1-y)} \leq z \leq \frac{\gamma y}{\gamma y + \delta}, 0 \leq y \leq y_0\right\}.$$

If $\alpha\gamma - \beta\delta > 0$, *then* Ω *is a trapping region and contains the heteroclinic orbit connecting* $(0,0)$ *and* (x_0, y_0).

4 Allowance for Incubation Period

There is a latent period that after infection, at the end of which the infected individual becomes infectious. If we take this fact into consideration, then (4) becomes

$$\begin{aligned} \frac{dy}{dt} &= \alpha z(t) - \alpha z(t) y(t-r) - \beta y(t-a) \\ \frac{dz}{dt} &= \gamma y(t-a) - \gamma z(t) y(t-a) - \delta z \end{aligned} \tag{8}$$

where r is the delay. The fixed points of (8) are the same as for (4).

The linearisation of (8) at the fixed point $(0,0)$ and (x_0, y_0) is, respectively, given by

$$\begin{bmatrix} \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \begin{bmatrix} -\beta & \alpha \\ \gamma & -\delta \end{bmatrix} \begin{bmatrix} y(t-r) \\ z \end{bmatrix} \tag{9}$$

and

$$\begin{bmatrix} \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \begin{bmatrix} \frac{-\gamma(\alpha+\beta)}{(\gamma+\delta)} & \frac{\alpha\beta(\gamma+\delta)}{\gamma(\alpha+\beta)} \\ \frac{\gamma\delta(\alpha+\beta)}{\alpha(\gamma+\delta)} & \frac{-\alpha(\gamma+\delta)}{(\alpha+\beta)} \end{bmatrix} \begin{bmatrix} y(t-r) \\ z \end{bmatrix} \tag{10}$$

The characteristic equations of the linearised equation for the first and the second fixed point is, respectively, given by

$$\lambda^2 + \delta\lambda + \beta\lambda e^{-\lambda r} + (\beta\delta - \alpha\gamma)e^{-\lambda r} = 0 \tag{11}$$

$$\lambda^2 + \frac{\alpha(\gamma+\delta)}{\alpha+\beta}\lambda + \frac{\gamma(\alpha+\beta)}{\gamma+\delta}\lambda e^{-\lambda r} + (\gamma\alpha - \beta\delta)e^{-\lambda r} = 0. \tag{12}$$

To find sufficient conditions for every roots of (11) and (12) to have negative real parts we use a result in Driver [1], and we state it as the content of the following Lemma.

Lemma 1 *Suppose that*

$$\lambda^2 + b\lambda + q\lambda e^{-\lambda r} + pe^{-\lambda r} = 0 \tag{13}$$

where b, q, p and r are nonnegative constants. If $b > q + pr$, then every solution of (13) has negative real part.

Applying Lemma 1 to (11) and (12), we have the following results.

Theorem 3

(i) *The fixed point $(0, 0)$ of (8) is locally stable if $(\beta\delta - \alpha\gamma) > 0$ and*

$$(\delta - \beta) > (\beta\delta - \alpha\gamma)r.$$

(ii) *The fixed point (x_0, y_0) of (8) is locally stable if $(\alpha\gamma - \beta\delta) > 0$ and*

$$\frac{\alpha(\gamma + \delta)^2 - \gamma(\alpha + \beta)^2}{(\alpha + \beta)(\gamma + \delta)} > (\alpha\gamma - \beta\delta)r.$$

5 Model with Relatively Small Series of Outbreak

Usually only a small series of outbreak of the dengue fever disease occur in a large community such as a housing estate. Hence consider a relatively small series of outbreaks in a population sufficiently large for the number of susceptibles and infective of the host population to remain effectively constant. By letting $y = n$, then (4) becomes a single equation

$$\frac{dz}{dt} = \gamma(1 - z)n - \delta z. \tag{14}$$

We thus have a population model for infective in the vector population with solution

$$z = \frac{q}{p} + z_0 e^{-pt}$$

where $p = (n\gamma + \delta)$ and $q = n\gamma$. As $t \to \infty$ we have

$$z = \frac{q}{p} = \frac{n\gamma}{(n\gamma + \delta)}.$$

6 Conclusion

In the host-vector model [2], the fixed point at the origin is globally stable if the threshold paramater is less than one; in this case the disease dies out. If the threshold parameter is greater than one, the endemic parameter is globally stable, which implies that the disease remains endemic. If we assume no immunity after recovery from the disease, the host-vector model is reduced to two-dimensional planar equation and the dynamic produce similar results to the result of the model with immunity.

For a relatively small series of outbreaks in a large population, the number of susceptibles and infective of the host population can be assumed to remain constant. The

two-dimensional model is reduced to a single population model for the infective vector population. In this case also the disease will not die out.

Taking the incubation period into consideration, the model without immunity gives rise to two-dimensional delay differential equations. The presence of the delay seems to destabilise the dynamics; however more study should be done for this case.

Since no vaccine is yet available for dengue fever, the efforts to control the disease should be focused on vector. Application of insectisides to kill the adult population of mosquitoes does not seem to be entirely viable, since the report in [3] indicates that after its application, the adult population of mosquitoes returned to the pretreatment levels within two weeks. Consequently, decreasing the carrying capacity of the environment for mosquitoes by frequent reduction of vector breeding sites, seems to be the more effective way of controlling the disease.

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