THE HIROTA'S DIRECT METHOD AND MULTI-SOLITON SOLUTIONS

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Dedicated to my parents, husband and my lovely children.

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ABSTRACT

The study of soliton theory is always a major source of mathematical and physical inspiration. For the past few decades, soliton theory has attracted considerable attention in diverse physical applications and the various mathematical methods of solution. This project discusses on the direct method of Hirota to obtain the multi-soliton solutions of physically significant nonlinear waves equations. This includes equations with single bilinear form and coupled system of bilinear form, together with the use of Hirota *D*-operator and various types of transformation. Singularity analysis is used to formulate the suitable transformation.

ABSTRAK

Kajian teori soliton senantiasa menjadi sumber utama bagi inspirasi fizikal dan matematik. Sejak beberapa dekad yang lalu, teori soliton telah mendapat banyak perhatian dalam pelbagai penggunaan fizikal dan berbagai kaedah penyelesaian bermatematik. Projek ini membincangkan kaedah langsung Hirota untuk memperoleh penyelesaian multi-soliton bagi persamaan gelombang tidak linear berkepentingan fizikal. Persamaan-persamaan ini termasuklah yang mempunyai bentuk bilinear tunggal dan sistem bentuk bilinear berpasangan, dengan penggunaan pengoperasi D Hirota dan berbagai jenis transformasi. Analisis kesingularan digunakan untuk memformulasi transformasi yang sesuai.

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CHAPTER I

INTRODUCTION

In this chapter, we present a broad view of soliton theory and its historical background. Then, we state briefly the background of the problem of finding exact solution for some nonlinear evolution equations. We also outline the objectives and the scope of the study. Finally, the organization of this dissertation is presented.

1.1 Soliton Theory and its Historical Background

The basic idea of a solitary wave is typically bell-shaped, plane wave pulse which translates in one space direction. That is, any bell-shaped function $u(x - \xi t)$ is a solitary wave translating with speed ξ along x (Bullough and Caudrey, 1980). A soliton is a self-reinforcing solitary wave (a wave packet or pulse) that preserves its shape and speed while it travels at constant speed even though after interaction with another solitary wave. Solitons behave both as particles and as waves and occur frequently in nature and now even can be produced in laboratories (Guo, 1995). Soliton theory is a very active research topic which is closely related to modern physics because this theory is applied to explain a lot of physical problems.

They are many important problems related to soliton theory especially in research field such as fluid mechanics, plasma physics, nonlinear optics, classical and quantum field etc. In particular, the solitons are essential to describe phenomena such as the propagation of some hydrodynamic waves, localized waves in astrophysical plasmas, the propagation of signals in optical fibres, and the dynamics of biological molecules such as DNA and proteins, (Dauxois and Peyrard, 2006). For these reasons, both mathematicians and physicists pay much attention to soliton theory. In twentieth century, the soliton theory has developed rapidly and becomes one of the most important subjects in applied mathematics and physics. Until now, the theory of soliton is continuously progressing and developing.

The history of soliton theory began in 1834 when John Scott Russell (1808-1882) accidently observed a 'great solitary wave' while he was riding his horse along a canal near Edinburgh. For detail of his discovery, see Dauxois and Peyrard (2006). He spent almost 10 years to study this solitary wave and try to recreate it on canals, rivers, lakes and water tank. Nevertheless, nobody expected the discovery was so great because the formula he had derived did not agree with the theory of shallow water waves. Therefore, his studies triggered a lot of controversy in the scientific community of his time, which assumed that nonlinear effects were not importance.

One had to wait until 1876, Joseph Valentine de Boussinesq (1842-1929) proposed another new theory of shallow water waves, which agreed with Russell's observation. In 1895, the existence of the solitary wave was first confirmed by the equation derived by Diederik Johannes Korteweg (1848-1941) and Gustav De Vries (1866-1934). This equation is then bears their names (abbreviated as the KdV equation) and becomes the typical nonlinear evolution equation. Anyway, the studies about solitary waves still ran aground because the stability of the solitary wave is yet to be determined. So then it is meaningless to study them.

1953 is another great year for the development of soliton theory. In this year, Enrico Fermi (1901-1954), John R. Pasta (1918-1981) and Stanislaw M. Ulam (1909-1984) carried out an experiment to study the thermalisation of a solid by using computer (Guo, 1995). They expected the energy introduced in their simulation would slowly drift to the modes with higher wave number due to the existence of nonlinearity. But the final result showed that the nonlinearity itself is not enough to guarantee the equipartition of energy. Almost all the energy comes back to the original distribution at the beginning after a long time. This is the famous FPU paradox which stimulated the interest in the study of the solitary waves.

In 1965, Norman J. Zabusky and Martin D. Kruskal investigated the nonlinear simulation in detail. They confirmed that the solitary waves maintain their shapes and velocity after the interaction. This is similar to the colliding property of particles and therefore, they named them ‘solitons’. Their work was an important milestone in the history of the soliton theory and had been accepted in general. After that, one began to have the great interests in solitons and to pay more attention to them. This is more than 130 years after the discovery of solitary wave by J.S. Russell.

1.2 Background of the problem

The very first exact soliton solution for the KdV equation was obtained by Gardner, Greene, Kruskal and Miura (1967). They reduced the nonlinear problem to the linear one, which was well known as the Sturm-Liouville eigenvalue problem, then discovered the inverse scattering transform (IST) method for solving the initial value problem of the KdV equation. The IST is a well-developed mathematical theory which can be used to solve the initial-value problems for a restricted class of evolution equations. However, it is very difficult to set up an appropriate inverse scattering problem which depends seemingly on the existence of an infinite number of independent conservation laws for the evolution equation.

In 1970s, Hirota (1971) developed an ingenious method that is geared to finding multi-soliton solutions to evolution equations directly. Although the method is less general than the IST method since it does not solve initial-value problem, but it has the advantage of being applicable to a wider class of nonlinear equations in a unified way. The method is now known as the ‘Hirota’s direct method’. Compare to IST, Hirota’s method is rather heuristic, but it is more straightforward. If one is only interested in finding multi-soliton solutions, the best tool is Hirota’s method.

There are three main objectives in this dissertation. First, we wish to investigate how the Hirota's method can be used to solve multi-soliton solutions for various physically significant nonlinear partial differential equations.

The next goal of the dissertation is to discuss the Hirota' method in solving some of the integrable nonlinear partial differential equations such as the most famous KdV equation, Kadomtsev-Petviashvili (KP) equation, modified KdV (mKdV) equation and sine-Gordon (sG) equation.

Finally, we wish to investigate how the suitable transformation can be determined by means of singularity analysis.

1.3 Statement of the Problem

The project is to study in detail the Hirota's method and its application in solving multi-soliton solutions for various physically significant nonlinear partial differential equations. At the same time, we investigate the use of singularity analysis in Hirota's method.

1.4 Objectives of Study

I. To study the Hirota's direct method with its bilinear form and Hirota's D-Operator and the corresponding exact solutions in terms of perturbation series.

II. Use the Hirota's method to obtain the multi-soliton solutions for certain class of physically significant nonlinear evolution equations.

III. To investigate the use of singularity analysis to determine the appropriate transformation in obtaining the bilinear forms of the nonlinear evolution equations.

1.5 Scope of Study

The study will focus only on Hirota's method in solving nonlinear dispersive wave equations which can be transformed into the Hirota's bilinear form.

1.6 Organization of the Dissertation

In Chapter II, we present the literature review regarding the development of the soliton theory after Zabusky and Kruskal introduced the word "soliton".

In Chapter III, we introduce the Hirota's direct method. First, make a proper substitution to express the nonlinear differential equation in a bilinear form. Then, express the bilinear form of the equation as polynomials of exponential before the perturbation expansion can be used to produce the exact soliton solution. All the steps mentioned above will be discussed in detail.

Chapter IV is mainly the application of the Hirota's method in obtaining the exact soliton solutions for various physically significant nonlinear evolution equations.

In Chapter V, we discuss the connection between the singularity analysis and Hirota's method.

Finally, in Chapter VI, we conclude our dissertation and give some ideas for future study. In the appendices, we include some formulas of the *D*-operator.

CHAPTER II

LITERATURE REVIEW

This chapter presents the literature review on the development of soliton theory for the last few decades.

2.1 The Birth of Solitons

The early history of solitons has been marked by long eclipses. From the discovery of solitary wave by Russell to the Korteweg de-Varies (KdV) equation was about 60 years. Then, this phenomenon was forgotten until the FPU paradox was created in 1953. It is only ten years later that an explanation could be provided by Zabusky and Kruskal (1965).

Zabusky and Kruskal (1965) had observed the unusual nonlinear interactions among "solitary-wave pulse" propagating in nonlinear dispersive media. These phenomena were observed in the numerical solutions of the KdV equation which can be used to describe the one-dimensional, long-time asymptotic behavior of small, but finite amplitude, i.e. shallow-water waves.

In a paper published in Physical Review Letters (Zabusky and Kruskal, 1965), they introduced the word "soliton". The concept of the "soliton" introduced by them had been accepted in general.