EXACT SOLUTIONS FOR UNSTEADY UNIDIRECTIONAL FLOWS OF SECOND GRADE FLUID

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ABSTRACT

Second grade fluid or second grade fluid is one of the most popular used model to describe non-Newtonian fluids. In study, the problems involving unsteady unidirectional flows of second grade fluid, flows of second grade fluid through a porous medium and flows of heated second grade fluid are discussed. The unidirectional flows of second grade fluid considered in this study are Couette flow, Stokes' first flow problem and plane Poiseuille flow. The exact solutions for Couette flow and Poiseuille flow are obtained using the Laplace transform while the Fourier transform method is used to obtain the exact solution of the Stokes' first flow problem. The momentum equation for the flows of second grade fluid through a porous medium is derived based on the modified Darcy's law. The momentum equation for flows of second grade fluid through a porous medium has an extra term, which is the viscous damping force caused by the micro-pore structures of the porous medium. It is found that the velocity of the flows decrease when the porosity, φ and permeability, K parameter are increased. For flows of heated second grade fluids, the velocity and temperature distribution are determined using the simple and double Fourier sine transforms.

ABSTRAK

Bendalir peringkat kedua atau bendalir gred kedua merupakan salah satu model yang sering digunakan untuk mewakili bendalir bukan Newtonan. Perbincangan meliputi masalah yang berkaitan aliran bendalir peringkat kedua, aliran bendalir peringkat kedua melalui media poros dan aliran bendalir peringkat kedua yang dipanaskan. Aliran bendalir yang dikaji ialah aliran Couette, aliran Poiseuille dan masalah aliran pertama Stoke. Penyelesaian tepat untuk aliran Couette dan Poiseuille diperolehi menggunakan teknik jelmaan Laplace manakala teknik jelmaan Fourier digunakan untuk mendapat penyelesaian tepat bagi masalah aliran pertama Stoke. Berdasarkan hukum Darcy, persamaan momentum untuk bendalir peringkat kedua melalui media poros diterbitkan. Persamaan momentum untuk bendalir peringkat kedua melalui media poros mempunyai satu fungsi yang di mana fungsi itu mengandungi daya kelikatan yang disebabkan struktur media yang poros. Dalam kajian yang dilakukan, didapati bahawa taburan halaju bagi aliran bendalir peringkat kedua melalui media poros menurun apabila nilai parameter resapan dan keporosan meningkat. Seterusnya, penyelesaian tepat untuk taburan halaju dan suhu bagi aliran bendalir panas peringkat kedua diperolehi dengan mengaplikasikan teknik jelmaan sin Fourier mudah dan berganda.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

1.1 Introduction

Fluids can be divided into Newtonian fluid and non-Newtonian fluid. Newtonian fluids are fluids for which a linear relation exists between stress and spatial variation of velocity, while non-Newtonian fluids are the otherwise, where the stress at a given temperature and pressure is not a linear function of the spatial variation of velocity. Therefore, a constant coefficient of viscosity cannot be defined.

Non-Newtonian fluids form an extremely wide class of different materials, whose only common features are fluidity and a failure to obey Newton's law of friction. Recently, the interest in flows of non-Newtonian fluids has grown considerably because of their vast applications in daily life. It is well-known in fact that non-Newtonian fluids are more appropriate than Newtonian fluids in industrial and technological applications. A large variety of consumer goods containing high concentration of glass or carbon fibres, paints and lubricants containing polymer additives, many foodstuffs and biological fluids are non-Newtonian.

Non-Newtonian biologically important fluids response is typically observed in concentrated suspensions and in high molecular-weight materials. The rheology of blood has received much study. Blood is rheologically complex on two counts: it is a suspension because erythrocytes with characteristic dimensions of several micrometers are present, and the suspending fluid itself exhibits non-Newtonian behavior because of the presence of high molecular weight protein. Flows of elastic solutions and of those containing long-chain polymers, including coatings, as well as flows in extruders, molds, and other processing equipment, dominate rheology today.

Many industrial problems involve rheological concerns. These include the need to understand the transport of foams and yield-stress fluids in oil drilling and enhanced oil recovery, and the importance of understanding the behavior of biological macromolecules in micro fluidic devices for lab-on-a-chip applications. Geoscientists apply fluid mechanics in studies of volcanism and the convection through Earth's mantle and outer core since they are dealing with non-Newtonian fluid; lava.

Many materials such as clay coatings, drilling muds, certain oils and greases, and many emulsions have been treated as non-Newtonian fluids. It is difficult to suggest a single model, which exhibits all properties of non-Newtonian fluids as, is done for the Newtonian fluids. They cannot be described in a simple model by the Navier-Stokes equation as for the Newtonian fluids and there has been much confusion over the classification of non-Newtonian fluids.

The class of Newtonian flows for which there exists analytic solution of the Navier-Stokes equations is very restricted. In fact, the resulting equations of non-Newtonian fluids are higher order, more non-linear and complicated than the Navier-Stokes equations. Thus, this makes the task of obtaining the accurate solutions is difficult. Being scientifically challenging, flows of non-Newtonian fluids have been studied extensively.

1.2 Background of Research

Non-Newtonian fluids may be classified as (i) fluids for which the shear stress depends only on the rate of shear; (ii) fluids for which the relation between shear stress and shear rate depends on time; (iii) the visco-elastic fluids, which possess both elastic and viscous properties [11-13]. Given this, the term non-Newtonian fluid is open to interpretation [23].

Because of the great diversity in the physical structure of non-Newtonian fluids, it is not possible to recommend single constitutive equations. By the term *constitutive equation*, it means an equation relating the stress on some material to the motion of the material [23]. For this reason, many non-Newtonian models or constitutive equations have been proposed and most of them are empirical or semi empirical. Although many constitutive equations have been suggested, many problems are still unsolved. Some of the models do not give satisfactory results in accordance with the available experimental data. For this reason, in many practical application, empirical or semi empirical equations have been applied.

For previous solved problems which the flow is slow enough in the visco-elastic sense, the results given by Oldroyd's constitutive equations will be substantially equal to those of the second- or third order Rivlin-Ericksen constitutive equations [11]. Thus, it would seem reasonable to use the second- or third-order constitutive equations in carrying out the calculations. This is particularly so in view of the fact that the calculation will generally simpler. For this reason, the second-order fluid model is used.

The constitutive relation in the following form is for to the second grade fluid [7]:

$$\underset{\sim}{\sigma} = -p\underset{\sim}{I} + \mu \underset{\sim}{A}_1 + \alpha_1 \underset{\sim}{A}_2 + \alpha_2 \underset{\sim}{A}_1^2$$

where $\underset{\sim}{\sigma}$ is the stress tensor and p is the pressure. This model has three coefficients where μ is the coefficient of viscosity, and α_1 and α_2 are the material moduli which are usually referred to as the visco-elasticity and cross-viscosity parameter respectively. While $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ are the kinematical tensors. There are some restrictions on these coefficients due to the Clasius-Duhem inequality and also due to the assumption that Helmholtz free energy is minimum in equilibrium (see Appendix). A comprehensive discussion on the restrictions for these three coefficients has been given by Dunn and Fosdick [8].

In the studies of non-Newtonian fluids, the non-linearities and in certain cases the order of the differential equations are increased. The equation of motion of incompressible second grade fluids is of higher order than the Navier-Stokes equation. The Navier-Stokes equation is a second order partial differential equation, but the equation of motion of a second grade fluid is a third order partial differential equation. Thus, in order to obtain a determinate solution in a second grade fluid one requires either a boundary (initial condition) or some similar assumption in addition to the boundary conditions required for the Navier-Stokes equations.

Unsteady flows of a second grade fluid in a bounded region have been studied by Ting (1963). He is the first author who studied the unsteady flows of a fluid of second grade [10]. Then Hayat et al. (2000) has carried out a study on unsteady unidirectional flows of a non-Newtonian fluid which is the second grade fluid. In their study, they obtained the exact analytic solutions for a class of unsteady unidirectional flows and the frictional forces of an incompressible second grade fluid. The solutions are obtained by the method of separation of variables.

Then Hayat et al. (2004) extended the study by Hayat et al. (2000) to obtain the solutions for both large and small times. As for solutions for small times, the Laplace transform technique is particularly well studied. By knowing the unidirectional flows'velocity, the volume flux and frictional forces can be calculated. They found that large and small times solutions are dependent on the coefficient of visco-elasticity, α_1. As the visco-elastic parameter increases, the value of velocity at the distance y decreases.

Erdogan and Imrak (2007) conducted a study about some unsteady flows of second grade fluid where the unsteady unidirectional flows are considered. The correct condition for a free surface is given in this study. In their study, they have obtained velocity and stress solutions for each flows. Then they compared the stress at $t=0$ on the stationary boundary for flows generated by the impulsive motion of a boundary for second grade fluid with stress at $t=0$ for Newtonian fluid. They found that this stress is finite for second grade fluid and infinite for a Newtonian fluid.

Erdogan and Imrak (2005) continued their study on unsteady unidirectional flows of a second grade fluid. They have discovered that for small values of time some interesting results occur. The stress at the boundary show a different character for flow generated by impulsive motion of a boundary compared to that for a Newtonian fluid. Another property of a second grade fluid is that the required time to attain the asymptotic value of a second grade is longer than that for a Newtonian fluid.

The solution of the governing equation for unsteady unidirectional flows of second grade fluids in unbounded regions obtained by the Laplace transform method shows some undesirable results. For this reason, in order to solve the governing equation the use of the other transform methods are discussed by Erdogan and Imrak (2007). It is shown in their study that the solution obtained by the Laplace transform method does not satisfy the initial condition. For this reason, the Fourier transform method for the unsteady flows of second grade fluids in unbounded regions has been used instead of the Laplace transform method.

Stokes first solved Stokes' first problem in 1856, which involved a viscous Newtonian fluid [15]. Jordan and Puri (2003) conducted a study on Stokes' first problem involving a second grade fluid in a porous half-space. The open half space of the original Stokes' first problem has been replaced with a porous half space of permeability and porosity. Based on the Darcy's law, the Stokes' first flow problem was solved.

Tan and Masuoka (2005) developed a modified Darcy model (see Appendix) for visco-elastic fluids in a porous media, and then investigated the Stokes' first problem for visco-elastic fluids in a porous half-space. They obtained exact solutions of the velocity and temperature using Fourier sine transforms.

In the same year, research about the unsteady Couette flow of a second grade fluid in a layer of porous medium were conducted by Hayat et al. (2005). In this study, the two Couette flows of a second grade fluid were discussed when (i) bottom plate moves suddenly, (ii) bottom plate oscillates. The flow problems were solved using the modified Darcy's law and Laplace transform method.

Bandelli (1995) has carried out a research on the heated boundaries of the unsteady unidirectional flows of second order fluids. He derived the energy equation for second grade fluids and obtained exact solution for some flows.

Fatecau (2002) has conducted the Rayleigh-Stokes problem for heated second grade fluids. The Stokes' first flow for a heated flat plate and Rayleigh-Stokes flow for a heated edge were considered in this paper. The temperature distribution of a second grade fluid subject to a linear flow on a heated flat plate is determined using the Fourier sine transform.

1.3 Statement of Problem

Recently the interest in non-Newtonian fluids has grown considerably because of their practical and fundamental importance associated with many technological applications. Therefore, this research is conducted to study one of the most popular models for non-Newtonian fluids is the model that is called the second grade fluid. This study will review what are the mathematical studies that have been done on second grade fluids and the methods used to obtain the solution of the problems involving the second grade fluid.

1.4 Objective of Research

The objectives of this project are:

1. To derive the governing equations of motions for flows of second grade fluids.
2. To derive the exact solution for the problems involving unsteady unidirectional flows of second grade fluids.
3. To solve the problems involving unsteady unidirectional flows of second grade fluids through a porous medium.
4. To solve the problems of unsteady unidirectional flows for heated second grade fluids.

1.5 Scope of Research

This work is a series of exact solutions of unidirectional flows of second grade fluid. The unsteady Couette flow and Stokes' first flow problem of a second grade fluid in a layer of porous medium and the Rayleigh-Stokes flow and the Poiseuille flow for heated second grade fluid will be reviewed.

1.6 Significance of Research

The study of non-Newtonian fluid dynamics is important in relation with engineering, geophysics, physiology and pharmaceutics. In engineering, it affects the production and use of polymeric materials, but plasticity theory has been similarly important for the design of metal forming processes. Many industrially important substances such as concrete, paint and chocolate have complex flow characteristics. Geophysics includes the flow of lava, but in addition measures the flow of solid Earth materials over long time scales: those that display viscous behavior, e.g. granite, are known as rheids. While in physiology, many bodily fluids have complex compositions and thus flow characteristics. In particular, there is a specialist study of blood flow called hemorheology. The term biorheology is used for the wider field of study of the flow properties of biological fluids. Food rheology is important in the manufacture and processing of food products.

1.7 Organization of Dissertation

Chapter 1 contains general introduction to non-Newtonian fluid. The background, objective, statement and the significance of this research are stated in this chapter too.

The derivation of the governing equation of second grade fluid will be shown in Chapter 2. The constitutive equation of an incompressible second grade fluid and the discussion on the material moduli are considered.

As going through to Chapter 3, the problems involving of the unsteady unidirectional flows of second grade fluids will be studied. The governing equation for the unsteady unidirectional flows of second grade fluid will be derived. The method how to obtain the velocity distribution, frictional force and volume flux for the respective flows will be reviewed in this chapter.

While in Chapter 4, the heated unsteady unidirectional flows of a second grade fluid will be discussed. The governing equations for second grade fluid, which are the momentum and energy equations will be derived. The method to get the velocity and temperature distributions for the Stokes' first flow problem will be reviewed

Chapter 5 contains the review of the unsteady unidirectional flows problems through a porous medium. The governing equation derived in Chapter 3 cannot be used to govern the unidirectional flows of second grade fluid through a porous medium. Therefore, the governing equation for the unsteady unidirectional flows of second grade fluid through a porous medium will be derived in this chapter. Then, the method to obtain the velocity distribution for the unsteady unidirectional flows will be reviewed.

Lastly is the conclusion chapter. In this chapter, it will conclude and summarize all the results obtained from the beginning until the end of the research.