Matematika, 2001, Jilid 17, bil. 2, hlm. 51–61

Some Applications of Two-Generator p-Groups of Nilpotency Class Two

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Abstract The classification of all two-generator groups of nilpotency class two have been done by Nor Haniza Sarmin in 2000. Earlier research on the finite case was introduced by Kappe *et. al* in 1999 and Bacon and Kappe in 1993. Later, in 2000, James Beuerle and Kappe focus their classification to infinite metacyclic groups and some of their applications. This paper will focus on the applications of two-generator p-groups of nilpotency class two including the nonabelian tensor squares and the exterior squares. We will also determine which of these groups are capable.

Keywords Nonabelian tensor squares, two-groups, two-generator, nilpotency class two.

Abstrak *Klasifikasi bagi semua kumpulan dengan dua penjana yang mempunyai kelas nilpoten dua telah diperolehi oleh Nor Haniza Sarmin dalam tahun 2000. Kajian yang terdahulu bagi kumpulan terhingga telah dilakukan oleh Kappe et. al dalam tahun 1999 dan Bacon dan Kappe dalam tahun 1993. Kemudian, dalam tahun 2000, James Beuerle dan Kappe telah menumpukan klasifikasi mereka kepada kumpulan metakitaran tak terhingga dan beberapa aplikasinya. Penyelidikan ini ditumpukan kepada beberapa aplikasi bagi kumpulan-p dengan dua penjana termasuk kuasa dua tensor yang tak abelan dan kuasa dua peluaran. Kajian ini juga akan menentukan kumpulan berupaya dari kumpulan tersebut.*

Katakunci *Kuasa dua tensor yang tak abelan, kumpulan-dua, dua penjana, kelas nilpoten dua.*

1 Introduction

The classification of all two-generator groups of nilpotency class two up to isomorphism have been done by Nor Haniza Sarmin [8]. Earlier research on the classification of two-generator p-groups of nilpotency class two was introduced by Bacon and Kappe [1] for the case p odd, and Kappe, *et. al* [7] for the case $p = 2$.

Recent work of Beuerle and Kappe in [3] focus the classification on infinite metacylic groups and some of their applications including the nonabelian tensor squares, exterior squares, the symmetric products and the second homology groups. They also showed that the only capable group of the nonabelian infinite metacylic group is the infinite dihedral group.

In this paper, we will combine the paper by Bacon and Kappe [1] and Kappe, *et. al* [7] to give the classification of two-generator p-groups of nilpotency class two, up to isomorphisms in Theorem 2.1. We will compute some applications of two-generator p-groups of nilpotency class two including their nonabelian tensor squares and exterior squares. Then, we will use the result of these applications to determine which of these groups are capable.

In 1938, Baer [2] initiated a systematic investigation to determine the conditions of a group G must fulfill in order to be the group of inner automorphisms of some group E (that is $G \cong E/Z(E)$). Such group is called a capable group. Baer already determined all finitely generated abelian capable groups which are direct sums of cyclic groups. Baer stated that such group are capable if and only if their two highest torsion coefficients agree. Ellis in [5] investigated the capability of other classes of groups, concentrating to some extent on nilpotent group of class two. He used the method that involves the third integral homology of a group. He showed that the quaternion group

$$Q_{2n} = \langle a, b | a^n = b^2, b^{-1}ab = a^{-1} \rangle$$

of order $4n$ is not capable. He also showed that any finitely generated (but non-cyclic) nilpotent group of class ≤ 2 with prime exponent is capable. These results will be used to determine which of the two-generator p-groups of nilpotency class two are capable.

2 The Classification of Two-Generator *p*-Groups of Nilpotency Class Two

In this section, we bring the results of the classification of two-generator p-groups of nilpotency class two up to isomorphism from the papers by Bacon and Kappe [1] and Kappe, *et. al* [7]. The proofs of the classification are omitted and the details can be referred from the papers stated.

Theorem 2.1 states the classification of two-generator p-groups of nilpotency class two up to isomorphism. Groups of type (2.1.1) - (2.1.3) is the classification for the case $p \neq 2$. Replacing the prime p by 2 to groups of type (2.1.1) - (2.1.3) together with groups of type (2.1.4) below yield the classification of two-generator 2-groups of nilpotency class two. An extra condition $\alpha + \beta > 3$ have been added in (2.1.2) for the case $p = 2$ to ensure that the dihedral group of order 8 is not included in both (2.1.1) and (2.1.2). Instead, it is characterized to be of type (2.1.1).

Theorem 2.1 *Let G be a finite nonabelian two-generator p-group of nilpotency class two. Then G is isomorphic to exactly one group of the following four types:*

(2.1.1) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, *where* $[a,b]=c, [a,c]=[b,c]=1, |a|=p^{\alpha}, |b|=p^{\beta}, |c|=p^{\gamma}, \alpha,\beta,\gamma \in \mathcal{N}, \alpha \geq \beta \geq \gamma$;

(2.1.2) $G \cong \langle a\rangle \rtimes \langle b\rangle$, *where* $[a,b]=a^{p^{\alpha-\gamma}}, |a|=p^{\alpha}, |b|=p^{\beta}, |[a,b]|=p^{\gamma}, \alpha,\beta,\gamma \in \mathcal{N}, \alpha \geq 2\gamma, \beta \geq \gamma$, *and if* $p=2$ *then* $\alpha+\beta>3$;

(2.1.3) $G \cong (\langle c\rangle \times \langle a\rangle) \rtimes \langle b\rangle$, *where* $[a,b]=a^{p^{\alpha-\gamma}}c, [c,b]=a^{-p^{p(\alpha-\gamma)}}c^{-p^{\alpha-\gamma}}, |a|=p^{\alpha}, |b|=p^{\beta}, |c|=p^{\sigma}, |[a,b]|=p^{\gamma}, \alpha,\beta,\gamma,\sigma \in \mathcal{N}, \beta \geq \gamma > \sigma, \alpha+\sigma \geq 2\gamma$;

(2.1.4) $G \cong (\langle c\rangle \times \langle a\rangle)\langle b\rangle$, *where* $|a|=|b|=2^{\gamma+1}, |[a,b]|=2^{\gamma}, |c|=2^{\gamma-1}, [a,b]=a^2c, [c,b]=a^{-4}c^{-2}, a^{2^{\gamma}}=b^{2^{\gamma}}, \gamma \in \mathcal{N}$.

The groups in the above list are pairwise nonisomorphic and have nilpotency class two precisely.

3 Some Preparatory Results

In this section we bring some results that will be used in Section 4. These results are important to compute the exterior square of a group and to determine which groups are capable. The proofs of some theorems and lemmas are omitted.

Definition 3.1 [3] **Exterior Square**

Let G be a group. Then we define the exterior square of the group as

$$G \wedge G = (G \otimes G)/\langle x \otimes x | x \in G\rangle.$$

Theorem 3.1 [3] *The exterior center of a group G is*

$$Z^{\wedge}(G) = \{g \in G | g \wedge x = 1_{\wedge}, \quad \forall x \in G\}.$$

Definition 3.2 Epicenter

Let G be a group. Then $Z^*(G) = \{z \in Z(G) | x \wedge z = 1_{\wedge}, \quad \forall x \in G\}$ is called the epicenter of G.

Definition 3.3, Theorem 3.2, Theorem 3.3 and Proposition 3.1 below can be used to determine which of the groups in the classification are capable.

Definition 3.3 [4] **Capable Group**

A group G is said to be capable if it is isomorphic to the inner automorphism group, $Inn(H)$ of some group H, that is

$$G \cong H/Z(H).$$

Theorem 3.2 [3] *A group G is capable if and only if its exterior center is trivial, that is*

$$Z^{\wedge}(G) = \{1\}.$$

Theorem 3.3 [6] *A group G is capable if and only if its epicenter is trivial, denoted as*

$$Z^{*}(G) = \{1\}.$$

Proposition 3.1 [5] *Any finitely generated (but non-cyclic) nilpotent group of class ≤ 2 and prime exponent is capable.*

In the lemma below, we prove that $\langle x \otimes x | x \in G\rangle$ is generated by $a \otimes a, b \otimes b$ and $(a \otimes b)(b \otimes a)$. This result will be used to compute the exterior squares of two-generator p-groups of nilpotency class two.

Lemma 3.1 *Let $G = \langle a, b\rangle$ be a two-generator p-group of class two. Then*

$$\langle x \otimes x; x \in G\rangle = \langle a \otimes a, b \otimes b, (a \otimes b)(b \otimes a)\rangle.$$

Proof Let $x = a^m b^n z^l, z = [a, b]$. Then by Proposition 3.5 in [1], with $m = m', n = n', l = l'$, we have

$$x \otimes x = (a \otimes a)^{\alpha_1}(b \otimes b)^{\alpha_2}(a \otimes b)^{\alpha_3}(b \otimes a)^{\alpha_4}(a \otimes z)^{\alpha_5}(b \otimes z)^{\alpha_6}, \tag{1}$$

where

$$\begin{aligned} \alpha_1 &= m^2, \quad \alpha_2 = n^2, \quad \alpha_3 = mn, \quad \alpha_4 = mn, \\ \alpha_5 &= n\tbinom{m}{2} - n\tbinom{m}{2} + (n-n)m^2 + ml - ml = 0, \\ \alpha_6 &= m\tbinom{n}{2} - m\tbinom{n}{2} + nl - nl = 0. \end{aligned}$$

Hence,

$$x \otimes x = (a \otimes a)^{m^2}(b \otimes b)^{n^2}((a \otimes b)(b \otimes a))^{nm}. \tag{2}$$

Thus $\langle x \otimes x; x \in G\rangle \subseteq \langle a \otimes a, b \otimes b, (a \otimes b)(b \otimes a)\rangle$. Let $m = 1, n = 0$ or $m = 0, n = 1$, then $a \otimes a, b \otimes b \in \langle x \otimes x; x \in G\rangle$. For $n = m = 1$ we have $(a \otimes a)(b \otimes b)(a \otimes b)(b \otimes a)$ $\in \langle x \otimes x; x \in G\rangle$. This, together with the above, yields

$$\langle a \otimes a, b \otimes b, (a \otimes b)(b \otimes a)\rangle \subseteq \langle x \otimes x; x \in G\rangle. \qquad \square$$

The next two lemmas will be used to determine the capability of two-generator p-groups of nilpotency class two using Proposition 3.1.

Lemma 3.2 *Let G be a group of type (2.1.1) - (2.1.4) of Theorem 2.1 with $p = 2$. Then,*

$$\exp(G) = \begin{cases} 2^{\gamma+1}, & \text{if } \alpha = \beta = \gamma, \\ 2^{\alpha}, & \text{if } \alpha = \beta > \gamma, \\ 2^{\alpha}, & \text{if } \alpha > \beta, \\ 2^{\beta}, & \text{if } \beta > \alpha. \end{cases} \tag{3}$$

Proof To prove this lemma we need to consider that

$$\begin{aligned} \binom{2^\delta}{2} &= \frac{2^\delta(2^\delta-1)(2^\delta-2)!}{2!(2^\delta-2)!} \\ &= \frac{2^\delta}{2}(2^\delta-1) \\ &= 2^{\delta-1}(2^\delta-1). \end{aligned}$$

Let $g \in G$. Then we can write $g = a^i b^j [a,b]^k$, where i, j, k are integers. Thus, for some $\delta \in \mathbb{N}$,

$$\begin{aligned} g^{2^\delta} &= (a^i b^j [a,b]^k)^{2^\delta} \\ &= (a^i b^j)^{2^\delta}[a,b]^{k2^\delta} \\ &= a^{i2^\delta} b^{j2^\delta}[a,b]^{ij\binom{2^\delta}{2}}[a,b]^{k2^\delta} \\ &= a^{i2^\delta} b^{j2^\delta}[a,b]^{ij\binom{2^\delta}{2}+k2^\delta} \\ &= a^{i2^\delta}[a,b]^{ij\binom{2^\delta}{2}+k2^\delta}. \end{aligned}$$

Then, there are three cases, according to the conditions of the classification :
(i) $\alpha > \beta$, (ii) $\beta > \alpha$, and (iii) $\alpha = \beta$,
where $G = \langle a, b\rangle, |a| = 2^\alpha, |b| = 2^\beta, |[a,b]| = 2^\gamma,$ and $\gamma \geq 1$.
(i) Assume $\alpha > \beta$. Then

$$\begin{aligned} g^{2^\alpha} &= a^{i2^\alpha} b^{j2^\alpha}[a,b]^{ij\binom{2^\alpha}{2}+k2^\alpha} \\ &= (a^{2^\alpha})^i (b^{2^\alpha})^j [a,b]^{2^{\alpha-1}(2^\alpha-1)ij+k2^\alpha} \\ &= 1, \end{aligned}$$

since $\alpha > \beta \geq \gamma$ implies $\alpha > \beta$ and $\alpha > \gamma$, and $a^{2^{\alpha-1}} \neq 1$. Thus $\exp(G) = 2^\alpha$.
(ii) Assume $\beta > \alpha$. Then we obtain

$$\begin{aligned} g^{2^\beta} &= a^{i2^\beta} b^{j2^\beta}[a,b]^{ij\binom{2^\beta}{2}+k2^\beta} \\ &= (a^{2^\beta})^i (b^{2^\beta})^j [a,b]^{2^{\beta-1}(2^\beta-1)ij+k2^\beta} \\ &= 1, \end{aligned}$$

since $\beta > \alpha > \gamma$ and $b^{2^{\beta-1}} \neq 1$. Thus $\exp(G) = 2^\beta$.
(iii) Assume $\alpha = \beta$. Then we have two subcases :
(a) $\alpha = \beta = \gamma$. Then

$$\begin{aligned} g^{2^{\alpha+1}} &= a^{i2^{\alpha+1}} b^{j2^{\alpha+1}}[a,b]^{ij\binom{2^{\alpha+1}}{2}+k2^{\alpha+1}} \\ &= (a^{2^{\alpha+1}})^i (b^{2^{\alpha+1}})^j [a,b]^{2^\alpha(2^{\alpha+1}-1)ij+k2^{\alpha+1}} \\ &= 1, \end{aligned}$$

since $\alpha = \beta = \gamma$. On the contrary, $(ab)^{2^\alpha} = a^{2^\alpha} b^{2^\alpha}[a,b]^{\binom{2^\alpha}{2}} = [a,b]^{\binom{2^\alpha}{2}} = [a,b]^{2^{\alpha-1}(2^\alpha-1)}$ $\neq 1$, since $[a,b]^{2^{\alpha-1}} \neq 1$. Thus $\exp(G) = 2^{\alpha+1} = 2^{\beta+1} = 2^{\gamma+1}$.

(b) $\alpha = \beta > \gamma$. We obtain

$$\begin{aligned} g^{2^\alpha} &= a^{i2^\alpha} b^{j2^\beta} [a,b]^{ij\binom{2^\alpha}{2}+k2^\alpha} \\ &= (a^{2^\alpha})^i (b^{2^\beta})^j [a,b]^{2^{\alpha-1}(2^\alpha-1)ij+k2^\alpha} \\ &= 1, \end{aligned}$$

since $\alpha = \beta > \gamma$ and $a^{2^{\alpha-1}} \neq 1, b^{2^{\alpha-1}} \neq 1$. Thus $\exp(G) = 2^\alpha = 2^\beta$. $\square$

Lemma 3.3 *Let G be a group of type (2.1.1) - (2.1.3) of Theorem 2.1 with $p \neq 2$. Then,*

$$\exp(G) = \begin{cases} p^\gamma, & \text{if } \alpha = \beta = \gamma, \\ p^\alpha, & \text{if } \alpha = \beta > \gamma, \\ p^\alpha, & \text{if } \alpha > \beta, \\ p^\beta, & \text{if } \beta > \alpha. \end{cases} \tag{4}$$

Proof To proof this lemma we need to consider that

$$\begin{aligned} \binom{p^\delta}{2} &= \frac{p^\delta(p^\delta-1)(p^\delta-2)!}{2!(p^\delta-2)!} \\ &= \frac{p^\delta}{2}(p^\delta - 1). \end{aligned}$$

For the case p odd, we can see that $p^\delta - 1$ is an even, so that it can be divided by 2. Then, we can factorize the power of the variables involved. Let $g \in G$. Then we can write $g = a^i b^j [a,b]^k$, where i, j, k are integers. Thus, for some $\delta \in \mathcal{N}$,

$$\begin{aligned} g^{p^\delta} &= (a^i b^j [a,b]^k)^{p^\delta} \\ &= (a^i b^j)^{p^\delta} [a,b]^{kp^\delta} \\ &= a^{ip^\delta} b^{jp^\delta} [a,b]^{ij\binom{p^\delta}{2}} [a,b]^{kp^\delta} \\ &= a^{ip^\delta} b^{jp^\delta} [a,b]^{ij\binom{p^\delta}{2}+kp^\delta} \\ &= a^{ip^\delta} [a,b]^{ij\binom{p^\delta}{2}+kp^\delta}. \end{aligned}$$

Then, there are three cases, according to the conditions of the classification :
(i) $\alpha > \beta$, (ii) $\beta > \alpha$, and (iii) $\alpha = \beta$,
where $G = \langle a, b \rangle, |a| = p^\alpha, |b| = p^\beta, |[a,b]| = p^\gamma,$ and $\gamma \geq 1$.
(i) Assume $\alpha > \beta$. Then

$$\begin{aligned} g^{p^\alpha} &= a^{ip^\alpha} b^{jp^\alpha} [a,b]^{ij\binom{p^\alpha}{2}+kp^\alpha} \\ &= (a^{p^\alpha})^i (b^{p^\alpha})^j [a,b]^{\frac{p^\alpha}{2}(p^\alpha-1)ij+kp^\alpha} \\ &= (a^{p^\alpha})^i (b^{p^\alpha})^j ([a,b]^{p^\alpha})^{\frac{1}{2}(p^\alpha-1)ij+k} \\ &= 1, \end{aligned}$$

since $\alpha > \beta \geq \gamma$ implies $\alpha > \beta$ and $\alpha > \gamma$, and $a^{p^{\alpha-1}} \neq 1$. Thus $\exp(G) = p^\alpha$.

(ii) Assume $\beta > \alpha$. Then we obtain

$$\begin{aligned} g^{p^\beta} &= a^{ip^\beta} b^{jp^\beta} [a,b]^{ij\binom{p^\beta}{2}+kp^\beta} \\ &= (a^{p^\beta})^i (b^{p^\beta})^j [a,b]^{\frac{p^\beta}{2}(p^\beta-1)ij+kp^\beta} \\ &= (a^{p^\beta})^i (b^{p^\beta})^j ([a,b]^{p\beta})^{\frac{1}{2}(p^\beta-1)ij+k} \\ &= 1, \end{aligned}$$

since $\beta > \alpha > \gamma$ and $b^{p^{\beta-1}} \neq 1$. Thus $\exp(G) = p^\beta$.
(iii) Assume $\alpha = \beta$. Then we have two subcases :
(a) $\alpha = \beta = \gamma$. Then

$$\begin{aligned} g^{p^\alpha} &= a^{ip^\alpha} b^{jp^\alpha} [a,b]^{ij\binom{p^\alpha}{2}+kp^\alpha} \\ &= (a^{p^\alpha})^i (b^{p^\alpha})^j [a,b]^{\frac{p^\alpha}{2}(p^\alpha-1)ij+kp^\alpha} \\ &= (a^{p^\alpha})^i (b^{p^\alpha})^j ([a,b]^{p^\alpha})^{\frac{1}{2}(p^\alpha-1)ij+k} \\ &= 1, \end{aligned}$$

since $\alpha = \beta = \gamma$. Thus $\exp(G) = p^\alpha = p^\beta = p^\gamma$.
(b) $\alpha = \beta > \gamma$. We obtain

$$\begin{aligned} g^{p^\alpha} &= a^{ip^\alpha} b^{jp^\beta} [a,b]^{ij\binom{p^\alpha}{2}+kp^\alpha} \\ &= (a^{p^\alpha})^i (b^{p^\beta})^j [a,b]^{\frac{p^\alpha}{2}(p^\alpha-1)ij+kp^\alpha} \\ &= (a^{p^\alpha})^i (b^{p^\alpha})^j ([a,b]^{p^\alpha})^{\frac{1}{2}(p^\alpha-1)ij+k} \\ &= 1, \end{aligned}$$

since $\alpha = \beta > \gamma$ and $a^{p^{\alpha-1}} \neq 1, b^{p^{\alpha-1}} \neq 1$. Thus $\exp(G) = p^\alpha = p^\beta$. $\square$

Now, we are ready to compute the exterior squares and determine which of the groups in Theorem 2.1 are capable.

4 The Exterior Squares And The Capability of Groups

In this section, we will arrive at our main objective to compute the exterior squares of some of the groups classified in Theorem 2.1 and to determine which of these groups are capable. Some results have been obtained by earlier papers. Baeyl [6] has already proved that an extra special p-group is capable if and only if it is either dihedral of order eight or of order p^3 and exponent $p > 2$. He also mentioned that all dihedral groups are capable. Beurle and Kappe [3], have computed the exterior square for the infinite metacyclic group. They have also determined that the only nonabelian infinite metacyclic group that is capable is $\mathcal{Z} \rtimes \mathcal{Z}_2$, that is the infinite dihedral group. Specifically, if $H = \mathcal{Z} \rtimes \mathcal{Z}_4$, then $\mathcal{Z} \rtimes \mathcal{Z}_2 \cong H/Z(H)$. Some of these results will be used to determine the capability of groups according to the classification in Theorem 2.1.

Here, we will compute the exterior squares of groups of type (2.1.2). We will show that groups of type (2.1.2) for the case $p \neq 2$ are not capable. We will also show that groups of type (2.1.1) for the case $p \neq 2$ and $\alpha = \beta = \gamma = 1$ are capable. The computation of exterior squares and the determination of the capability of groups for the case $p \neq 2$ and $p = 2$ have to be done separately since they have different nonabelian tensor squares and conditions of parameters. At the end of this section, we will also show that D_4 is capable since $D_4 \cong D_8/Z(D_8)$.

Theorem 4.1 below is needed to compute the exterior square of groups of type (2.1.2). For the complete proof of the theorem we refer to the paper by Bacon and Kappe [1].

Theorem 4.1 [1] *Let G be a group of type (2.1.2) with $p \neq 2$, that is $G \cong \langle a \rangle \rtimes \langle b \rangle$, where α, β, γ are integers with $[a,b] = a^{p^{\alpha-\gamma}}, |a| = p^{\alpha}, |b| = p^{\beta}, |[a,b]| = p^{\gamma}, \alpha \geq 2\gamma, \beta \geq \gamma \geq 1$. Then*

$$G \otimes G \cong \mathcal{Z}^2_{p^{\beta}} \times \mathcal{Z}_{p^{\alpha-\gamma}} \times \mathcal{Z}_{min\{p^{\alpha-\gamma}, p^{\beta}\}},$$

generated by $(a \otimes a), (b \otimes b), (a \otimes b)(b \otimes a)$ and $(b \otimes a)$. The orders of the generators are,

$$\begin{aligned} |(a \otimes a)| &= p^{\alpha-\gamma}, \\ |(b \otimes b)| &= p^{\beta}, \\ |(a \otimes b)(b \otimes a)| &= p^{\beta}, \\ |(b \otimes a)| &= \min\{p^{\alpha-\gamma}, p^{\beta}\}. \end{aligned}$$

Next, we will compute the exterior square of groups of type (2.1.2) for the case $p \neq 2$.

Proposition 4.1 *Let G be a group of type (2.1.2) with $p \neq 2$, that is $G \cong \langle a \rangle \rtimes \langle b \rangle$, where $[a,b] = a^{p^{\alpha-\gamma}}, |a| = p^{\alpha}, |b| = p^{\beta}, |[a,b]| = p^{\gamma}, \alpha, \beta, \gamma \in \mathcal{N}, \alpha \geq 2\gamma, \beta \geq \gamma$. Then,*

$$G \wedge G = (G \otimes G)/\langle x \otimes x | x \in G \rangle \cong \langle b \otimes a \rangle$$

where $\langle b \otimes a \rangle$ has the order of $\min(p^{\alpha-\gamma}, p^{\beta})$.

Proof By Theorem 4.1, $G \otimes G$ is generated by $a \otimes a, b \otimes b, (a \otimes b)(b \otimes a)$ and $b \otimes a$ where their orders are $p^{\alpha-\gamma}, p^{\beta}, p^{\beta}$ and $\min\{p^{\alpha-\gamma}, p^{\beta}\}$, respectively. By Lemma 3.1, we have $\langle x \otimes x | x \in G \rangle = \langle a \otimes a, b \otimes b, (a \otimes a)(b \otimes a) \rangle$. Thus, we conclude that

$$G \wedge G \cong \langle b \otimes a \rangle$$

with the order of min $\{p^{\alpha-\gamma}, p^{\beta}\}$. □

Now we state the tensor square for groups of type (2.1.2) for the case $p = 2$, which will be used to compute their exterior squares.

Theorem 4.2 [7] *Let G be a group of type (2.1.2) with $p = 2$, that is $G \cong \langle a \rangle \rtimes \langle b \rangle$, where α, β, γ are integers with $[a,b] = a^{2^{\alpha-\gamma}}, |a| = 2^{\alpha}, |b| = 2^{\beta}, |[a,b]| = 2^{\gamma}, \alpha \geq 2\gamma, \beta \geq \gamma \geq 1, \alpha + \beta > 3$. Then*

$$G \otimes G \cong \mathcal{Z}_{2^{\alpha-\gamma+1}} \times \mathcal{Z}_{2^{\beta}} \times \mathcal{Z}_{2^{\min\{\alpha-\gamma\}}} \times \mathcal{Z}_{2^{\min\{\alpha,\beta\}}},$$

generated by $(a \otimes a), (b \otimes b), (a \otimes b)(b \otimes a)$ and $(b \otimes a)$. The orders of the generators are,

$$\begin{aligned} |(a \otimes a)| &= 2^{\alpha-\gamma+1}, \\ |(b \otimes b)| &= 2^{\beta}, \\ |(a \otimes b)(b \otimes a)| &= 2^{\min\{\alpha,\beta,\}} \\ |(b \otimes a)| &= 2^{\min\{\alpha,\beta\}}. \end{aligned}$$

In the next proposition, we will compute the exterior squares of groups of type (2.1.2) for the case $p = 2$.

Proposition 4.2 *Let G be a group of type (2.1.2) with $p = 2$, that is $G \cong \langle a \rangle \rtimes \langle b \rangle$, where $[a, b] = a^{2^{\alpha-\gamma}}, |a| = 2^{\alpha}, |b| = 2^{\beta}, |[a, b]| = 2^{\gamma}, \alpha, \beta, \gamma \in \mathcal{N}, \alpha \geq 2\gamma, \beta \geq \gamma, \alpha + \beta > 3$. Then,*

$$G \wedge G = (G \otimes G)/\langle x \otimes x | x \in G \rangle \cong \langle b \otimes a \rangle$$

where $\langle b \otimes a \rangle$ has the order of $2^{\min\{\alpha,\beta\}}$.

Proof By Theorem 4.1, $G \otimes G$ is generated by $a \otimes a, b \otimes b, (a \otimes b)(b \otimes a)$ and $b \otimes a$ where their orders are $2^{\alpha-\gamma+1}, 2^{\beta}, 2^{\min\{\alpha-\gamma,\beta\}}$ and $2^{\min\{\alpha,\beta\}}$, respectively. By Lemma 3.1, we have $\langle x \otimes x | x \in G \rangle = \langle a \otimes a, b \otimes b, (a \otimes a)(b \otimes a) \rangle$. Thus, we conclude that

$$G \wedge G \cong \langle b \otimes a \rangle$$

with the order of $2^{\min\{\alpha,\beta\}}$. $\square$

Now, we are ready to determine the capability of groups of type (2.1.2) using Theorem 3.3.

Proposition 4.3 *Let $G = \langle a, b \rangle$ be a two-generator p-group ($p \neq 2$) of class two and type (2.1.2), that is $G \cong \langle a \rangle \rtimes \langle b \rangle$, where $[a, b] = a^{p^{\alpha-\gamma}}, |a| = p^{\alpha}, |b| = p^{\beta}, |[a, b]| = p^{\gamma}$, $\alpha, \beta, \gamma \in \mathcal{N}, \alpha \geq 2\gamma, \beta \geq \gamma$. Then*

$$Z^*(G) \neq 1, \text{ and thus } G \text{ is not capable.}$$

Proof By Proposition 4.1, $G \wedge G$ is a cyclic group of order $\min(p^{\alpha-\gamma}, p^{\beta})$. We will show that $1 \neq a^{p^{\alpha-\gamma}} = [a, b] \in Z^*(G)$. Let $g = a^m b^n [a, b]^k$. We have

$$\begin{aligned} g \otimes [a, b] &= a^m b^n [a, b]^k \otimes [a, b] \\ &= (a^m b^n \otimes [a, b])([a, b]^k \otimes [a, b])([a^m b^n, [a, b]^k] \otimes [a, b])([a, b]^k \otimes [a^m b^n, [a, b]]). \end{aligned}$$

We already know that $[a, b]^k \otimes [a, b] = [a^k, b] \otimes [a, b] = 1_{\otimes}$, and obviously $[a^m b^n, [a, b]] = 1$. Then

$$\begin{aligned} g \otimes [a, b] &= a^m b^n \otimes [a, b] \\ &= (a^m \otimes [a, b])(b^n \otimes [a, b])([a^m, b^n] \otimes [a, b])(b^n \otimes [a^m, [a, b]]) \\ &= (a^m \otimes [a, b])(b^n \otimes [a, b]) \\ &= (a^m \otimes a^{p^{\alpha-\gamma}})(b^n \otimes a^{p^{\alpha-\gamma}}). \end{aligned}$$

So we obtain

$$\begin{aligned} g \otimes [a, b] &= (a^m \otimes a^{p^{\alpha-\gamma}})(b^n \otimes a^{p^{\alpha-\gamma}}) \\ &= (a^m \otimes a^{p^{\alpha-\gamma}})(b \otimes a)^{np^{\alpha-\gamma}} (a \otimes [b, a])^{n\binom{p^{\alpha-\gamma}}{2}} (b \otimes [b, a])^{p^{\alpha-\gamma}\binom{n}{2}} \\ &= (a^m \otimes a^{p^{\alpha-\gamma}})(b \otimes a)^{np^{\alpha-\gamma}} (a \otimes a^{-p^{\alpha-\gamma}})^{n\binom{p^{\alpha-\gamma}}{2}} (b \otimes [b, a])^{p^{\alpha-\gamma}\binom{n}{2}}. \end{aligned}$$

Obviously,

$$\begin{aligned} a^m \otimes a^{p^{\alpha-\gamma}} &= (a\otimes a)^{mp^{\alpha-\gamma}}(a\otimes[a,a])^{m\binom{p^{\alpha-\gamma}}{2}}(a\otimes[a,a])^{p^{\alpha-\gamma}\binom{m}{2}} \\ &= (a\otimes a)^{mp^{\alpha-\gamma}} \\ &= 1_\otimes, \end{aligned}$$

since by Theorem 4.1 $|a\otimes a| = p^{\alpha-\gamma}$ and

$$\begin{aligned} a\otimes a^{-p^{\alpha-\gamma}} &= (a\otimes a)^{-p^{\alpha-\gamma}}(a\otimes[a,a])^{\binom{-p^{\alpha-\gamma}}{2}}(a\otimes[a,a])^{-p^{\alpha-\gamma}\binom{1}{2}} \\ &= (a\otimes a)^{-p^{\alpha-\gamma}} \\ &= ((a\otimes a)^{p^{\alpha-\gamma}})^{-1} \\ &= 1_\otimes. \end{aligned}$$

It follows that, $g\otimes[a,b] = (b\otimes a)^{np^{\alpha-\gamma}}(b\otimes[b,a])^{p^{\alpha-\gamma}\binom{n}{2}}$ mod $\langle x\otimes x | x\in G\rangle$. By Theorem 4.1, $(b\otimes a)^{np^{\alpha-\gamma}} = 1_\otimes$, and $(b\otimes[b,a])^{p^{\alpha-\gamma}} = b\otimes[b,a]^{p^{\alpha-\gamma}} = b\otimes 1 = 1_\otimes$, since $[b,a]^{p^{\alpha-\gamma}} = (a^{-p^{\alpha-\gamma}})^{p^{\alpha-\gamma}} = 1$.

Thus $g\otimes[a,b] \equiv 1_\otimes$ mod $\langle x\otimes x | x \in G\rangle$. Therefore, the epicenter for this type of group is not trivial. Hence, by Theorem 3.3 G is not capable. $\square$

In our next proposition, we will prove that groups of type (2.1.1) for the case $p\neq 2$ and $\alpha=\beta=\gamma=1$ is capable by Proposition 3.1.

Proposition 4.4 *Let $G=\langle a,b\rangle$ be a two-generator p-group ($p\neq 2$) of class two and type (2.1.1) that is, $G\cong(\langle c\rangle\times\langle a\rangle)\rtimes\langle b\rangle$, where $[a,b]=c,[a,c]=[b,c]=1,|a|=p^\alpha,|b|=p^\beta,|c|=p^\gamma,\alpha,\beta,\gamma\in\mathcal{N},\alpha\geq\beta\geq\gamma$. If $\alpha=\beta=\gamma=1$, then G is capable.*

Proof By Lemma 3.3, we obtain

$$\exp(G) = p^\gamma = p,$$

since $\alpha=\beta=\gamma=1$. Thus, by Proposition 3.1, G is capable. $\square$

As an example, the Burnside group, $B(2,3)$, is capable using this proposition. The Burnside group is a group of type (2.1.1) in Theorem 2.1 with $p=3$ and $\alpha=\beta=\gamma=1$, presented by

$$B(2,3) = \langle a,b | a^3=b^3=(ab)^3=(a^{-1}b)^3=1\rangle.$$

Obviously, according to Lemma 3.2 and Lemma 3.3, $\exp(G)$ for other cases is a power of p which is not a prime. Thus, we cannot use Proposition 3.1 to determine the capability of other types of groups classified in Theorem 2.1.

5 Conclusion

We have succeed to compute the exterior squares for groups of type (2.1.2). We have proved in Proposition 4.3 that the groups of type (2.1.2) where $p\neq 2$ is not capable. Finally, in Proposition 4.4, we have shown one case for groups of type (2.1.1) of Theorem 2.1 to be capable.

6 Acknowledgements

The authors would like to thank the Research and Management Center (RMC), UTM for the support in this work under Vot 71718.

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