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CLASSIFICATION OF FIXED POINTS OF POTTS–BETHE MAPPING OF DEGREE FOUR ON $\mathbb{Q}_5$

(Pengelasan Titik Tetap Pemetaan Potts–Bethe Darjah Empat pada $\mathbb{Q}_5$)

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ABSTRACT

The Potts–Bethe mapping is a rational function arises in the study of the Potts model on the Cayley tree (or Bethe lattice). In this paper, the Potts–Bethe mapping of degree four is considered over the field $\mathbb{Q}_5$ of 5-adic numbers. In some regimes (a condition appear in the study of p-adic Potts model), the fixed points are found and their stability are determined. It is done by solving some quartic equation over $\mathbb{Q}_5$ and calculating the value of derivative at each fixed points. This is the continuation of the previous work where contraction and chaos are found, but here other property is realized such as 1-Lipschitz.

Keywords: rational function; p-adic number; fixed point

ABSTRAK

Pemetaan Potts-Bethe ialah fungsi nisbah yang timbul dalam kajian model Potts pada pokok Cayley (atau kekisi Bethe). Dalam artikel ini, pemetaan Potts–Bethe darjah empat dipertimbangkan di atas medan $\mathbb{Q}_5$ bagi nombor 5-adic. Dalam sesetengah rejim (keadaan yang muncul dalam kajian model Potts p-adic), titik tetap ditemui dan tingkah lakunya dikelaskan. Ia dilakukan dengan menyelesaikan persamaan kuartik pada $\mathbb{Q}_5$ dan mengira nilai pembezaan di setiap titik. Ini adalah kesinambungan kerja sebelumnya di mana pengecutan dan kekacauan ditemui, tetapi di sini sifat lain ditemui seperti 1-Lipschitz.

Kata kunci: fungsi nisbah; nombor p-adic; titik tetap

1. Introduction

Rational function is one of the most studied functions especially in discrete dynamical system. The iconic rational functions is the Möbius transformation (or homographic map), that is,

$$\phi(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0.$$

As iterations in discrete dynamical system, this function is well studied, for instances, over the field $\mathbb{C}$ of complex number and the field $\mathbb{Q}_p$ of p-adic numbers (Fan *et al.* 2014). Such studies involve for examples the investigation on Julia and Fatou sets. The Fatou set is the maximal open set on which ϕ is equicontinuous, whereas the Julia set is the complement of the Fatou set. The following is the rational function that we consider in this paper. This rational function arises in the investigation of (p-adic) Gibbs measures of the (p-adic) Potts model on the Cayley trees (Ahmad *et al.* 2018, 2019).

Definition 1.1. The Potts–Bethe mapping of degree positive integer k is defined as follows

$$f_{a,b,k}(x) = \left(\frac{ax+b}{x+a+b-1}\right)^k, \qquad a \neq 1, a+b \neq 0, \qquad k \geq 1.$$

If the degree k is one, then the Potts–Bethe mapping is reduced to a special case of the Möbius transformation. Fan *et al.* (2014) has studied the dynamics of this transformation in which it has a 1-Lipschitz property. Whereas, for $k = 2$ and $k = 3$, the dynamics of the Potts–Bethe mapping was analysed by Mukhamedov and Khakimov (2016) and Ahmad *et al.* (2018) respectively. In this paper, the Potts–Bethe mapping of degree four over the field $\mathbb{Q}_5$ of 5-adic numbers is considered, that is, $f_{a,b,4} : \mathbb{Q}_5 \to \mathbb{Q}_5$,

$$f_{a,b,4}(x) = \left(\frac{ax + b}{x + a + b - 1}\right)^4, \qquad a, b \in \mathbb{Q}_5, \qquad a \neq 1,\ a + b \neq 0. \tag{1}$$

More general case is done by Mukhamedov and Khakimov (2018), and Khakimov and Mukhamedov (2022).

Throughout, $f_{a,b}$ means $f_{a,b,4}$. The fixed points of $f_{a,b}$, that is, $f_{a,b}(x) = x$ has a connection with the p-adic Gibbs measure of the p-adic Potts model (Ahmad *et al.* 2018), and more general context in (Mukhamedov & Khakimov 2016, 2018, 2021, 2023; Mukhamedov *et al.* 2023; Rozikov *et al.* 2022; Rahmatullaev & Tukhtabaev 2023). There, the parameters a, b satisfy the condition $|a - 1|_5 < 1$ and $|b + 1|_5 < 1$. In order to find the fixed points, one has to solve some polynomial equation which comes from the fixed point equation. Solving this equation is an old problem and this problem is discussed recently over $\mathbb{Q}_p$ (Mukhamedov *et al.* 2014; Saburov & Ahmad 2015a,b, 2018; Saburov *et al.* 2021) for lower degree polynomial.

2. Preliminaries

For a fixed prime number p, the field $\mathbb{Q}_p$ of p-adic numbers is the completion of the set $\mathbb{Q}$ of rational numbers with respect to the p-adic absolute value $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ given by

$$|x|_p = \begin{cases} p^{-\kappa}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

where $x = p^{\kappa}\left(\frac{m}{n}\right)$ with $k, m \in \mathbb{Z}$, $n \in \mathbb{N}$ and $p \nmid mn$. For any $x \in \mathbb{Q}_p$, we have $x = \frac{x^*}{|x|_p}$ such that

$$x^* = x_0 + x_1 \cdot p + x_2 \cdot p^2 + \dots$$

for $x_0 \in \{1, 2, \dots, p - 1\}$ and $x_i \in \{0, 1, 2, \dots, p - 1\}$, $\forall i \in \mathbb{N} = \{1, 2, 3, \dots\}$. Here x^* is the element of the p-adic unit

$$\mathbb{Z}_p^* = \{x \in \mathbb{Q}_p : |x|_p = 1\}$$

We denote the set of all fixed points of $f_{a,b}$ by

$$\mathbf{Fix}\{f_{a,b}\} = \{x \in \mathbb{Q}_5 : f_{a,b}(x) = x\}.$$

It can be easily checked that $\mathbf{x}^{(0)} = 1 \in \mathbf{Fix}\{f_{a,b}\}$. Then it follows from $f_{a,b}(x) - 1 = x - 1$ that

$$\frac{(x-1)(a-1)}{(x+a+b-1)^4}\big((ax+b)^3 + (ax+b)^2(x+a+b-1) + (ax+b)(x+a+b-1)^2 + (x+a+b-1)^3\big) = (x-1).$$

From here, any other fixed point $x \neq x^{(0)}$ is the root of the following quartic equation

$$(a-1)\left((ax+b)^3+(ax+b)^2(x+a+b-1)+(ax+b)(x+a+b-1)^2 +(x+a+b-1)^3\right)=(x+a+b-1)^4 \qquad (2)$$

Let $x = x^{(\infty)} + (a-1)y$ where $x^{(\infty)} = 1-a-b$. Then, the quartic equation, Eq. (2), can be written as

$$y^4-(1+a+a^2+a^3)y^3+(3a^2+2a+1)(a+b)y^2-(3a+1)(a+b)^2y+(a+b)^3=0. \qquad (3)$$

Then we have the following consequences.

Proposition 2.1. (Ahmad *et al.* 2019) Let $|a-1|_5 < 1$ and $|b+1|_5 < 1$. Then the quartic equation (3) always has four roots $y^{(1)}$, $y^{(2)}$, $y^{(3)}$ and $y^{(4)}$ such that

(i) $\left|y^{(1)}\right|_5 = 1$ and $\left|y^{(1)}-4\right|_5 < 1$,

(ii) $y^{(2)} = \dfrac{1}{|a+b|_5}\left(y_0^{(2)} + y_1^{(2)}5 + \dots\right)$ where $2y_0^{(2)} - (a+b)^* \equiv 0 \pmod 5$,

(iii) $y^{(3)} = \dfrac{1}{|a+b|_5}\left(y_0^{(3)} + y_1^{(3)}5 + \dots\right)$ and $y^{(4)} = \dfrac{1}{|a+b|_5}\left(y_0^{(4)} + y_1^{(4)}5 + \dots\right)$ where $y_0^{(3)}$ and $y_0^{(4)}$ are roots of the congruence

$$2t^2 - 2(a+b)^*t + ((a+b)^*)^2 \equiv 0 \pmod 5. \qquad (4)$$

Thus we have

$$\mathbf{Fix}\{f_{a,b}\} = \left\{x^{(0)},\ x^{(1)},\ x^{(2)},\ x^{(3)},\ x^{(4)}\right\}$$

where $x^{(0)} = 1, x^{(\infty)} = 1-a-b$, and $x^{(i)} = x^{(\infty)} + (a-1)y^{(i)}$ for $i = 1,2,3,4$.

Since there the derivative for the Potts–Bethe mapping exists, we have the following definition.

Definition 2.2. Let $\lambda = f'_{a,b}(x_0)$. The fixed point x_0 is called attracting, indifferent, and repelling fixed point if $|\lambda|_p < 1$, $|\lambda|_p = 1$, and $|\lambda|_p > 1$ respectively. Moreover, the fixed point is stable if $|\lambda|_p < 1$.

The following is the definition of 1-Lipschitz property.

Definition 2.3. Let $N \subset \mathbb{Q}_p$. Then the function $f_{a,b}$ has 1-Lipschitz property on N if for any $x, y \in N$,

$$|f_{a,b}(x) - f_{a,b}(y)|_p = |x-y|_p.$$

3. Classification of the Fixed Points

Theorem 3.1. *Consider the fixed point set* $\mathbf{Fix}\{f_{a,b}\} = \left\{x^{(0)},\ x^{(1)},\ x^{(2)},\ x^{(3)},\ x^{(4)}\right\}$ *of the Potts–Bethe mapping (Eq. (1)).*

(1) Let $0 < |a-1|_5 < |b+1|_5 < 1$. *Then*

(i) $x^{(0)}$ *is an attracting fixed point;*

(ii) $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, *and* $x^{(4)}$ *are repelling fixed points.*

(2) Let $0 \leq |b+1|_5 < |a-1|_5 < 1$. *Then*

(i) $x^{(0)}$ *and* $x^{(1)}$ *are indifferent fixed points;*
(ii) $x^{(2)}$, $x^{(3)}$ *and* $x^{(4)}$ *are repelling fixed points.*

(3) Let $0 < |a+b|_5 < |a-1|_5 = |b+1|_5 < 1$. *Then*

(i) $x^{(1)}$ *is an attracting fixed point;*
(ii) $x^{(0)}$, $x^{(2)}$, $x^{(3)}$ *and* $x^{(4)}$ *are repelling fixed points.*

(4) Let $0 < |a+b|_5 = |a-1|_5 = |b+1|_5 < 1$. *Then*

(i) $x^{(0)}$ *and* $x^{(1)}$ *are indifferent fixed points;*
(ii) $x^{(2)}$, $x^{(3)}$ *and* $x^{(4)}$ *are repelling fixed points.*

Proof. For $0 \leq i \leq 4$, we have

$$f'\left(x^{(i)}\right) = \frac{4(a-1)(a+b)(ax^{(i)}+b)^3}{(x^{(i)}+a+b-1)^5} = \frac{4(a-1)(a+b)x^{(i)}}{(ax^{(i)}+b)(x^{(i)}+a+b-1)}. \tag{5}$$

One could calculate

$$\left|f'\left(x^{(0)}\right)\right|_5 = \begin{cases} \dfrac{|a-1|_5}{|b+1|_5} < 1, & \text{if } 0 < |a-1|_5 < |b+1|_5 < 1; \\ \dfrac{|a-1|_5}{|a-1|_5} = 1, & \text{if } 0 \leq |b+1|_5 < |a-1|_5 < 1; \\ \dfrac{|a-1|_5}{|a+b|_5} = 1, & \text{if } 0 < |a+b|_5 = |a-1|_5 = |b+1|_5 < 1; \\ \dfrac{|a-1|_5}{|a+b|_5} > 1, & \text{if } 0 < |a+b|_5 < |a-1|_5 = |b+1|_5 < 1. \end{cases}$$

Next, we substitute $x^{(i)} = x^{(\infty)} + (a-1)y^{(i)}$ for $i = 1, 2, 3, 4$ into (5) and obtain

$$f'\left(x^{(i)}\right) = \frac{4(a+b)x^{(i)}}{(a-1)y^{(i)}\left(ay^{(i)} - (a+b)\right)}. \tag{6}$$

By Proposition 2.1-(i), it can be easily derived

$$\left|f'\left(x^{(1)}\right)\right|_5 = \begin{cases} \dfrac{|b+1|_5}{|a-1|_5} > 1, & \text{if } 0 < |a-1|_5 < |b+1|_5 < 1; \\ \dfrac{|a-1|_5}{|a-1|_5} = 1, & \text{if } 0 \leq |b+1|_5 < |a-1|_5 < 1; \\ \dfrac{|a-1|_5}{|a+b|_5} = 1, & \text{if } 0 < |a+b|_5 = |a-1|_5 = |b+1|_5 < 1; \\ \dfrac{|a+b|_5}{|a-1|_5} < 1, & \text{if } 0 < |a+b|_5 < |a-1|_5 = |b+1|_5 < 1. \end{cases}$$

Furthermore, by using Proposition 2.1-(ii) and -(iii), we get

$$f'\left(x^{(i)}\right) = \frac{4(a+b)|a+b|_5^2 x^{(i)}}{(a-1)\left(y^{(i)}\right)^*\left(a\left(y^{(i)}\right)^* - (a+b)^*\right)}, \qquad i = 2, 3, 4. \tag{7}$$

Moreover, one can easily checked that $a\left(y^{(i)}\right)^* \equiv \left(y^{(i)}\right)^* \not\equiv (a+b)^* \pmod 5$ for $i = 2, 3, 4$. Consequently,

$$\left|f'\left(x^{(i)}\right)\right|_5 = \begin{cases} \dfrac{1}{|a-1|_5|b+1|_5} > 1, & \text{if } 0 < |a-1|_5 < |b+1|_5 < 1; \\ \dfrac{1}{|a-1|_5^2} > 1, & \text{if } 0 \leq |b+1|_5 < |a-1|_5 < 1; \\ \dfrac{1}{|a-1|_5|a+b|_5} > 1, & \text{if } 0 < |a+b|_5 = |a-1|_5 = |b+1|_5 < 1; \\ \dfrac{1}{|a-1|_5|a+b|_5} > 1, & \text{if } 0 < |a+b|_5 < |a-1|_5 = |b+1|_5 < 1. \end{cases}$$

By these calculations, we conclude the proof. □

Remark that a part of this theorem, Theorem 3.1-(1), is the results from Ahmad *et al.* (2019). Ahmad *et al.* (2019) characterise the dynamics of the Potts–Bethe mapping (Eq. (1)) as the contraction and chaos using the methods by Fan *et al.* (2007) and Fan and Liao (2018). Meanwhile in this paper we classify all cases, that is, for $|a-1|_5 < 1$ and $|b+1|_5 < 1$. Ahmad *et al.* (2019) did not consider these cases because their aim is to show the existence of chaos. In the meantime, in this paper, we want to further study the dynamics of the Potts–Bethe mapping for all cases. Here we have the situation where the fixed points are indifferent, Theorem 3.1-(2) and 3.1-(4). This shows that the Potts–Bethe mapping has 1-Lipschitz property over some domian. It follows directly where for any $x, y \in N$

$$|f_{a,b}(x) - f_{a,b}(y)|_p = \left|f'\left(x^{(i)}\right)\right|_p |x-y|_p = |x-y|_p.$$

for $i = 0, 1$. The domain N is the largest neighborhood of $x^{(i)}$ which satifies 1-Lipschitz property. In future work, using this 1-Lipshitz property, one can characterize completely the dynamics, comparing to Fan *et al.* (2014, 2017).

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