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A CLASS OF TWO-SEX POPULATION QUADRATIC STOCHASTIC OPERATORS: *b*-BISTOCHASTIC-VOLTERRA

(Kelas Operator Kuadratik Stokastik bagi Populasi Dua Jantina : b-Bistokastik-Volterra)

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ABSTRACT

This paper aims to investigate the simplest non-linear Markov operators which is the quadratic. Study of quadratic stochastic operators (QSOs) is not an easy task as linear operators. Thus, researchers introduced classes of QSOs such as Volterra QSOs, strictly non-Volterra QSOs, Orthogonal preserving QSOs, Centered QSOs etc. However, all the introduced classes were not yet cover the whole set of QSOs. Therefore, we introduce a new class of QSOs, namely b-bistochastic-Volterra QSOs or simply bV-QSOs. In this paper, we describe the canonical form of bV-QSO on two-dimensional simplex. This helps understanding the dynamical behaviours of bV-QSOs for future works.

Keywords: quadratic operator; Markov operator; b-bistochatic; Volterra; two-sex population; bisexual population

ABSTRAK

Fokus kajian ini adalah untuk menyiasat pengendali Markov bukan linear yang paling mudah iaitu kuadratik. Kajian tentang operator kuadratik stokastik (QSO) bukan satu tugas mudah seperti operator linear. Oleh itu, penyelidik memperkenalkan beberapa kelas QSO seperti QSO Volterra, QSO bukan Volterra, QSO mengawet ortogon, QSO berpusat dan lain-lain. Walau bagaimanapun, semua kelas tersebut tidak meliputi keseluruhan set QSO. Oleh itu, kami memperkenalkan satu kelas baru QSO iaitu b-bistokastik QSO Volterra atau secara ringkas bV-QSO. Dalam kajian ini, kami memperkenalkan bentuk berkanun bV-QSO yang ditakrifkan pada simpleks dua dimensi. Hal ini membantu memahami tingkah laki dinamik bV-QSO bagi kerja masa hadapan.

Kata kunci: operator kuadratik; operator Markov; b-bistokastik; Volterra; populasi dua jantina; populasi dwi-jantina

1. Introduction

Markov processes is an evolution of the probability distribution in a linear fashion under the action of a stochastic operator. Markov operators have been used extensively, especially in the fields of mathematics and physics. These operators, however, might not be appropriate model for a system that needs various interactions between components, such as disease dynamics and population genetics. For instance, the mathematical model that should be applied in population dynamics must be nonlinear since it involves the recombination of genes that occurs due to their pairing.

Quadratic is the most simplest non-linear case, yet numerous problems are still open in this direction. Tracing down to history, Bernstein (1924)'s work on population genetic was the first considered such kind of operators. Moreover, Kesten (1970) associated a dynamical systems induced by quadratic stochastic operators (QSOs) of asexual (multi-type) populations and sex-linked systems. By constraining the values of heredity coefficients of QSOs, he proved that such operators has only single fixed point. In addition to that, biological interpretation of various mating rules are given together with Mendelian genetics model. In short, the works written by Kesten provide a crucial contribution to comprehend population processes mathematically.

Comprehensive references, open problems and recent achievements can be referred to Mukhamedov and Ganikhodjaev (2015); Ganikhodzhaev *et al.* (2011); Jamilov *et al.* (2023); Embong and Mukhamedov (2023); Eshmatov *et al.* (2023); Abdurakhimova and Rozikov (2022); Rozikov and Xudayarov (2022); Hamzah *et al.* (2022). Studying the QSOs in general setting is not direct as linear case, hence researchers considered some particular cases of QSOs such as Volterra QSOs, QSO corresponding to permutation, Orthogonal preserving QSOs, Lebesgue QSO, and etc (see (Badocha and Bartoszek, 2018; Mukhamedov and Fadillah Embong, 2021; Jamilov *et al.*, 2020)). Unfortunately, all the introduced classes does not cover the set of all QSOs. In this sense, one can introduce another class of QSO which directly will help the development of our understanding of non-linear operators

Thus, in this paper, we are going to consider a new class of QSO namely $b-$bistochasic-Volterra QSOs or in short written as bV-QSOs that was first introduced in Chye *et al.* (2021). This paper is organized as follows:In the first part, we highlight the history of QSOs. In Section 2, preliminaries results and notations that are required in this paper will be given. Next section is devoted as the main results of this paper which on the description of bV-QSO on two dimensional simplex. This paper ends with a conclusion in the last section.

2. Preliminaries

As usual, the simplex represent the set of all probability distribution

$$S^{n-1} = \left\{ \mathbf{x} = (x_i)_{i=1}^n \in \mathbb{R}^n | x_i \geq 0, \sum_{i \in E} x_i = 1 \right\}. \tag{1}$$

Denote $S^{n-1} \times S^{v-1}$ the Cartesian product between these two simplices S^{n-1} and S^{v-1}. Define an operator V on $S^{n-1} \times S^{v-1}$ by

$$V(\mathbf{x},\mathbf{y}) = \begin{cases} \mathbf{x}'_k = \left(\sum\limits_{i,j=1}^{n,v} P^{(f)}_{ij,k} x_i y_j \right)_{k=1}^{n}, \\ \\ \mathbf{y}'_l = \left(\sum\limits_{i,j=1}^{n,v} P^{(m)}_{ij,l} x_i y_j \right)_{l=1}^{v}. \end{cases} \tag{2}$$

where $(\mathbf{x},\mathbf{y}) \in S^{n-1} \times S^{v-1}$, $\mathbf{x} = (x_1, \dots, x_n) \in S^{n-1}$ and $\mathbf{y} = (y_1, \dots, y_v) \in S^{v-1}$. Moreover, the heredity coefficients $P^{(f)}_{ij,k}$ and $P^{(m)}_{ij,l}$ satisfy

$$P^{(f)}_{ij,k} \geq 0, \quad \sum_{k=1}^{n} P^{(f)}_{ij,k} = 1, \quad P^{(m)}_{ij,l} \geq 0, \quad \sum_{l=1}^{v} P^{(m)}_{ij,l} = 1. \tag{3}$$

In the sense of population dynamics, these heredity coefficients means the probability of a female offspring being type k and, respectively a male offspring being type l, when the parental pair is i, j $(i, k = 1, 2, ..., n;$ and $j, l = 1, 2, ..., v)$. These values account for a number of variables, including the recombination process, gamete selection, mutations, and differential birth rate, among others. Indeed, the operator V is well-defined i.e., it maps from $S^{n-1} \times S^{v-1}$ into itself. In this paper, we consider $P_{ij,k}$ is symmetrical, or else a new coefficient can be defined as

$$\tilde{P}_{ij,k} = \frac{P_{ij,k} + P_{ji,k}}{2},$$

where the symmetricalness is satisfied. The operator given by 2 is known as QSO of a two-sex population. In this direction, some works have been done as for examples Rozikov and Zhamilov (2011); Castanos *et al.* (2018); Ganikhodzhaev *et al.* (2014); Ganikhodjaev *et al.* (2021); Dzhumadil'daev *et al.* (2016); Ganikhodjaev and Jamilov (2015); Boxonov (2023). In case of $V : S^{n-1} \to S^{n-1}$, then this operator is called Quadratic Stochastic Operator. Note, b-order was defined by Mukhamedov and Embong (2015) as follows:

Definition 2.1. (Mukhamedov & Embong 2015; Parker & Ram 1996) Let us define a functional $u_k : \mathbb{R}^n \to \mathbb{R}$,

$$u_k(x_1, ..., x_n) = \sum_{i=1}^{k} x_i,$$

where $k = 1, 2, ..., n-1$. Let $\mathbf{x}, \mathbf{y} \in S^{n-1}$, we said that $\mathbf{x}$ is b-ordered by $\mathbf{y}$ if ,

$$\mathbf{x} \leq^b \mathbf{y} \iff u_k(\mathbf{x}) \leq u_k(\mathbf{y}), \text{ for all } k = 1, 2, ..., n-1.$$

The original idea of $b-$order came from the concept of *majorization* that was introduced by Hardy *et al.* (1952). To differentiate between these two orders, recall that for any $\mathbf{x} = (x_1, x_2, \ldots x_n) \in S^{n-1}$, we rearrange $\mathbf{x}$ and denote it by $\mathbf{x}_{[\downarrow]} = (x_{[1]}, x_{[2]}, \ldots x_{[n]})$, where

$$x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}.$$

Similar meaning is applied to the notion $\mathbf{x}_{[\uparrow]}$. For any $\mathbf{x}, \mathbf{y} \in S^{n-1}$, if the condition $\mathbf{x}_{[\downarrow]} \leq^b \mathbf{y}_{[\downarrow]}$ holds, then we say that $\mathbf{x}$ *is majorized by* $\mathbf{y}$ (or $\mathbf{y}$ $majorates$ $\mathbf{x}$). In short, $\mathbf{x} \prec \mathbf{y}$ or $\mathbf{y} \succ \mathbf{x}$. A complete references on this topic can be referred to Hardy *et al.* (1952). On the other hand, $b-$order does not required any rearrangement of $\mathbf{x}$.

Ganikhodzhaev (1993a) defined any QSO V that satisfies $V(\mathbf{x}) \prec \mathbf{x}$ is called doubly QSOs. It turns out that, this class of QSOs has it own application, as for instance Abdulghafor *et al.* (2020) used doubly QSOs to model consensus problem which is one of the main part in Multi-Agent Systems. Thus, it is nature to define a class of QSOs by using $b-$order, namely any QSO V that satisfies $V(\mathbf{x}) \leq^b \mathbf{x}$ for all $\mathbf{x} \in S^{n-1}$ is called $b-$bistochastic QSO. The study of this class of QSOs is well-established on lower dimensions (see Mukhamedov and Embong (2015, 2016a,b, 2018, 2017); Embong *et al.* (2023)).

Next, a QSO is call Volterra QSO if $P_{ij,k} = 0$ for any $k \notin \{i, j\}$. One can use the definition of QSOs to reduce a Volterra QSO into:

$$\mathbf{x}_k^{'} = x_k \left(1 + \sum_{i=1}^{n} a_{ki} x_i\right), \quad k \in \{1, 2, \ldots, n\}. \tag{4}$$

In this case, one obtains $a_{ki} = 2P_{ik,k} - 1$, for $i \neq k$, $a_{ki} = -a_{ik}$, and $|a_{ki}| \leq 1$. In biological context, if k and i, j represent child and parents, respectively, then Volterra QSO could be interpreted as the child's trait only inherit from their parents. Further, studying the dynamical systems of Volterra QSO for the associated biological population, one actually describes the survival of each genotypes. Some of the trait could survive, but some of them could vanish. The paper written by Embong and Mukhamedov (2023); Ganikhodzhaev (1993b); Ganikhodjaev *et al.* (2015) are some of the published papers that devoted to Volterra QSOs.

In Rozikov and Zhamilov (2011); Castanos *et al.* (2018); Chye *et al.* (2021); Embong and

Chye@ Mohd Hairie Lim (2022) a new class of QSO of two-sex population were defined, thus in this paper we want to continue the research that have been done in Chye *et al.* (2021); Embong and Chye@ Mohd Hairie Lim (2022). Recall that a QSO $V : S^{n-1} \times S^{v-1} \to S^{n-1} \times S^{v-1}$ is called bV-QSO if

$$\mathbf{x}_k^{'} \leq^b \mathbf{x},$$

and the heredity coefficients satisfy $P_{ij,k}^{(m)} = 0$ for any $k \notin \{i, j\}$. Here, we consider $n = v = 3$.

3. Two-Dimensional $b-$Bistochastic Volterra

This section provides the canonical form of bV-QSO on two-dimensional simplex. In this case i.e., $n = 3$, the simplex given in Eq. (1) can be written as follows:

$$S^2 = \left\{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 | x_1, x_2, x_3 \geq 0, x_1 + x_2 + x_3 = 1\right\}. \tag{5}$$

For simplicity, the domain $S^2 \times S^2$ is denoted as

$$\mathbb{D}^2 = \{(x_1, x_2, y_1, y_2) : x_1, x_2, y_1, y_2 \in [0, 1], x_1 + x_2 \leq 1, y_1 + y_2 \leq 1\}.$$

The following lemma is presented in order to prove the main result in this paper.

Lemma 3.1. *Let $f : \mathbb{D}^2 \mapsto \mathbb{R}$ be a quadratic polynomial where $\mathbb{D}^2 = \{(x_1, x_2, y_1, y_2) : x_1, x_2, y_1, y_2 \in [0, 1], x_1 + x_2 \leq 1, y_1 + y_2 \leq 1\}$. If $f(x, y, z, w) \leq 0$ on any boundaries of $\mathbb{D}^2$ and*

(1) if the critical point $(x_0, y_0, z_0, w_0) \in \mathbb{D}^2$ with
- *(i) (x_0, y_0, z_0, w_0) is a maximum point and $f(x_0, y_0, z_0, w_0) \leq 0$, then for all $(x, y, z, w) \in \mathbb{D}^2$ one has $f(x, y, z, w) \leq 0$,*
- *(ii) (x_0, y_0, z_0, w_0) is a saddle point and $f(x_0, y_0, z_0, w_0) \leq 0$, then for all $(x, y, z, w) \in \mathbb{D}^2$ one has $f(x, y, z, w) \leq 0$,*
- *(iii) (x_0, y_0, z_0, w_0) is a minimum point, then for all $(x, y, z, w) \in \mathbb{D}^2$ one has $f(x, y, z, w) \leq 0$,*

(2) if the critical point $(x_0, y_0, z_0, w_0) \notin \mathbb{D}^2$, then for any $(x, y, z, w) \in \mathbb{D}^2$ one has $f(x, y, z, w) \leq 0$.

Proof. The proof is obvious since the given function is either paraboloid or saddle roof, hence the maximum could happen either at the extrema or at the boundaries.□

Let $\{\overset{(f)}{P}_{ij,k}\}$, $\{\overset{(m)}{P}_{ij,k}\}$ be the heredity coefficients for a given bV-QSO. In order to simplify the written canonical form of 2D bV-QSO, the following notations are defined:

$$\begin{array}{llllll} \overset{(f)}{P}_{11,1} = A_1, & \overset{(f)}{P}_{13,1} = C_1, & \overset{(f)}{P}_{23,1} = E_1, & \overset{(f)}{P}_{11,2} = A_2, & \overset{(f)}{P}_{13,2} = C_2, & \overset{(f)}{P}_{23,2} = E_2, \\ \overset{(f)}{P}_{12,1} = B_1, & \overset{(f)}{P}_{22,1} = D_1, & \overset{(f)}{P}_{33,1} = F_1, & \overset{(f)}{P}_{12,2} = B_2, & \overset{(f)}{P}_{22,2} = D_2, & \overset{(f)}{P}_{33,2} = F_2, \end{array} \tag{6}$$

and

$$\begin{array}{llllll} \overset{(m)}{P}_{11,1} = a_1, & \overset{(m)}{P}_{13,1} = c_1, & \overset{(m)}{P}_{23,1} = e_1, & \overset{(m)}{P}_{11,2} = a_2, & \overset{(m)}{P}_{13,2} = c_2, & \overset{(m)}{P}_{23,2} = e_2, \\ \overset{(m)}{P}_{12,1} = b_1, & \overset{(m)}{P}_{22,1} = d_1, & \overset{(m)}{P}_{33,1} = f_1, & \overset{(m)}{P}_{12,2} = b_2, & \overset{(m)}{P}_{22,2} = d_2, & \overset{(m)}{P}_{33,2} = f_2. \end{array} \tag{7}$$

To keep further in this section, let us get some auxialiary results. Note that the boundaries of $\mathbb{D}^2$ can be described as follows:

$$\begin{aligned} &\partial_1 = \{(x_1,x_2,y_1,y_2) \in \mathbb{D}^2; x_1 = 0\}, && \partial_2 = \{(x_1,x_2,y_1,y_2) \in \mathbb{D}^2; x_2 = 0\},\\ &\partial_3 = \{(x_1,x_2,y_1,y_2) \in \mathbb{D}^2; x_2 = 1-x_1\}, && \partial_4 = \{(x_1,x_2,y_1,y_2) \in \mathbb{D}^2; y_1 = 0\},\\ &\partial_5 = \{(x_1,x_2,y_1,y_2) \in \mathbb{D}^2; y_2 = 0\}, && \partial_6 = \{(x_1,x_2,y_1,y_2) \in \mathbb{D}^2; y_2 = 1-y_1\}. \end{aligned} \tag{8}$$

Let $f : \mathbb{D}^2 \to \mathbb{R}$ is defined as follows:

$$f(x_1,x_2,y_1,y_2) = A_1x_1y_1 + A_2x_1y_1 + B_2(x_1y_2 + x_2y_1) + D_2x_2y_2 - x_1 - x_2. \tag{9}$$

The next proposition shows that f takes negative values on $\mathbb{D}^2$.

Proposition 3.2. Let f be given by

$$f(x_1,x_2,y_1,y_2) = A_1x_1y_1 + A_2x_1y_1 + B_2(x_1y_2 + x_2y_1) + D_2x_2y_2 - x_1 - x_2,$$

then for any $(x_1,x_2,y_1,y_2) \in \mathbb{D}^2$, $f(x_1,x_2,y_1,y_2) \le 0$.

Proof. To prove this proposition, Lemma 3.1 will be applied. Thus the first part of this proof, we will show that each side of $\mathbb{D}^2$ (see Eq. (8)), the functional f takes negative values. In particular, we prove for ∂_1 and ∂_3 only, since the other follows the same steps. Second, the critical point is computed and turns out that it is outside of the interested domain.

Now, let us take any $(x_1,x_2,y_1,y_2) \in \partial_1$, then by definition $x_1 = 0$. Since $y_1 + y_2 \le 1$ and stochasticity of the heredity coefficients imply that

$$f(0,x_2,y_1,y_2) = x_2(B_2y_1 + D_2y_2 - 1), \le x_2(y_1 + y_2 - 1) \le 0.$$

Hence, f takes negative values on ∂_1. Next, consider $(x_1,x_2,y_1,y_2) \in \partial_3$, which implies $x_2 = 1 - x_1$, then equation becomes,

$$f(x_1,1-x_1,y_1,y_2) = x_1y_1(A_1 + A_2) + B_2(x_1y_2 + y_1(1-x_1)) + D_2y_2(1-x_1) - 1.$$

Using the same argument as the first case, f can be further estimated by

$$f(x_1,1-x_1,y_1,y_2) \le x_1y_1 + x_1y_2 + y_1 - y_1x_1 + y_2 - y_2x_1 - 1 = y_1 + x_2 - 1 \le 0.$$

The other sides can be done in similar manners.

Further, let us show that the critical point of f is outside of the domain $\mathbb{D}^2$. By applying partial derivative with respect to x and y on f, the following equations are obtained,

$$\begin{aligned} f_{x_1} &= y_1(A_1 + A_2) + B_2y_2 - 1, & f_{x_2} &= B_2y_1 + D_2y_2 - 1,\\ f_{y_1} &= x_1(A_1 + A_2) + B_2x_2, & f_{y_2} &= B_2x_1 + D_2x_2. \end{aligned}$$

Finding the critical point is equivalance solving $f_{x_1} = 0, f_{x_2} = 0, f_{y_1} = 0$ and $f_{y_2} = 0$. By assuming $x_1 \in [0,1]$ (or else, we get a solution outside of the $\mathbb{D}^2$), then clearly $x_2 = -\frac{B_2x_1}{D_2}$ is the solution for $f_{y_2} = 0$ in which outside of the interested domain. By applying Lemma 3.1 $f(x_1,x_2,y_1,y_2) \le 0$ for any (x_1,x_2,y_1,y_2) in the interior domain. Thus, all the cases above prove the proposition.□

Theorem 3.3. *Let V be a QSO defined on $S^2 \times S^2$. Then V is a bV-QSO if and only if it has the following form:*

(i) $x_1' = A_1x_1y_1,$

(ii) $x'_2 = A_2x_1y_1 + B_2(x_1y_2 + x_2y_1) + D_2x_2y_2$,
(iii) $y'_1 = x_1y_1 + b_1(x_1y_2 + x_2y_1) + c_1(x_1(1 - y_1 - y_2) + y_1(1 - x_1 - x_2))$,
(iv) $y'_2 = b_2(x_1y_2 + x_2y_1) + x_2y_2 + e_2(x_2(1 - y_1 - y_2) + y_2(1 - x_1 - x_2))$.

Proof. The proof will start with if part first. Suppose that a QSO, V is a bV-QSO. From the definition of bV-QSO, one has

$$V(\mathbf{x},\mathbf{y}) = \begin{cases} x'_1 = \overset{(f)}{P}_{11,1}x_1y_1 + \overset{(f)}{P}_{12,1}x_1y_2 + \overset{(f)}{P}_{13,1}x_1y_3 + \overset{(f)}{P}_{21,1}x_2y_1 + \overset{(f)}{P}_{22,1}x_2y_2 \\ \quad + \overset{(f)}{P}_{23,1}x_2y_3 + \overset{(f)}{P}_{31,1}x_3y_1 + \overset{(f)}{P}_{32,1}x_3y_2 + \overset{(f)}{P}_{33,1}x_3y_3 \\ x'_2 = \overset{(f)}{P}_{11,2}x_1y_1 + \overset{(f)}{P}_{12,2}x_1y_2 + \overset{(f)}{P}_{13,2}x_1y_3 + \overset{(f)}{P}_{21,2}x_2y_1 + \overset{(f)}{P}_{22,2}x_2y_2 \\ \quad + \overset{(f)}{P}_{23,2}x_2y_3 + \overset{(f)}{P}_{31,2}x_3y_1 + \overset{(f)}{P}_{32,2}x_3y_2 + \overset{(f)}{P}_{33,2}x_3y_3 \\ x'_3 = \overset{(f)}{P}_{11,3}x_1y_1 + \overset{(f)}{P}_{12,3}x_1y_2 + \overset{(f)}{P}_{13,3}x_1y_3 + \overset{(f)}{P}_{21,3}x_2y_1 + \overset{(f)}{P}_{22,3}x_2y_2 \\ \quad + \overset{(f)}{P}_{23,3}x_2y_3 + \overset{(f)}{P}_{31,3}x_3y_1 + \overset{(f)}{P}_{32,3}x_3y_2 + \overset{(f)}{P}_{33,3}x_3y_3 \\ y'_1 = \overset{(m)}{P}_{11,1}x_1y_1 + \overset{(m)}{P}_{12,1}x_1y_2 + \overset{(m)}{P}_{13,1}x_1y_3 + \overset{(m)}{P}_{21,1}x_2y_1 + \overset{(m)}{P}_{22,1}x_2y_2 \\ \quad + \overset{(m)}{P}_{23,1}x_2y_3 + \overset{(m)}{P}_{31,1}x_3y_1 + \overset{(m)}{P}_{32,1}x_3y_1 + \overset{(m)}{P}_{33,1}x_3y_3 \\ y'_2 = \overset{(m)}{P}_{11,2}x_1y_1 + \overset{(m)}{P}_{12,2}x_1y_2 + \overset{(m)}{P}_{13,1}x_1y_3 + \overset{(m)}{P}_{21,2}x_2y_1 + \overset{(m)}{P}_{22,2}x_2y_2, \\ \quad + \overset{(m)}{P}_{23,2}x_2y_3 + \overset{(m)}{P}_{31,2}x_3y_1\overset{(m)}{P}_{32,2}x_3y_2 + \overset{(m)}{P}_{33,2}x_3y_3 \\ y'_3 = \overset{(m)}{P}_{11,3}x_1y_1 + \overset{(m)}{P}_{12,3}x_1y_2 + \overset{(m)}{P}_{13,3}x_1y_3 + \overset{(m)}{P}_{21,3}x_2y_1 + \overset{(m)}{P}_{22,3}x_2y_2, \\ \quad + \overset{(m)}{P}_{23,3}x_2y_3 + \overset{(m)}{P}_{31,3}x_3y_1\overset{(m)}{P}_{32,3}x_3y_2 + \overset{(m)}{P}_{33,3}x_3y_3 \end{cases} \tag{10}$$

where $(\mathbf{x},\mathbf{y}) = ((x_1, x_2, x_3), (y_1, y_2, y_3)) \in S^2 \times S^2$. By assumption that V is a bV-QSO the following inequalities are obtained:

$$V(\mathbf{x},\mathbf{y})_1 \leq x_1, \tag{11}$$
$$V(\mathbf{x},\mathbf{y})_1 + V(\mathbf{x},\mathbf{y})_2 \leq x_1 + x_2, \tag{12}$$

which satisfy for all $(\mathbf{x},\mathbf{y}) \in S^2 \times S^2$. From Eq. (11) and the properties of heredity coefficients imply the following inequality:

$$\begin{aligned} \overset{(f)}{P}_{11,1}x_1y_1 + \overset{(f)}{P}_{12,1}(x_1y_2 + x_2y_1) + \overset{(f)}{P}_{13,1}\,(x_1(1 - y_1 - y_2) + y_1(1 - x_1 - x_2)) \\ +\overset{(f)}{P}_{22,1}x_2y_2 + \overset{(f)}{P}_{23,1}x_2(1 - y_1 - y_2) + y_2(1 - x_1 - x_2) \\ +\overset{(f)}{P}_{33,1}(1 - x_1 - x_2)(1 - y_1 - y_2) - x_1 \leq 0. \end{aligned} \tag{13}$$

Take $((0, 0, 1), (0, 0, 1)) \in S^2 \times S^2$ then Eq. (13) becomes $\overset{(f)}{P}_{33,1} \leq 0$. Due to positivity of the heredity coefficients, one infers that $\overset{(f)}{P}_{33,1} = 0$. Using similar step, one can choose $((0, 0, 1), (0, 1, 0))$, $((0, 0, 1), (1, 0, 0))$, $((0, 1, 0), (1, 0, 0))$ and $((0, 1, 0), (0, 1, 0))$ to get $P^{(f)}_{23,1} =$

$0, P^{(f)}_{13,1} = 0$, $P^{(f)}_{12,1} = 0$ and $P^{(f)}_{22,1} = 0$, respectively. Therefore x'_1 takes the following form:

$$x'_1 = \overset{(f)}{P}_{11,1} x_1 y_1, \tag{14}$$

which proves (i). Next, from Eqs. (12) and (14) yield

$$\begin{aligned} &\overset{(f)}{P}_{11,1} x_1 y_1 + \overset{(f)}{P}_{11,2} x_1 y_1 + \overset{(f)}{P}_{12,2} x_1 y_2 + \overset{(f)}{P}_{13,2} x_1 y_3 + \overset{(f)}{P}_{21,2} x_2 y_1 + \overset{(f)}{P}_{22,2} x_2 y_2 \\ &\quad + \overset{(f)}{P}_{23,2} x_2 y_3 + \overset{(f)}{P}_{31,2} x_3 y_1 + \overset{(f)}{P}_{32,2} x_3 y_2 + \overset{(f)}{P}_{33,2} x_3 y_3 - x_1 - x_2 \leq 0. \end{aligned} \tag{15}$$

By choosing the approriate points in $S^2 \times S^2$, one reaches to a conclusion that $\overset{(f)}{P}_{33,2} = 0$, $\overset{(f)}{P}_{23,2} = 0$ and $\overset{(f)}{P}_{13,2} = 0$. Hence, substitute these coefficients into Eq. (10) to get

$$x'_2 = \overset{(f)}{P}_{11,2} x_1 y_1 + \overset{(f)}{P}_{12,2} (x_1 y_2 + x_2 y_1) + \overset{(f)}{P}_{22,2} x_2 y_2, \tag{16}$$

which proves (ii). Moreover, based on the definition of bV-QSO we obtain the following results:

$$\begin{aligned} \overset{(m)}{P}_{11,1} &= \overset{(m)}{P}_{22,2} = 1, \\ \overset{(m)}{P}_{11,2} = \overset{(m)}{P}_{13,2} = \overset{(m)}{P}_{22,1} = \overset{(m)}{P}_{23,1} = \overset{(m)}{P}_{33,1} &= \overset{(m)}{P}_{33,2} = 0. \end{aligned}$$

Therefore y'_1 and y'_2 take the following forms:

$$y'_1 = x_1 y_1 + \overset{(m)}{P}_{12,1} (x_1 y_2 + x_2 y_1) + \overset{(m)}{P}_{13,1} (x_1 (1 - y_1 - y_2) + y_1 (1 - x_1 - x_2)), \tag{17}$$

$$y'_2 = \overset{(m)}{P}_{12,2} (x_1 y_2 + x_2 y_1) + x_2 y_2 + \overset{(m)}{P}_{23,2} (x_2 (1 - y_1 - y_2) + y_2 (1 - x_1 - x_2)). \tag{18}$$

By applying the notations defined in Eqs. (6) and (7), the forms in (i), (ii), (iii), and (iv) are obtained.□

Next, suppose that a QSO, V takes the form given by (i), (ii), (iii), and (iv). The aim is to show that the given V is a bV-QSO. From (i), one has $V(\mathbf{x}, \mathbf{y})_1 = A_1 x_1 y_1$. Taking into account $A_1, x_1, y_1 \in [0, 1]$ then one infers that $A_1 x_1 y_1 \leq x_1$. Hence,

$$V(\mathbf{x}, \mathbf{y})_1 \leq x_1. \tag{19}$$

Next, one needs to show

$$V(\mathbf{x}, \mathbf{y})_1 + V(\mathbf{x}, \mathbf{y})_2 \leq x_1 + x_2. \tag{20}$$

One can see that $V(\mathbf{x}, \mathbf{y})_1 + V(\mathbf{x}, \mathbf{y})_2 - x_1 - x_2$ is equivalent to $f(x_1, x_2, y_1, y_2)$ where $(x_1, x_2, y_1, y_2) \in \mathbb{D}^2$ given by Eq. (9). Therefore based on Proposition 3.2, one concludes $f(x_1, x_2, y_1, y_2) \leq 0$ which implies Eq. (20). The statements (iii) and (iv) are direct implications of the definition of Volterra QSO.

Remark 3.4. This class of bV-QSO differ from Volterra QSOs. For instance, in Volterra QSOs, the value for $P^{(f)}_{22,2} = 1$, but in case of bV-QSO, $P^{(f)}_{22,2}$ takes any value from the interval $[0, 1]$.

4. Conclusion

In this paper, we consider a class of two-sex population namely b-bistochastic-Volterra QSOs or simply bV-QSOs. Full description of the canonical form bV-QSOs on two-dimensional simplex were given. For future study, one can use the form to construct the suitable Lyapunov Function associated with the operators which helps the study of their dynamical behaviour. Moreover, this will be a first step for further study on any finite dimension of such operators.

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