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PERFECT CODES IN INDUCED SUBGRAPH OF UNIT GRAPH ASSOCIATED WITH SOME COMMUTATIVE RINGS

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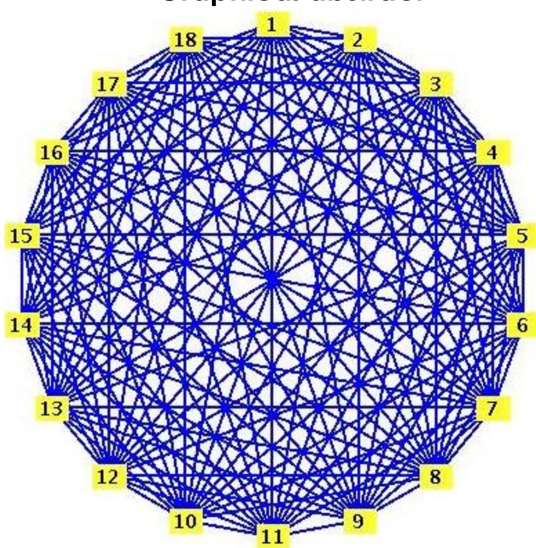
Graphical abstract



Abstract

The unit graph associated with a ring R is the graph whose vertices are elements of R, and two different vertices x and y are adjacent if and only if $x+y\in U(R)$, where $U(R)$ is the set of unit elements of R. The aim of this paper is to present the perfect codes in induced subgraph of unit graph associated with some commutative rings with unity in which its vertex set is $U(R)$. We characterize some families of commutative rings with induced subgraphs of unit graphs accepting the non-trivial perfect codes, and some other families of commutative rings with induced subgraphs of unit graphs which do not accept perfect codes.

Keywords: Commutative ring, unit of ring, unit graph, induced subgraph, perfect code

Abstrak

Graf unit yang dikaitkan dengan gelanggang R adalah graf yang mana bucunya adalah unsur bagi R, dan dua bucu yang berbeza x dan y adalah bersebelahan jika dan hanya jika $x+y\in U(R)$, yang mana $U(R)$ adalah set bagi unsur- unsur unit bagi R. Tujuan makalah ini adalah untuk memberikan kod-kod sempurna dalam subgraf teraruh bagi graf unit bersekutu dengan beberapa gelanggang kalis tukar tertib dengan identiti yang mana set bucunya ialah $U(R)$. Kami mencirikan beberapa keluarga gelanggang kalis tukar tertib dengan subgraf teraruh bagi graf-graf unit yang menerima kod-kod sempurna bukan remeh, dan beberapa keluarga gelanggang kalis tukar tertib lain dengan subgraf teraruh bagi graf-graf unit yang tidak menerima kod sempurna.

Kata kunci: Gelanggang kalis tukar tertib, unit bagi gelanggang, graf unit, subgraf teraruh, kod sempurna

1.0 INTRODUCTION

In this paper, R indicates the commutative ring with unity, $O(R)$ shows the order of R, and $U(R)$ indicates the set of units of R. The induced subgraph of unit graph with vertex set $U(R)$ is denoted by $\Gamma(R)$ and the order of the perfect code C is denoted by $|C|$.

Ashrafi *et al*. [1] defined the unit graph of ring R and proved some results on the properties of the unit graph in terms of connectivity, planarity, girth and diameter. Later, Akbari *et al*. [2] characterized the non-commutative ring having unit graphs as *r*-partite graph. Furthermore, some other properties of the unit graphs have introduced by many authors which include the Hamiltonian property in [3], dominating number in [4], girth in [5] and the diameter in [6]. In 2020, Hashemi *et al*. [7] showed that the dominating number of the unit graph is $\gamma(R)=1$ if the ring R is a division ring and $\gamma(R)=2$ if R is a local ring with a non-zero maximal ideal.

Coding theory proposed in the twentieth century as a problem in engineering concerning the efficient transmission of information. In [8], Shannon introduced the concept of channel capacity. The channel capacity is a measure of the amount of information that can be conveyed between the input x and output y of the channel. Shannon also proved that there exists channel coding schemes that can achieve arbitrary very low probability of error as long as the transmission rate is below the channel capacity. The goal of coding theory is to design such an error-correcting codes with rates as close to the channel capacity that can achieve low probability of errors. In general, electronic information can be considered as strings containing zeroes and ones. From this point of view, coding theory was widely studied using binary field as its alphabet.

Perfect code plays a pivotal role in the fast growing of error-correcting codes theory. In [9], Hamming constructed perfect binary single-error-correcting codes of length $2n-1$ for some integer n. In a graph Γ, a subset $C \subseteq V(\Gamma)$ is a single-error-correcting code if the distance between any of two vertices of Γ is at least three. The research on perfect codes in graph was started by Biggs [10] to generalize the classical notion of the perfect Hamming error-correcting codes. Furthermore, the author conducted a research to initiate the connectivity between perfect codes and graphs. He investigated the non-trivial perfect codes in distance transitive graph. Later, Kratochvil [11] proved that the second power of a graph, Γ^2 admits a 1-perfect code of order $|V(\Gamma)|$ if $\Gamma \cong \bar{\Gamma}$. In another paper, Kratochvil [12] proved the existence of perfect codes in Cartesian product of graphs, $\Gamma \times \Gamma'$. Also, it was proven that a complete bipartite graph Γ does not accept the non-trivial perfect codes if Γ contains at least three vertices. Dvořáková-Rulićová [13] studied the *t*-perfect codes in regular graphs and proved the existence of *t*-perfect codes in a (*d*+*1*)-regular graph containing an induced subgraph with maximum degree *d*. Later on, Cull and Nelson [14] proved that infinite classes of graphs can be defined by the concept of Tower of Hanoi Puzzle so that a perfect single-error-correcting code is supported by each graph.

A subset C of the vertices in a graph Γ is said to be a perfect code if the closed neighbourhoods of the code words $c \in C$ partition the vertex set [12]. Recently, there are some numbers of researches focused on perfect codes in graphs associated with groups. Ma *et al*. [15] found the lower and upper bounds for the order of the subset H of a group G to be a perfect code in the power graph of G. In addition, they characterized the group G in which the enhanced power graph of G admits the trivial perfect codes. In [16], Ma established some results in determining the perfect codes and total perfect codes accepted by proper reduced power graphs associated with finite groups. However, Chen *et al.* [17] characterized the finite groups in which their associated Cayley graphs do not admit the non-trivial subgroup perfect codes. In addition, results on perfect codes in Cayley graphs associated with groups can be found in [18-20].

Furthermore, in 2021, Zaid *et al.* [21] determined the *k*-perfect codes in commuting zero divisor graphs associated with the ring of 2×2 matrices over $\mathbb{Z}_3$ and $\mathbb{Z}_4$. Since there are limited numbers of researches on the perfect codes in graphs associated with rings, therefore in this paper, we aim to determine the perfect codes in induced subgraph of unit graph associated with some commutative rings with unity.

In Section 3, we provide some examples and remarks on induced subgraphs of unit graphs to determine the perfect codes in these graphs. In Section 4, we characterize the commutative rings with unity to determine whether their associated induced subgraphs of unit graphs are accepting the perfect codes or not. We illustrate that if R is an integral domain having exactly two unit elements, then an order 2 perfect code exists on induced subgraph of unit graph (Proposition 4.1). We further illustrate that if R is a division ring of prime order or a division ring of characteristics 0 or any other commutative rings with $Char(R)=O(R)=3p$, p is prime, then no perfect code exists on induced subgraph of unit graph (Theorem 4.1). Moreover, we prove the existence of an order 2 perfect code in induced subgraph of unit graph associated with a local ring with $Char(R) = O(R) = 3^m, m \geq 2$ (Theorem 4.2). Lastly, we show that if R is a ring with $Char(R) = O(R) = 2k,\ k \geq 1$, then a perfect code of order $U(R)$ exists in induced subgraph of unit graph (Theorem 4.3).

3.0 PRELIMINARIES

This section provides some basic definitions which are used in this research. The definition of a unit graph is first given.

Definition 2.1 [1] Unit Graph
Let R be a ring and $U(R)$ the set of unit elements of R. The unit graph of R, denoted by $\Gamma(R)$, is a graph obtained by setting all elements of R to be the vertices and defining distinct vertices x and y to be adjacent if and only if $x+y \in U(R)$.

Definition 2.2 [22] Induced Subgraph
A graph K is said to be a subgraph of a graph Γ if $V(K) \subseteq V(\Gamma)$ and $E(K) \subseteq E(\Gamma)$. The subgraph K is called an induced subgraph of Γ if whenever x and y are vertices of K and $xy \in E(\Gamma)$, then this implies that $xy \in E(K)$.

Thus, an induced subgraph of a unit graph with vertex set $U(R)$ is a graph which is obtained by removing all non-unit vertices.

Definition 2.3 [10] Closed Neighbourhood of a Vertex
Let $x \in V(\Gamma)$ and k a non-negative integer, then the closed neighborhood of the vertex x with distance k, $S_k(x)$ is defined to be the set of vertices of Γ whose distance from x is not greater than k.

Definition 2.4 [10] *K*-Perfect Code
A k-perfect code in Γ is a subset C of $V(\Gamma)$ such that the sets $S_k(x)$, as x runs through C, form a partition of $V(\Gamma)$.

This is equivalent to say that $S_k(x) \cap S_k(y) = \phi$ for all distinct $x, y \in C$ and $\cup S_k(x) = V(\Gamma)$ for all $x \in C$. If $k=1$, then a 1-perfect code is simply called a perfect code.

In the next section, some examples and remarks are provided to visualize the concept of a perfect code in induced subgraph of a unit graph with vertex set unit elements of a commutative ring with unity.

3.0 EXAMPLES AND REMARKS

In this section, we provide three different examples and three remarks to illustrate the perfect codes in induced subgraphs of unit graphs associated with some commutative rings with unity.

Example 3.1 Let $R = \mathbb{Z}_{19}$. Then, $\Gamma(R)$ consists of a component of complete 9-partite graph, given in Figure 1.

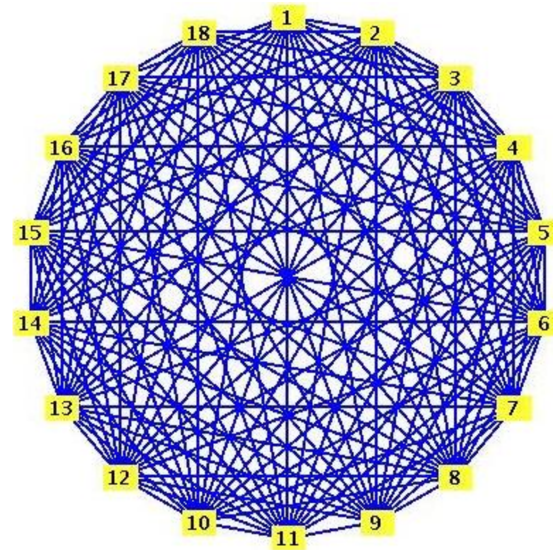


Figure 1 The induced subgraph of unit graph of ring $R = \mathbb{Z}_{19}$.

Let C denote the perfect code, then no code C of order $1 \leq |C| \leq 18$ is perfect in $\Gamma(R)$.

Example 3.2 Let $R = \mathbb{Z}_{27}$. Then, $\Gamma(R)$ consists of two components of complete graph K_9, given in Figure 2.

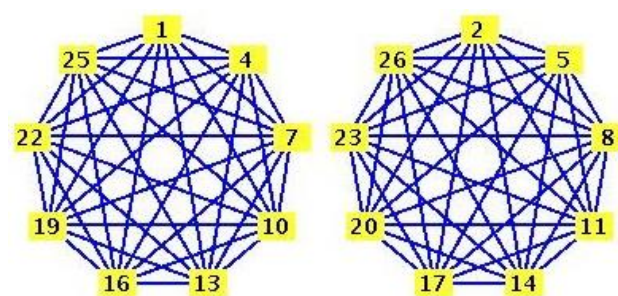


Figure 2 The induced subgraph of the unit graph of ring $R = \mathbb{Z}_{27}$.

By the definition of a perfect code, $C = \{r_i, r_j\}$ such that $r_i + r_j \notin U(R)$ is a perfect code in $\Gamma(R)$.

Example 3.3 Let $R = \mathbb{Z}_{33}$. Then, $\Gamma(R)$ consists of two components of complete 5-partite graph, given in Figure 3.

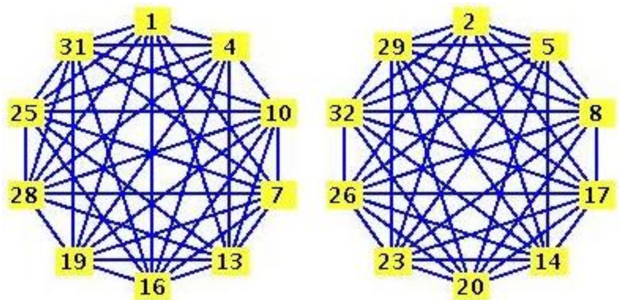


Figure 3 The induced subgraph of unit graph of ring $R = \mathbb{Z}_{33}$

Let $C \subseteq V(\Gamma(R))$ be a code, then no code C of any order is a perfect code in $\Gamma(R)$.

From the above examples we established the following remarks.

Remark 3.1 If $Char(R) = O(R) = 3^k$, where $k \geq 2$, then
(i) $\Gamma(R) = 2K_m$, where $m = 3^{k-1}$.
(ii) $C = \{r_i, r_j\}$, where $r_i + r_j \notin U(R)$.

Remark 3.2 If $O(R) = p$, where $p \geq 5$ is prime, then
(i) $\Gamma(R)$ consists of a component of complete $\frac{p-1}{2}$-partite graph with partite set of order 2.
(ii) $\Gamma(R)$ does not accept the perfect code.

Remark 3.3 If $Char(R) = O(R) = 3p$, where $p \geq 5$ is prime, then
(i) $\Gamma(R) = 2K_{m_1, m_2, \ldots, m_k}$, where $K_{m_1, m_2, \ldots, m_k}$ is a complete multipartite graph with partite set of order 2.
(ii) $\Gamma(R)$ does not accept the perfect code.

4.0 RESULTS AND DISCUSSIONS

In this section, some results are proven to characterize the families of commutative rings with induced subgraphs of unit graphs accepting the non-trivial perfect codes, and some other families of commutative rings with induced subgraphs of unit graphs which do not accept the perfect codes.

Proposition 4.1 Let $\Gamma(R)$ be the induced subgraph of a unit graph associated with an integral domain R with $|U(R)| = 2$. Then $|C| = 2$.

Proof: Assume R is an integral domain, thus R contains the identity element e. Since $|U(R)| = 2$, this implies $V(\Gamma(R)) = \{e, r\}$. By the definition of a unit graph, $e + r \notin U(R)$, therefore $\Gamma(R)$ is a $\bar{K}_2$ graph. Next, assume that $C \subseteq V(\Gamma(R))$ is a code, then $C = \{e, r\}$ is the order 2 perfect code in $\Gamma(R)$, since $S_1(e)$ and $S_1(r)$ partition $V(\Gamma(R))$ into two disjoint sets, that is $S_1(e) \cap S_1(r) = \phi$ and $S_1(e) \cup S_1(r) = V(\Gamma(R))$.

Theorem 4.1 Let $\Gamma(R)$ be the induced subgraph of a unit graph associated with R. Then $\Gamma(R)$ does not accept the perfect code if R is any of the following rings:
(i) R is a division ring with $O(R) = p$, where $p \geq 5$ is prime.
(ii) R is a commutative ring with $Char(R) = O(R) = 3p$, where $p \geq 5$ is prime.
(iii) R is a division ring with $Char(R) = 0$.

Proof: (i) Assume R is a division ring with $O(R) = p$, where $p \geq 5$ is prime. By the definition of a division ring, every $0 \neq r_i \in R$ is a unit. According to $\Gamma(R)$, $V(\Gamma(R)) = U(R)$ and two distinct vertices r_i and r_j form an edge if and only if $r_i + r_j \in U(R)$, otherwise r_i is not adjacent to r_j. Since $|V(\Gamma(R))| = p - 1$, then $r_i + r_j \notin U(R)$ for $i = 1, 2, 3, \ldots, p-1$ and $j = p - i$. This yields that $\Gamma(R)$ consists of a component of complete $\frac{p-1}{2}$-partite graph with partite set of order 2 as $\Gamma(R) = K_{m_1, m_2, \ldots, m_k}$. Next, let $C \subseteq V(\Gamma(R))$ be a code. Then there is no code C of order m, where $1 \leq m \leq p-1$ to be a perfect code in $\Gamma(R)$, since
(a) if $m = 1$, then $S_1(r_k) \neq V(\Gamma(R))$, for $r_k \in C$.
(b) if $1 < m \leq p-1$, then $S_1(r_k) \cap S_1(r_l) \neq \phi$ for all distinct $r_k, r_l \in C$.

(ii) Assume that R is a commutative ring with $Char(R) = O(R) = 3p$, where $p \geq 5$ is prime, then $|U(R)| = 12k - 4$ or $|U(R)| = 12k$, where $k \geq 1$. By $\Gamma(R)$, $V(\Gamma(R)) = U(R)$ and two distinct vertices r_i and r_j form an edge if and only if $r_i + r_j \in U(R)$. Since, $|V(\Gamma(R))| = 2(2(3k-1))$ or $|V(\Gamma(R))| = 2(2(3k))$, this shows that $\Gamma(R)$ is a graph consisting of two components of either complete $(3k-1)$-partite graph or complete $3k$-partite graph with partite set of order 2. Next, suppose that $C \subseteq V(\Gamma(R))$ is a code, then no code

C is perfect in $\Gamma(R)$ because of the following reasons:
If $|C|=1$, then $S_1(r_k)\neq V(\Gamma(R))$ for $r_k\in C$, since the vertices in the same partite set are not adjacent.
If $|C|>1$, then $S_1(r_k)\cap S_1(r_l)\neq\phi$ for all distinct $r_k,r_l\in C$, since the vertices in different partite sets are adjacent in $\Gamma(R)$.
(iii) Assume R is a division ring with $Char(R)=0$, that is $R=\{r_i\}_{i=1}^{\infty}$. By the definition of a division ring, every $0\neq r_i\in R$ is a unit. Let $U(R)$ denote the set of all $0\neq r_i\in R$, then $V(\Gamma(R))=U(R)$ and $E(\Gamma(R))=\{\{r_i,r_j\}: r_i+r_j\in U(R)\}$. Since R is a division ring with $Char(R)=0$, then $r_i+r_j\notin U(R)$ only if $r_i+r_j=0$. This implies that $\Gamma(R)$ is an infinite regular graph. Next, let $C\subseteq V(\Gamma(R))$ be a code, then no code C is perfect in $\Gamma(R)$, since if $|C|=1$, then $S_1(r_k)=\{r_i\in V(\Gamma(R)): d(r_k,r_i)\leq 1\}\neq V(\Gamma(R))$. However, if $|C|>1$, then $S_1(r_k)\cap S_1(r_l)\neq\phi$ for all distinct $r_k,r_l\in C$.

Theorem 4.2 Let R be a local ring with $Char(R)=O(R)=3^m$, $m\geq 2$ and $\Gamma(R)$ be the induced subgraph of $\Gamma(R)$. If C is the perfect code in $\Gamma(R)$, then $|C|=2$.
Proof: Assume R is a local ring with $Char(R)=O(R)=3^m$, where $m\geq 2$, that is $R=\{r_i: i=1,2,3,\ldots,3^m\}$. Further, assume that $U(R)$ is the set of units of R, then by the definition of a unit, it can be obtained that $|U(R)|=2\cdot 3^{m-1}$. By the induced subgraph of unit graph, $|V(\Gamma(R))|=2\cdot 3^{m-1}$ and two distinct vertices r_i and r_j form an edge if and only if $r_i+r_j\in U(R)$. It follows that the vertices $r_{3i-2}+r_{3j-2}\in U(R)$ and $r_{3i-1}+r_{3j-1}\in U(R)$, but $r_{3i-2}+r_{3i-1}\notin U(R)$ for all $1\leq i<j\leq 3^{m-1}$. Therefore, $\Gamma(R)$ consists of two components of complete graph of order 3^{m-1}, i.e, $\Gamma(R)=\cup_{k=1}^{2}K_{3^{m-1}}=2K_{3^{m-1}}$, $m\geq 2$. Suppose that $\Gamma_1(R)=K_{3^{m-1}}$ and $\Gamma_2(R)=K_{3^{m-1}}$ are the components of $\Gamma(R)$, then $d(r_i,r_j)=1$ for distinct vertices r_i and r_j in $\Gamma_1(R)$ and $\Gamma_2(R)$. Let $C\subseteq V(\Gamma(R))$ be a code, then the code $C=\{r_i,r_j\}$, where $r_i\in V(\Gamma_1(R))$ and $r_j\in V(\Gamma_2(R))$ is perfect in $\Gamma(R)$, since the calculation of the closed neighbourhood of the code words r_i and r_j with distance not more than 1 gives that $S_1(r_i)\cap S_1(r_j)=\phi$ and $S_1(r_i)\cup S_1(r_j)=V(\Gamma(R))$ for $r_i,r_j\in C$. Hence, $|C|=2$.

Theorem 4.3 Let R be a ring with $Char(R)=O(R)=2k$, where $k\geq 1$ and $\Gamma(R)$ be the induced subgraph of $\Gamma(R)$. If C is the perfect code in $\Gamma(R)$, then $|C|=|U(R)|$.
Proof: Assume $Char(R)=O(R)=2k$, where $k\geq 1$. Then, $U(R)=\{r_i: r_i\cdot r_j=e, \forall i\leq j\}$. This gives that $r_i+r_j\notin U(R)$ for any distinct elements $r_i,r_j\in U(R)$. Hence, $\Gamma(R)$ is a graph with $V(\Gamma(R))=U(R)$ and $E(\Gamma(R))=\phi$. Suppose that $|V(\Gamma(R))|=m$, then $\Gamma(R)=\bar{K}_m$. Let $C\subseteq V(\Gamma(R))$ be a code, then $C=\{r_i: r_i\in U(R)\}$ is perfect in $\Gamma(R)$, since $S_1(r_i)\cap S_1(r_i)=\phi$ for all distinct $r_i,r_j\in C$ and $\cup_{i=1}^{m}S_1(r_i)=V(\Gamma(R))$ for all $r_i\in C$. Hence, $|C|=m=|U(R)|$.

5.0 CONCLUSION

Perfect codes have been studied as an important objects in coding theory since the beginning of information theory. This paper is devoted in studying perfect codes in induced subgraph of unit graph associated with some commutative rings R with unity. The findings of this paper classify the commutative rings R according to their associated induced subgraphs of unit graph which accept the perfect codes or not. Therefore, the results are established and illustrated that an order two subset of R with the set of units of R are perfect codes. However, some results have shown that there are no subset of R that can be a perfect code. In short, the findings of this research are summarized in Table 1.

Table 1 Summary of the research

Commutative Ring	Theorem	Type of $\Gamma(R)$	$\lvert C\rvert$
Integral domain with $\lvert U(R)\rvert=2$	Proposition 4.1	$\bar{K}_2$	2
Division ring with $O(R)=p$, $p\geq 5$ is prime.	Theorem 4.1 Example 3.1	Complete $\frac{p-1}{2}$ – partite graph	No perfect code exists in $\Gamma(R)$

Commutative Ring	Theorem	Type of $\Gamma(R)$	$\lvert C \rvert$
Ring with $Char(R)=O(R)=3p$, $p \geq 5$ is prime	Theorem 4.1 Example 3.3	Complete $(3k-1)-$partite graph or Complete $3k-$partite graph	No perfect code exists in $\Gamma(R)$
Division ring with $Char(R)=0$	Theorem 4.1	Infinite regular graph	No perfect code exists in $\Gamma(R)$
Local ring with $Char(R)=O(R)=3^m$, $m \geq 2$	Theorem 4.2 Example 3.2	$2K_{3^{m-2}}$, $m \geq 2$	2
Ring with $Char(R)=O(R)=2k$, $k \geq 1$	Theorem 4.3	$\overline{K}_m$, $m=\lvert U(R)\rvert$	m

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