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## GLOBAL STABILITY OF A FRACTIONAL-ORDER ECO-EPIDEMIOLOGICAL MODEL WITH INFECTED PREDATOR: THEORETICAL ANALYSIS

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**Abstract.** A theoretical knowledge of the global stability of an eco-epidemiological model is not only important in itself but is also important in understanding the results of numerical simulations. In this paper the global stability of a fractional-order eco-epidemiological model with infected predator and harvesting is investigated using the Lyapunov function.

**Keywords:** fractional-order model; global stability; eco-epidemiological.

**2020 AMS Subject Classification:** 92D25, 26A33.

### 1. INTRODUCTION

Mathematical models of the relationship between predator and prey in the presence of infectious diseases, which play an important role in the dynamics, are called eco-epidemiological

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models. There have been various studies of such models with disease being present in the constituent populations. These studies include [1]-[9].

Harvesting can influence the dynamics of eco-epidemiological models. In recent years the demand for greater resources has resulted in over-exploitation. Therefore there is a need for a sustainable strategy to protect ecosystems [10].

Mathematical models incorporating fractional differential equations have attracted much attention in recent years. Such models are believed to be more suitable for models that depend on past history [11, 12]. Further, such models are more realistic and less prone to errors [13]. Studies on fractional-order eco-epidemiological models include [14]-[21]. Ghosh et al. [1] studied a fractional-order eco-epidemiological model incorporating fear, treatment, and hunting cooperation effects to explore the memory effect in an ecological system through Caputo-type fractional-order derivative. In the work by Mukherjee [21], the author investigated a fractional-order predator-prey system with fear effect. Moustafa et al. [14] described the dynamical behavior of fractional-order Rosenzweig-MacArthur model allowing for a prey refuge. The influence of an infectious disease on a prey-predator model equipped with a fractional-order derivative is studied in [3]. The effect of fractional-order derivative on a prey-predator model with infection and harvesting is discussed by Moustafa et al. [16]. However, these papers did not deal with fractional-order eco-epidemiological model with infected predator and harvesting as such.

This paper is a theoretical study of the global stability of a fractional-order eco-epidemiological model with infected predator and harvesting. There has, so far as we are aware, been no theoretical studies of such a model.

## 2. MODEL DESCRIPTION

This paper investigates the global dynamic properties of a generalisation of the integer-order eco-epidemiological model introduced in [22]. The Caputo fractional derivative of order  $q$  ( ${}^c D^q$ ) is introduced and harvesting ( $H$ ) is included. This generalised(fractional) model can be written as:

$${}^c D^q x(t) = rx \left(1 - \frac{x}{k}\right) - \frac{c_1 xy}{a+x} - \frac{c_2 xz}{a+x}, \quad x(0) = x_0,$$

$$(1) \quad \begin{aligned} {}^c D^q y(t) &= \frac{m_1 xy}{a+x} - \frac{\lambda yz}{b+y} - d_1 y, \quad y(0) = y_0, \\ {}^c D^q z(t) &= \frac{m_2 xz}{a+x} + \frac{\lambda yz}{b+y} - d_2 z - Hz, \quad z(0) = z_0, \end{aligned}$$

where  $q \in (0, 1)$ . The population is divided into: prey population density ( $x$ ), susceptible predator population density ( $y$ ) and infected predator population density ( $z$ ).

### 3. EQUILIBRIUM POINTS AND GLOBAL STABILITY

So as to evaluate the equilibrium points of model (1), let

$${}^c D^q x(t) = 0, \quad {}^c D^q y(t) = 0 \text{ and } {}^c D^q z(t) = 0.$$

Thus, the equilibrium points of model (1) are as follows:

- (1)  $E_0 = (0, 0, 0)$ , which always exists.
- (2)  $E_1 = (k, 0, 0)$ , which always exists.
- (3)  $E_2 = \left( \frac{ad_1}{m_1 - d_1}, \frac{rd_1(k+a)}{c_1 k(m_1 - d_1)} (\mathfrak{R}_{01} - 1)(a + x_2), 0 \right)$ , which exists if  $\mathfrak{R}_{01} > 1$  and  $m_1 > d_1$ , where,  $x_2 = \frac{ad_1}{m_1 - d_1}$ .
- (4)  $E_3 = \left( \frac{a\zeta}{m_2 - \zeta}, 0, \frac{r\zeta(k+a)}{c_2 k(m_2 - \zeta)} (\mathfrak{R}_{02} - 1)(a + x_3) \right)$ , which exists if  $\mathfrak{R}_{02} > 1$  and  $m_2 > \zeta$ , where,  $x_3 = \frac{a\zeta}{m_2 - \zeta}$ .
- (5)  $E_4 = (x_4, y_4, z_4)$  where

$$y_4 = \frac{b(\zeta(a + x_4) - m_2 x_4)}{m_2 x_4 + (\lambda - \zeta)(a + x_4)}, \quad z_4 = \frac{b(m_1 x_4 - d_1(a + x_4))}{m_2 x_4 + (\lambda - \zeta)(a + x_4)},$$

and that  $x_4$  needs to be a positive root of the following cubic polynomial:

$$(2) \quad x_4^3 + v_1 x_4^2 + v_2 x_4 + v_3 = 0,$$

where

$$\begin{aligned} v_1 &= a - k + \frac{a(\lambda - \zeta)}{\lambda - \zeta + m_2}, \\ v_2 &= \frac{a(a - 2k)r\lambda + \zeta(bkc_1 - a(a - 2k)r) - bkc_2(d_1 - m_1) - k(ar + bc_1)m_2}{r(\lambda - \zeta + m_2)}, \\ v_3 &= \frac{ak(ar(\zeta - \lambda) + b(c_1\zeta - c_2d_1))}{r(\lambda - \zeta + m_2)}. \end{aligned}$$

In accordance with Theorem 3.4 in [23], the analytical conditions about the existence of the equilibrium point  $E_4$  can be illustrated in Table 1, Table 2 and Table 3.

TABLE 1.  $\Theta > 0$ .

Conditions	Equilibria of model (1)
$v_1 < 0, v_2 \in \mathbb{R}, v_3 > 0$	Two distinct positive equilibria
$v_1 \geq 0, v_2 < 0, v_3 > 0$	Two distinct positive equilibria
$v_1 \geq 0, v_2 \in \mathbb{R}, v_3 < 0$	One positive equilibrium
$v_1 < 0, v_2 > 0, v_3 < 0$	Three distinct positive equilibria
$v_1 < 0, v_2 \leq 0, v_3 < 0$	One positive equilibrium

TABLE 2.  $\Theta = 0$ .

Conditions	Equilibria of model (1)
$v_1 < 0, v_2 \in \mathbb{R}, v_3 > 0$	Two same positive equilibria
$v_1 \geq 0, v_2 < 0, v_3 > 0$	Two same positive equilibria
$v_1 \geq 0, v_2 \in \mathbb{R}, v_3 < 0$	One positive equilibrium
$v_1 < 0, v_2 > 0, v_3 < 0$	Two same positive equilibria
$v_1 < 0, v_2 \leq 0, v_3 < 0$	One positive equilibrium

TABLE 3.  $\Theta < 0$ .

Conditions	Equilibria of model (1)
For any value of $v_1, v_2 \in \mathbb{R}, v_3 \geq 0$	No positive equilibrium exists
For any value of $v_1, v_2 \in \mathbb{R}, v_3 < 0$	One positive equilibrium.

with

$$\Theta = 18v_1v_2v_3 - 4v_1^3v_3 + v_1^2v_2^2 - 4v_2^3 - 27v_2^2.$$

The following theorems investigate the global stability of the equilibrium points  $E_1, E_2, E_3$  and  $E_4$ .

**Theorem 1.** *The equilibrium point  $E_1$  of model (1) is globally asymptotically stable if  $d_1 > \min \left\{ \frac{m_1k}{a+k}, \frac{c_1k}{a} \right\}$  and  $\zeta > \min \left\{ \frac{c_2k}{a}, \frac{m_2k}{a+k} + \lambda \right\}$ .*

*Proof.* The following positive definite Lyapunov function can be considered:

$$V = x - k - k \ln\left(\frac{x}{k}\right) + \frac{1}{2}y^2 + \frac{1}{2}z^2 + y + z.$$

Calculating the  $q$ -order derivative of  $V$  along the solution of model (1) and using Lemma 3.1 in [24],

$$\begin{aligned} {}^c D^q V &\leq \left(\frac{x-k}{x}\right) {}^c D^q x + y {}^c D^q y + z {}^c D^q z + {}^c D^q y + {}^c D^q z \\ &= (x-k) \left( r \left(1 - \frac{x}{k}\right) - \frac{c_1 y}{a+x} - \frac{c_2 z}{a+x} \right) \\ &\quad + \left( \frac{m_1 x}{a+x} - \frac{\lambda z}{b+y} - d_1 \right) y^2 + \left( \frac{m_2 x}{a+x} + \frac{\lambda y}{b+y} - \zeta \right) z^2 \\ &\quad + \frac{m_1 x y}{a+x} + \frac{m_2 x z}{a+x} - d_1 y - \zeta z \\ &= -\frac{r}{k} (x-k)^2 - \frac{c_1 x y}{a+x} - \frac{c_2 x z}{a+x} + \frac{c_1 k y}{a+x} + \frac{c_2 k z}{a+x} \\ &\quad + \left( \frac{m_1 x}{a+x} - \frac{\lambda z}{b+y} - d_1 \right) y^2 + \left( \frac{m_2 x}{a+x} + \frac{\lambda y}{b+y} - \zeta \right) z^2 \\ &\quad + \frac{m_1 x y}{a+x} + \frac{m_2 x z}{a+x} - d_1 y - \zeta z \\ &\leq -\frac{r}{k} (x-k)^2 + \left( \frac{m_1 k}{a+k} - d_1 \right) y^2 + \left( \frac{m_2 k}{a+k} + \lambda - \zeta \right) z^2 \\ &\quad + \left( \frac{c_1 k}{a} - d_1 \right) y + \left( \frac{c_2 k}{a} - \zeta \right) z. \end{aligned}$$

Thus,  ${}^c D^q V \leq 0$  if  $d_1 > \min\left\{\frac{m_1 k}{a+k}, \frac{c_1 k}{a}\right\}$  and  $\zeta > \min\left\{\frac{c_2 k}{a}, \frac{m_2 k}{a+k} + \lambda\right\}$ . By Lemma 4.6 in [25], it is proof that the equilibrium point  $E_1$  is globally asymptotically stable.  $\square$

**Theorem 2.** *The equilibrium point  $E_2$  of model (1) is globally asymptotically stable if  $y_2 < \frac{r(a+x_2)a}{c_1 k}$  and  $\frac{\lambda y_2}{b} + \frac{m_2 k}{a+k} + \frac{L c_2 x_2}{a} < \zeta$ .*

*Proof.* The following positive definite Lyapunov function is considered.

$$V = L \left( x - x_2 - x_2 \ln\left(\frac{x}{x_2}\right) \right) + y - y_2 - y_2 \ln\left(\frac{y}{y_2}\right) + z.$$

By calculating the  $q$ -order derivative of  $V$  along the solution of model (1) and using Lemma 3.1 in [24],

$$\begin{aligned}
{}^c D^q V &\leq L(x-x_2) \left( r - \frac{rx}{k} - \frac{c_1 y}{a+x} - \frac{c_2 z}{a+x} \right) \\
&\quad + (y-y_2) \left( \frac{m_1 x}{a+x} - \frac{\lambda z}{b+y} - d_1 \right) + \frac{m_2 x z}{a+x} + \frac{\lambda y z}{b+y} - \zeta z \\
&\leq L(x-x_2) \left( \frac{rx_2}{k} + \frac{c_1 y_2}{a+x_2} - \frac{rx}{k} - \frac{c_1 y}{a+x} - \frac{c_2 z}{a+x} \right) \\
&\quad + (y-y_2) \left( \frac{m_1 x}{a+x} - \frac{\lambda z}{b+y} - \frac{m_1 x_2}{a+x_2} \right) + \frac{m_2 x z}{a+x} + \frac{\lambda y z}{b+y} - \zeta z \\
&\leq L \left( \frac{c_1 y_2}{(a+x)(a+x_2)} - \frac{r}{k} \right) (x-x_2)^2 + \frac{(x-x_2)(y-y_2)}{(a+x)(a+x_2)} (am_1 - aLc_1 - Lc_1 x_2) \\
&\quad + \frac{\lambda y_2 z}{b+y} + \frac{m_2 x z}{a+x} - \frac{Lc_2 z x}{a+x} + \frac{Lc_2 x_2 z}{a+x} - \zeta z \\
&\leq L \left( \frac{c_1 y_2}{a(a+x_2)} - \frac{r}{k} \right) (x-x_2)^2 + \frac{(x-x_2)(y-y_2)}{(a+x)(a+x_2)} (am_1 - aLc_1 - Lc_1 x_2) \\
&\quad + z \left( \frac{\lambda y_2}{b} + \frac{m_2 k}{a+k} + \frac{Lc_2 x_2}{a} - \zeta \right).
\end{aligned}$$

Suppose  $L = \frac{am_1}{c_1(a+x_2)}$ . Thus,  ${}^c D^q \leq 0$  when  $y_2 < \frac{r(a+x_2)a}{c_1 k}$  and  $\frac{\lambda y_2}{b} + \frac{m_2 k}{a+k} + \frac{Lc_2 x_2}{a} < \zeta$ . Hence the theorem is proved.  $\square$

**Theorem 3.** *The equilibrium point  $E_3$  of model (1) is globally asymptotically stable if  $z_3 < \frac{r(a+x_3)a}{c_2 k}$  and  $\frac{m_1 k}{a+k} + \frac{Mc_1 x_3}{a} - \frac{\lambda z_3}{b} < d_1$ .*

*Proof.* It can be used the following positive definite Lyapunov function.

$$V = M \left( x - x_3 - x_3 \ln \left( \frac{x}{x_3} \right) \right) + y + z - z_3 - z_3 \ln \left( \frac{z}{z_3} \right).$$

Computing the time derivative of  $V$  along the solution of model (1) and utilizing Lemma 3.1 in [24],

$$\begin{aligned}
{}^c D^q V &\leq M(x-x_3) \left( r - \frac{rx}{k} - \frac{c_1 y}{a+x} - \frac{c_2 z}{a+x} \right) \\
&\quad + \frac{m_1 xy}{a+x} - \frac{\lambda yz}{b+y} - d_1 y + (z-z_3) \left( \frac{m_2 x}{a+x} + \frac{\lambda y}{b+y} - \zeta \right) \\
&\leq M(x-x_3) \left( \frac{rx_3}{k} - \frac{rx}{k} + \frac{c_2 z_3}{a+x_3} - \frac{c_1 y}{a+x} - \frac{c_2 z}{a+x} \right) \\
&\quad + \frac{m_1 xy}{a+x} - \frac{\lambda yz}{b+y} - d_1 y + (z-z_3) \left( \frac{m_2 x}{a+x} - \frac{m_2 x_3}{a+x_3} + \frac{\lambda y}{b+y} \right)
\end{aligned}$$

$$\begin{aligned}
 &\leq M \left( \frac{c_2 z_3}{(a+x)(a+x_3)} - \frac{r}{k} \right) (x-x_3)^2 + \frac{(x-x_3)(z-z_3)}{(a+x)(a+x_3)} (am_2 - aMc_2 - Mc_2 x_3) \\
 &\quad - \frac{\lambda y z_3}{b+y} + \frac{m_1 xy}{a+x} - \frac{Mc_1 xy}{a+x} + \frac{Mc_1 x_3 y}{a+x} - d_1 y \\
 &\leq M \left( \frac{c_2 z_3}{a(a+x_3)} - \frac{r}{k} \right) (x-x_3)^2 + \frac{(x-x_3)(z-z_3)}{(a+x)(a+x_3)} (am_2 - aMc_2 - Mc_2 x_3) \\
 &\quad + y \left( \frac{m_1 k}{a+k} + \frac{Mc_1 x_3}{a} - \frac{\lambda z_3}{b} - d_1 \right).
 \end{aligned}$$

Suppose  $M = \frac{am_2}{c_2(a+x_3)}$ . Thus,  ${}^c D^q \leq 0$  when  $z_3 < \frac{r(a+x_3)a}{c_2 k}$  and  $\frac{m_1 k}{a+k} + \frac{Mc_1 x_3}{a} - \frac{\lambda z_3}{b} < d_1$ . Hence the theorem is proved.  $\square$

**Theorem 4.** *The equilibrium point  $E_4$  of model (1) is globally asymptotically stable if  $\xi v^2 + \lambda v^2 z_4 + \xi y_4 z_4 + \lambda y_4^2 z_4 - \xi \rho z_4 - \lambda \rho y_4 z_4 - \xi y_4 \rho - \lambda \rho y_4 z_4 < 0$  and  $r > k(c_1 y_4 + c_2 z_4)$ .*

*Proof.* The following positive definite Lyapunov function can be used.

$$V = N_1 \left( x - x_4 - x_4 \ln \left( \frac{x}{x_4} \right) \right) + y - y_4 - y_4 \ln \left( \frac{y}{y_4} \right) + N_2 \left( z - z_4 - z_4 \ln \left( \frac{z}{z_4} \right) \right).$$

Computing the time derivative of  $V$  along the solution of model (1) and utilizing using Lemma 3.1 in [24],

$$\begin{aligned}
 {}^c D^q V &\leq N_1 (x - x_4) \left( r - \frac{rx}{k} - \frac{c_1 y}{a+x} - \frac{c_2 z}{a+x} \right) + (y - y_4) \left( \frac{m_1 x}{a+x} - \frac{\lambda z}{b+y} - d_1 \right) \\
 &\quad + N_2 (z - z_4) \left( \frac{m_2 x}{a+x} + \frac{\lambda y}{b+y} - \zeta \right) \\
 &\leq N_1 (x - x_4) \left( \frac{rx_4}{k} - \frac{rx}{k} + \frac{c_1 y_4}{a+x_4} - \frac{c_1 y}{a+x} + \frac{c_2 z_4}{a+x_4} - \frac{c_2 z}{a+x} \right) \\
 &\quad + (y - y_4) \left( \frac{m_1 x}{a+x} - \frac{m_1 x_4}{a+x_4} + \frac{\lambda z_4}{b+y_4} - \frac{\lambda z}{b+y} \right) \\
 &\quad + N_2 (z - z_4) \left( \frac{m_2 x}{a+x} - \frac{m_2 x_4}{a+x_4} + \frac{\lambda y}{b+y} - \frac{\lambda y_4}{b+y_4} \right) \\
 &\leq N_1 \left( \frac{c_1 y_4 + c_2 z_4}{(a+x)(a+x_4)} - \frac{r}{k} \right) (x - x_4)^2 + (am_1 - N_1 c_1 a - N_1 c_1 x_4) \frac{(x - x_4)(y - y_4)}{(a+x)(a+x_4)} \\
 &\quad + (N_2 m_2 a - N_1 c_2 a - N_1 c_2 x_4) \frac{(x - x_4)(z - z_4)}{(a+x)(a+x_4)} - \frac{b\lambda (y - y_4)(z - z_4)}{(b+y)(b+y_4)} \\
 &\quad + \frac{\lambda z_4 (y - y_4)^2}{(b+y)(b+y_4)} - \frac{\lambda y_4 (y - y_4)(z - z_4)}{(b+y)(b+y_4)} + \frac{N_2 \lambda b (y - y_4)(z - z_4)}{(b+y)(b+y_4)}
 \end{aligned}$$

$$\begin{aligned}
&\leq N_1 \left( c_1 y_4 + c_2 z_4 - \frac{r}{k} \right) (x - x_4)^2 + (am_1 - N_1 c_1 a - N_1 c_1 x_4) \frac{(x - x_4)(y - y_4)}{(a + x)(a + x_4)} \\
&\quad + (N_2 m_2 a - N_1 c_2 a - N_1 c_2 x_4) \frac{(x - x_4)(z - z_4)}{(a + x)(a + x_4)} \\
&\quad + (N_2 \lambda b - \lambda b - \lambda y_4) \frac{(y - y_4)(z - z_4)}{(b + y)(b + y_4)} + \frac{\lambda z_4 (y - y_4)^2}{(b + y)(b + y_4)} \\
&\leq N_1 \left( c_1 y_4 + c_2 z_4 - \frac{r}{k} \right) (x - x_4)^2 + (am_1 - N_1 c_1 a - N_1 c_1 x_4) \frac{(x - x_4)(y - y_4)}{(a + x)(a + x_4)} \\
&\quad + (N_2 m_2 a - N_1 c_2 a - N_1 c_2 x_4) \frac{(x - x_4)(z - z_4)}{(a + x)(a + x_4)} \\
&\quad + \frac{\xi y z + \lambda y^2 z_4 + \xi y_4 z_4 + \lambda y_4^2 z_4 - \xi y z_4 - \lambda y y_4 z_4 - \xi y_4 z - \lambda y y_4 z_4}{(b + y)(b + y_4)} \\
&\leq N_1 \left( c_1 y_4 + c_2 z_4 - \frac{r}{k} \right) (x - x_4)^2 + (am_1 - N_1 c_1 a - N_1 c_1 x_4) \frac{(x - x_4)(y - y_4)}{(a + x)(a + x_4)} \\
&\quad + (N_2 m_2 a - N_1 c_2 a - N_1 c_2 x_4) \frac{(x - x_4)(z - z_4)}{(a + x)(a + x_4)} \\
&\quad + \frac{\xi v^2 + \lambda v^2 z_4 + \xi y_4 z_4 + \lambda y_4^2 z_4 - \xi \rho z_4 - \lambda \rho y_4 z_4 - \xi y_4 \rho - \lambda \rho y_4 z_4}{(b + y)(b + y_4)},
\end{aligned}$$

where  $\xi = N_2 \lambda b - \lambda b - \lambda y_4$ . Suppose  $N_1 = \frac{am_1}{c_1(a+x_4)}$ ,  $N_2 = \frac{m_1 c_2}{m_2 c_1}$ , and  $v < z$ ,  $y < \rho$ . Thus,  ${}^c D^q V(x, y, z) \leq 0$ , when  $\xi v^2 + \lambda v^2 z_4 + \xi y_4 z_4 + \lambda y_4^2 z_4 - \xi \rho z_4 - \lambda \rho y_4 z_4 - \xi y_4 \rho - \lambda \rho y_4 z_4 < 0$  and  $r > k(c_1 y_4 + c_2 z_4)$ . By Lemma 4.6 in [25], it is proof that  $E_4$  is globally asymptotically stable.  $\square$

Now, the proof of the existence of transcritical bifurcation around the equilibrium point  $E_1(k, 0, 0)$  is given by using Sotomayor's theorem.

**Theorem 5** (Transcritical bifurcation around  $E_1$ ). *The fractional-order model (1) undergoes a transcritical bifurcation with respect to the bifurcation parameter  $H$  around  $E_1(k, 0, 0)$  when  $H = H_{tr1} = \frac{m_2 k}{a+k} - d_2$  and keeping  $\mathfrak{R}_{01} < 1$ , while no saddle-node bifurcation can occur.*

*Proof.* The Jacobian matrix for the model (1) around  $E_1$  when  $H = H_{tr1}$  is as follows:

$$(3) \quad J(E_1) = \begin{pmatrix} -r & -\frac{c_1 k}{a+k} & -\frac{c_2 k}{a+k} \\ 0 & d_1 (\mathfrak{R}_{01} - 1) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$



A straightforward computation indicates that the Jacobian matrix (3) has zero eigenvalue  $\mu_3$ . Here,  $\mu_1 = -r < 0$  and  $\mu_2 = d_1(\mathfrak{R}_{01} - 1) < 0$  when  $\mathfrak{R}_{01} < 1$ . Let  $V = (v_1, v_2, v_3)^T = (\frac{-c_2kv_3}{r(a+k)}, 0, v_3)^T$  and  $W = (\tau_1, \tau_2, \tau_3)^T = (0, 0, \tau_3)^T$  be the two eigenvectors corresponding to the zero eigenvalue of the matrices  $J(E_1)$  and  $(J(E_1))^T$ , respectively. Where  $v_3$  and  $\tau_3$  are any non zero real numbers. Therefore,

$$W^T(F_H(E_1, H_{tr1})) = 0,$$

$$W^T(DF_H(E_1, H_{tr1})V) = -v_3\tau_3 \neq 0,$$

$$W^T(D^2F(E_1, H_{tr1})(V, V)) = \left( \frac{2b^2\lambda v_2v_3}{b^3} + \frac{2am_2v_1v_3}{(a+k)^2} \right) \tau_3 \neq 0.$$

By Sotomayor's Theorem for local bifurcation [26], the fractional-order model (1) has a transcritical bifurcation around  $E_1$  when  $H = H_{tr1} = \frac{m_2k}{a+k} - d_2$  as  $H$  passes through the value  $H_{tr1}$ , while no saddle-node bifurcation can occur.  $\square$

#### 4. CONCLUSION

In this paper, a fractional-order eco-epidemiological model with infected predator and harvesting has been formulated and analyzed. The equilibrium points were identified and their global properties were investigated. The existence of transcritical bifurcation was shown using Sotomayor's theorem. The threshold parameters ( $\mathfrak{R}_{01}$  and  $\mathfrak{R}_{02}$ ) were used to determine the existence conditions of the equilibrium points.

#### CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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