ASM Sc. J., 18, 2023

https://doi.org/10.32802/asmscj.2023.1155

Thermally Radiation of MHD Two-Phase Williamson Fluid with Newtonian Heating (NH)

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Due to the unique qualities in its behaviours that study the solid and fluid aspects, a study on two-phase flow (solid and liquid) is deemed to be supplementary trustworthy in describing the application of liquid in industrial sectors. Over the past few decades, several non-Newtonian fluid models have been found, but the Williamson model stands out as the most intriguing. The Williamson flow model will be more fascinating to study when the existing particles are considered. Therefore, the purpose of this article is to investigate Williamson fluid with dust particles under the existing MHD and heat radiation. To make inquiry more intriguing, the analysed model also included a Newtonian heating (NH) condition. After employing similarity transformation, the resultant equations, as in Ordinary Differential Equations (ODEs), are solved using the Runge-Kutta Fehlberg (RKF45) method. The findings showed that the presence of fluid-particle interaction (FPI) influenced the fluid velocity, subsequent in a declining the fluid movement and an increase in particle motion.

Keywords: Williamson Fluid; Two-Phase Flow; Runge-Kutta Fehlberg (RKF45) method

I. INTRODUCTION

The two-phase fluid flow, which is also known as a dusty fluids, is significant in many engineering problems due to its applicability in industries applications. Generally, the two-phase flow cover the phases of solid and liquid, liquid with liquid, gas and solid, as well as gas to liquid. The common two-phase, which involves in many natural processes combine between fluid and solid. For instance, the lymph cells, which containing the blood present as corpuscles (solids) suspended in the liquid of the human body. Additionally, the process in geothermal reservoirs and condensation flow of cold surfaces in foam insulators under heating pipes are applied the two-phase flow characteristics. In this case, the non-Newtonian type of fluid being the fluid phase at which the stress level and its strain is non-linear due to the deformation depends on viscosity. Other studies focused on non-Newtonian fluid were highlighted in Isa *et al.* (2016), Butt *et al.* (2017), Arifin *et al.* (2017), Siddiqa *et al.* (2017), Abbas *et al.* (2019), Kumar *et al.* (2021) and Zokri *et al.* (2018). The Williamson model explains the behaviour of pseudoplastic fluids, which is the most frequently bump into non-Newtonian fluids. The endeavour on the Williamson model was in progress since 1929 by Williamson (1929), where the studied also supported by experimental output. Further research on this model has been done by many researchers and documented in Dapra and Scarpi (2007), Kumari *et al.* (2012), Nadeem *et al.* (2013), Megahed (2019), Khan *et al.* (2018) and Dada and Onwubuoya (2020). Reports

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under the topic of Williamson liquid are also documented under Raza *et al.* (2021) and Salahuddin *et al.* (2021), which concentrate on the case with radiation and exponential viscosity, respectively. The concentration of investigation has been given on a single phase (fluid phase). Some output on the extension model to two-phase (fluid-solid) are highlighted in Dinesh *et al.* (2021), Bilal *et al.* (2021), Kasim *et al.* (2020), and Arifin *et al.* (2019). The article's goal is to provide the results of a numerical investigation of a two-phase Williamson fluid flowing across a sheet where the condition is stretched vertically. The flow field was integrated with the presence of heat radiation, MHD and NH.

II. MATHEMATICAL ANALYSIS

The steady, two-dimensional incompressible flow of Williamson fluid with solid particles (dust phase) over a vertical stretching sheet is studied.

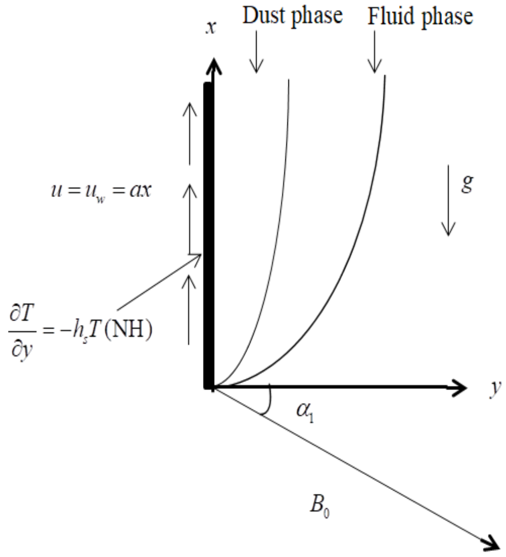

Figure 1. Physical Flow

The flow is embedded with MHD and thermal radiation, which bounded by boundary condition of NH. To facilitate the problem formulation, the particles considered are under assumption of in spherical shape, size is uniform and having no interaction among the particles. The physical configuration can be found in Figure 1.

The model that governs the flow is divided into fluid and dust phase. The both phases are shown as Equation (1) to (3) and (4) to (6) respectively. It is worth to mentioned the model has been adopted in the study by Dasman *et al.* (2021), which studied in dusty micropolar fluid.

Fluid phase:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + \sqrt{2}\nu\Gamma\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} + \frac{\rho_p}{\rho\tau_v}(u_p - u) - \frac{\sigma u B_0{}^2}{\rho}\sin^2\alpha_1, \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} + \frac{\rho_p c_s}{\tau_T \rho c_p}\left(T_p - T\right) - \frac{1}{\rho c_p}\frac{\partial q_r}{\partial y} \tag{3}$$

Dust phase:

$$\frac{\partial}{\partial x}(u_p) + \frac{\partial}{\partial y}(v_p) = 0, \tag{4}$$

$$\rho_p\left(u_p\frac{\partial u_p}{\partial x} + v_p\frac{\partial u_p}{\partial y}\right) = \frac{\rho_p}{\tau_v}(u - u_p), \tag{5}$$

$$\rho_p c_s\left(u_p\frac{\partial T_p}{\partial x} + v_p\frac{\partial T_p}{\partial y}\right) = -\frac{\rho_p c_s}{\gamma_T}(T_p - T) \tag{6}$$

The NH are introduced on the surface of sheet and can be expressed as

Fluid Phase

$$u = u_w(x) = ax,\ \ v = 0,\ \ \frac{\partial T}{\partial y} = -h_s T \quad \text{at} \quad y = 0$$
$$u \to 0,\ \ T \to T_\infty,\ \ \text{at} \ \ y \to \infty \tag{7}$$

Dust Phase

$$u_p \to 0,\ \ v_p \to v,\ \ T_p \to T_\infty \quad \text{at} \quad y \to \infty$$

Remark that the expressions of q_r obeying the Rosseland's theory take the form from Gireesha *et al.* (2013).

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y}, \tag{8}$$

The summarisation on the parameters involved in Equations (1) - (8) are as follows:

(u,v)	velocities components of the fluid along x and y axes
(u_p,v_p)	velocities components of the particle along x and y axes
μ	viscosity of the fluid
ρ	density of fluid
ρ_p	density of dust
α_1	aligned angle
$\tau_v = 1/k$	relaxation time of particles phase
k	Stoke's resistance (drag force)
c_p	specific heat of fluid
c_s	specific heat of dust particle
T	temperature of fluid
T_p	temperature of particle
γ_T	thermal relaxation time
q_r	radiative heat flux.
a	positive constant
h_s	heat transfer parameter
σ^*	Stefan-Boltzmann constant
k^*	Absorption parameter
T^4	coefficient linear function of temperature

Applying the Taylor's series, the T^4 is expand to

$$T^4 = 4T_\infty^{\ 3}T - 3T_\infty^{\ 4} \tag{9}$$

It is worth to claim the higher order terms were ignored. The equation (3) is switched to

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = k\left(1 + \frac{16T_\infty^{\ 4}\sigma^*}{3kk_1}\right)\frac{\partial^2 T}{\partial y^2} + \frac{\rho_p c_s}{\tau_T \rho c_p}\left(T_p - T\right). \tag{10}$$

The similarity transformation that is suited for the considered model are

$$\begin{gathered} u = axf'(\eta),\ v = -(av)^{1/2} f(\eta),\ \eta = \left(\frac{a}{v}\right)^{1/2} y,\ \theta(\eta) = \frac{T - T_\infty}{T_\infty} \\ u_p = axF'(\eta),\ \ v_p = -(av)^{1/2} F(\eta),\ \ \theta_p(\eta) = \frac{T_p - T_\infty}{T_\infty}, \end{gathered} \tag{11}$$

The term $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$ is declared at which the ψ indicates as stream function.

The transformed governing Equations (1)-(6) are as follows

$$\begin{gathered} f'''(\eta) + f(\eta)f''(\eta) - \left(f'(\eta)\right)^2 + \lambda_3 f''(\eta)f'''(\eta) \\ +\beta N\left(F'(\eta) - f'(\eta)\right) - M\sin^2\alpha_1 f'(\eta) = 0, \end{gathered} \tag{12}$$

$$\left(1 + \frac{4}{3}R\right)\theta''(\eta) + \Pr f(\eta)\theta'(\eta) + \frac{2}{3}\beta N\left(\theta_p(\eta) - \theta(\eta)\right) = 0 \tag{13}$$

$$\left(F'(\eta)\right)^2 - F(\eta)F''(\eta) + \beta\left(F'(\eta) - f'(\eta)\right) = 0, \tag{14}$$

$$\theta_p{}'(\eta)F(\eta) + \frac{2}{3}\frac{\beta}{\Pr\gamma}\left(\theta(\eta) - \theta_p(\eta)\right) = 0 \tag{15}$$

The Equation (7) switched to

$$\begin{gathered} \text{Fluid Phase} \\ f(0) = 0,\ f'(0) = 1,\ \ \theta'(0) = -b\left(1 + \theta(0)\right) \text{ at } \eta = 0 \\ f'(\eta) \to 0, \theta(\eta) \to 0 \text{ at } \eta \to \infty \\ \text{Dust Phase} \\ F(\eta) \to 0, F(\eta) \to f(\eta), \theta_p(\eta) \to 0 \text{ at } \eta \to \infty \end{gathered} \tag{16}$$

where the term ($'$) signifies differentiation with respect to η. The parameter obtained are in dimensionless form as listed below:

$N = \rho_p / \rho$	mass concentration of particle phase
$M = \sigma B_0^2 / \rho a$	magnetic field parameter
$\beta = 1/a\tau_v$	fluid-particle interaction
$\Pr = \mu c_p / k$	Prandtl number
$\gamma = c_s / c_p$	specific heat ratio of mixture
$b = -h_s\left(v/a\right)^{1/2}$	conjugate parameter for NH
$\lambda_3 = \sqrt{2a^3/v}\Gamma x$	Williamson parameter
$R = -4\sigma^* T_\infty^{\ 3}/kk^*$	radiation parameter

The numerical values of the physical quantities can be acquired by expressions

$$C_f \operatorname{Re}_x^{\frac{1}{2}} = \left(f''(0) + \frac{\lambda_3}{2}\left(f''(0)\right)^2\right),\ Nu_x \operatorname{Re}_x^{-\frac{1}{2}} = b\left(1 + \frac{1}{\theta(0)}\right) \tag{17}$$

III. NUMERICAL PROCEDURE

Equations (12) - (16) are solved using RKF45 as this method is stable, less complex and self-starting nature. For velocity and temperature profiles to meet the boundary criteria, the ultimate boundary layer thickness $\eta_\infty = 6$ is set up. Results are presented in graphical and tabular form.

IV. RESULT AND DISCUSSION

To validate the correctness of numerical solutions, a direct comparative study is performed with Salleh *et al.* (2010) and also the exact values by Turkyilmazoglu (2016) for the limiting case in which all parameters are set to be in consistent to each other. The results is shown in Table 1. A strong agreement of present results and analytic solution is achieved and thus validate the present numerical results.

Table 1. Comparison of $\theta(0)$ when $M=\lambda_1=\beta=N=0$, $\gamma\to\infty$ and $b=1$

Pr	Salleh *et al.* (2010)	Turkyilmazoglu (2016)	Present
3	6.02577	6.05158546	6.051585531
5	1.76594	1.76039543	1.760395438
7	1.13511	1.11681524	1.116818808
10	0.76531	0.76452369	0.76452521
100	0.16115	0.14780542	0.147805745

For the computation, the value of $R=0.2$, $\alpha_1=\pi/4$, $M=1$, $\beta=N=0.5$, $b=0.3$, $\mathrm{Pr}=10$, $\lambda_1=0.1$ and $\gamma=0.1$ are fixed throughout the study except the studied parameters.

Figures 2 and 3 exhibit the variation of FPI parameter, β on distribution of velocity and temperature correspondingly. The fluid phase experience the deceleration while contrary behaviour for dust phase. Theoretically, at strong β, the relaxation, τ_v deteriorate where at this situation the dust particle incontroled of stabilisation of fluid motion. A similar trend can be discovered in temperature profile for both phases when β enhanced. It is worth to notice the dust phases show a noticeable change compared to fluid phase in all profiles.

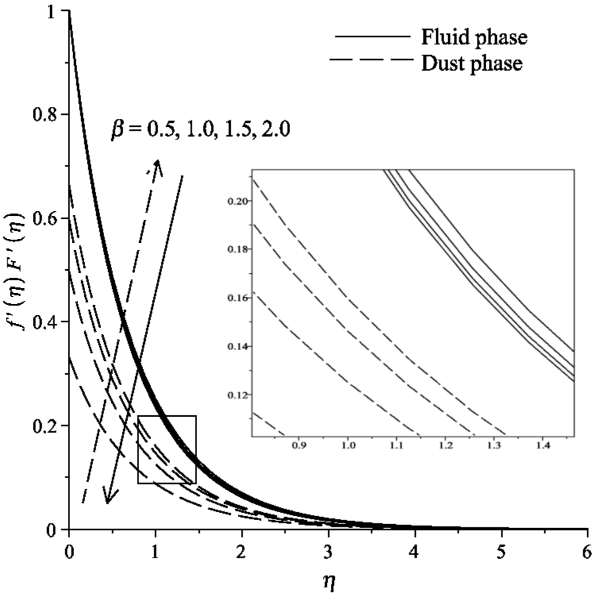

Figure 2. Velocity profile in various β

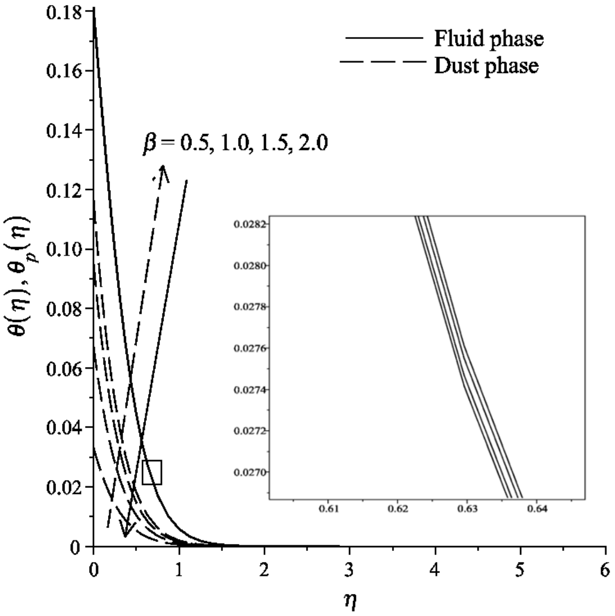

Figure 3. Temperature profile in various β

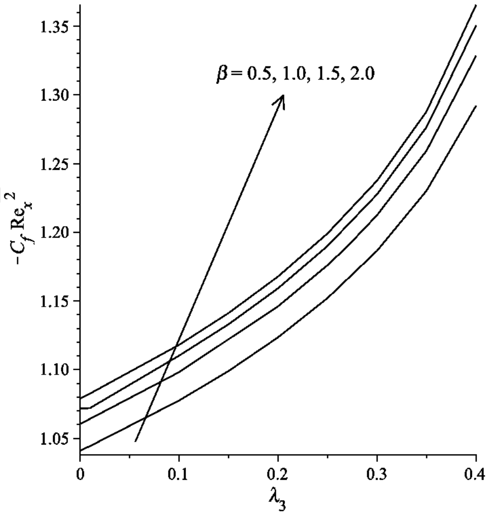

Figure 4. Skin friction coefficient

The influence of λ_3 on $C_f\,\mathrm{Re}_x^{1/2}$ and $Nu_x\,\mathrm{Re}_x^{-1/2}$ with several values of β is displayed in Figures 4 and 5. The improving trend of $C_f\,\mathrm{Re}_x^{1/2}$ is established as its magnitude developed as λ_3 and β enhanced. This indicates the generation of drag-like force from interaction of fluid and dust following in a reduction of the fluid's velocity. For the physical quantity $Nu_x\,\mathrm{Re}_x^{-1/2}$, its value shows a declining tendency in response to the effect of λ_3. Nonetheless, the numerical value shows an increment trend when β

increased. Figure 6 and 7 illustrates the distribution of velocity and temperature for both phase under various value of λ_3 . The boosted in value of λ_3 led to deteriorate the velocity of fluid and enhanced the rate of fluid's temperature.

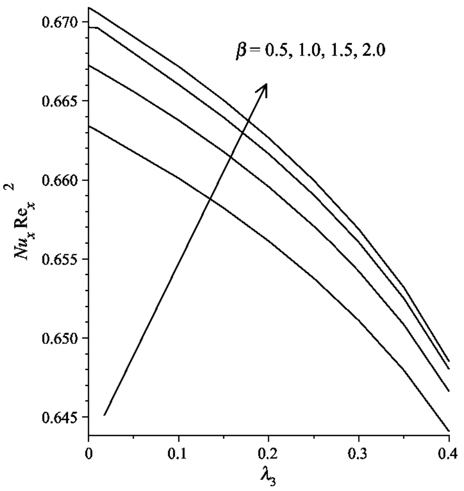

Figure 5. Nusselt number

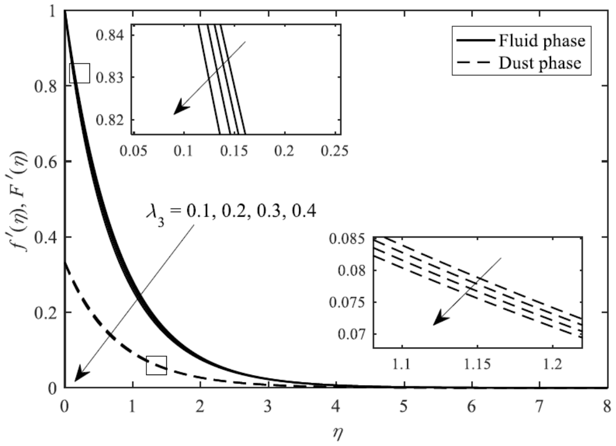

Figure 6. Velocity profile in various λ_3

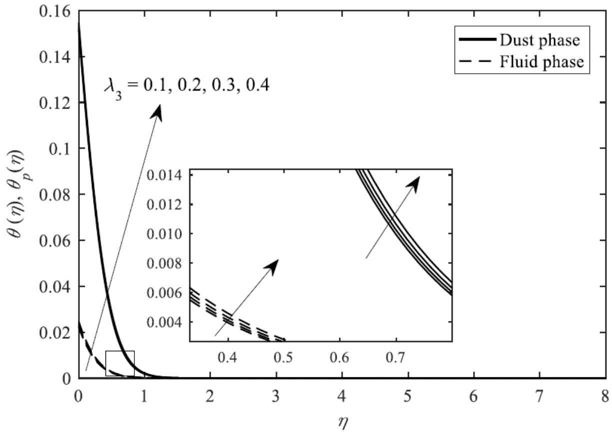

Figure 7. Temperature profile in various λ_3

V. CONCLUSION

Analysis is done on the dusty Williamson fluid with embedded of radiation, MHD and NH. Using the Maple software, the RKF45 technique is used to solve the model. According to the reported results, it was discovered that Williamson fluid is greatly impacted by dust particles existense, which modifies its inclination to flow. The results obtained here will help to understand the complex flow in multiple phase under mathematical point of view.

VI. ACKNOWLEDGEMENT

This project is funded by UMPSA through RDU223015. An appreciation also goes to UTM, UiTM Johor and UiTM Pahang.

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