Modelling Period Effect of Lee-Carter Mortality Model with SETAR Model

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Abstract

Mortality dynamics plays an important role in understanding mortality and life expectancy that will affect on economy of the countries. Many studies have considered Lee Carter method with time index as an indicator for forecasting. In order to forecast the mortality rates, the period index in the Lee Carter model is applied to the random walk with drift model. Despite its performance on the forecasting ability, it is lack in term of time varying parameter that leads to higher error and less accurate forecasting result. This is because the random walk with drift model is only adequate to data with linear series. In this study, the concept of non-linear time series model is used and a self-exciting threshold autoregressive (SETAR) will proposed to the period index. It shows that our model outperformed the random walk with drift (RWD) model for forecasting accuracy when Malaysian mortality data from 1980-2010 are considered. Long term forecasting analysis up to 2017 comparing the two models are then performed.

Keywords*:* Drift model, Lee Carter model, Random Walk with Drift, Self-exciting Threshold Autoregressive.

1. Introduction

In the last few decades, the quality of human life has been improving. This can be seen by the decline in mortality and increase life expectancy. While there might be slow decline in mortality in poorer areas and countries, in general it is improving year upon year. This leads to growing number of elderly populations recently. Financially, it could present a burden to a country's economics in terms of providing prevalent healthcare and bigger amount of pension especially to government pensioners. Thus, it is important to monitor the trend continuously and to understand mortality dynamics.

In Europe, the pension system of countries like Germany and Finland are very much associated to life expectancy. The UK's retirement age is also expected to be raised in the next few years.

In Malaysia, the retirement age has been revised three times (Stoeldraijer et al., 2013). As people live longer, it is expected that they can also work up to higher age before retiring. Retirement also relates to longevity risk. There is a risk that a person will outlive his pension fund or an insurance company might have to pay more to policy holders as they live a lot longer.

As a result, there are growing number of research being conducted in modelling and forecasting mortality. The breakthrough came in 1992, when Lee and Carter proposed Lee Carter (LC) model (Shapovalov et al., 2019). The model was first introduced using mortality data from the United States (US). It is a stochastic model that incorporates age and year factor in a log bilinear form. The model comprises of age specific constants a_x and $b_{x,}$ and a time varying index, k_t. The age specific constants would explain the general trend of mortality and also the speed of change in death rate in relation to time varying index. The time varying index is an indicator of mortality level. The log of mortality would be fit into this model and to get the forecast, the time varying index would be projected a few steps into the future. According to Lee and Carter, they tried to find the best ARIMA model to fit k_t. Following Box and Jenkins procedures, they found that the best model fit to be random walk with drift and forecast is carried out using this model. This model is used extensively by the researchers because of its simplicity and robustness (Augustine & Saratha, 2015).

Other researchers have also extended this model to further improve its effectiveness in the model and forecast by introducing a new estimation protocol (Clements & Krolziq, 1998). They carefully assessed LC model and performed slight changes to each part of the model and test if any changes could improve the model. A few researchers like Cairns et al. (2009), Renshaw and Haberman (2006) have included cohort as a parameter in the model. This would then enhance the forecast ability of the model (Tong, 1990; Hansen & Seo, 2002). Others like De Jong and Tickle (2006) and Wan Zakiyatussariroh et al. (2014) have used state space in trying to represent LC model so that model error can be reduced (Cuthbertson, 2004; Clements & Smith, 1997). Meanwhile, CBD model introduced by Cairns, Blake and Dowd focused on the higher age population especially 55 years and above (Di Narzo et al., 2009). CBD model uses logit of the mortality whereby the mortality is defined as death probabilities instead of death rate.

Despite the success of LC model, there is a shortcoming. The LC's time varying parameter, k_t was modelled using a linear ARIMA model. The assumption made by LC is that the k_t of US population was "roughly" linear. They even admitted that linearity would not be reasonable if the time series was adjusted back to the centuries preceding the analysis (Shapovalov et al., 2019). Shapovalov et al. (2019) mentioned that the random walk with drift model only hold if the linear trend is continuous. Analysis of LC model on Malaysia male and female mortality data shows that there is declining trend of the time varying parameter, k_t. However, the series is not entirely linear. This could result in higher error when fitted with random walk with drift and the forecast would be less accurate.

To achieve more reasonable fit and accurate forecast, this study suggests that the time varying parameter to be modelled using non-linear time series model. Most of non-linear time series model is applied in economics and finance. They have been used in particular to model exchange rates and stock exchanges for example comparing random walk with drift and feed forward neural network models (Hyndman et al., 2011). The study showed that non-linear models was better than linear models in fitting and forecasting.

In this study, we are going to model and forecast the time varying parameter, k_t using self-exciting threshold autoregressive (SETAR) model and compare the performance with random walk with drift model. Threshold autoregression (TAR) was introduced by Tong in 1978. The model contains Linear autoregression in two or more regime or series and it is governed by a threshold value. The model is called self-exciting when the threshold is the lag value of time series (Clements & Krolziq, 1998).

The model will be discussed further in section 2. In section 3, we will discuss our findings and discussion. Section 4 will be the conclusion.

2. Materials and Methods

In this section, to illustrate the application of Lee Carter model and the time varying parameter which will be modelled using random walk with drift and SETAR model, this study used the mortality data from Department of Statistics Malaysia from year 1980 to 2017. The log death rate for each age specific can be seen in Figure 1. Overall, Malaysian mortality shows declining trend in every age group across the years. It can be noted that there is a vast improvement in mortality for population of ages between 5-10 years and 25 years. The improvement for lower ages might be due to the improvement in health services that the government implemented to curb child and infant mortality. While improvements in young adults might be due to lower number of casualties in accident (Tong, 1990). Our analysis will use the Malaysian mortality data that was separated into two parts. One part is data from 1980 to 2010 is used as in sample fit. Another part of data which is from 2011 to 2017 was used form out of sample fit.

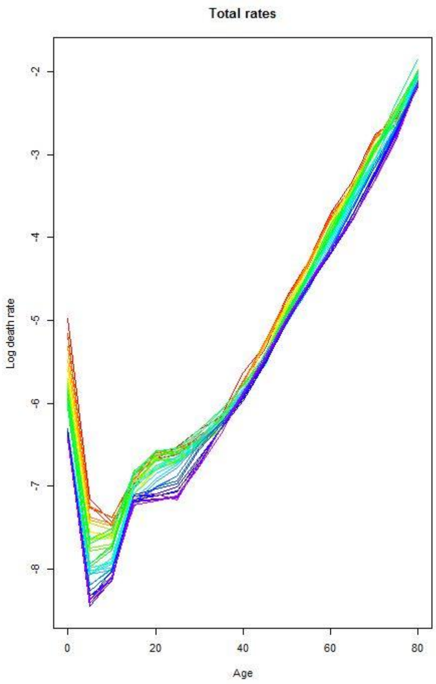


Figure 1. Age specific log death rate for Malaysian population from 1980-2017. Earlier years log death rates are marked with red, yellow, and green lines. Most recent year log death rates are marked with light blue and purple lines

2.1 Lee-Carter mortality model

The LC model was introduced in 1992. This model is chosen as it is simple and consists of very few parameters. It would be more parsimony as compared to model with a lot more parameters. The structure of model is given by Eq. (1):

$$log(m_{x,t}) = a_x + b_x k_t + \varepsilon_{x,t} \tag{1}$$

where $m_{x,t}$ is the central death rate at age x and year t, a_x and b_x are the age specific parameters, $\varepsilon_{x,t}$ is the error term at age x and time t which has 0 mean and variance σ^2. k_t is the time varying parameter or mortality index at time t.

Lee and Carter imposed two constraints to get unique solution to parameter estimate. Those were $\sum_t k_t = 0$and $\sum_x b_x = 1$. The parameters were estimated using Maximum Likelihood Estimation (MLE). The likelihood function on the Poisson number of death can be represented in Eq. (2):

$$L_{x,t}(\theta; d_{x,t}) = \frac{\lambda_{x,t}^{d_{x,t}} e^{-\lambda_{x,t}}}{d_{x,t}!} \tag{2}$$

where $\mu_{x,t} = e^{a_x + b_x k_t}$, $\lambda_{x,t} = E_{x,t}\mu_{x,t}$ and the simplified log likelihood function is given by Eq. (3):

$$\ln L_{x,t}(\theta; d_{x,t}) = d_{x,t}\ln(\lambda_{x,t}) - \lambda_{x,t} - \ln(d_{x,t}!) \tag{3}$$

2.2 Time varying parameter k_t modelling

In this section, the time varying parameter are evaluated by using random walk with drift and self-exciting threshold autoregressive. Random walk is used to forecast error that is in linear and self-exciting threshold autoregressive is used to forecast error that are non-linear.

2.2.1 Random walk with drift

Lee and Carter searched for the best ARIMA model fit k_t and through procedures by Box and Jenkins found random walk with drift to be suitable model. The model is in the form:

$$k_t = k_{t-1} + d + \varepsilon_t$$

where $k_{t-1}$is the lag for k_t, d is the drift parameter, andε_t is error term. A random walk is a process where the value at time t is equal to its past value plus white noise or an error term. The error term is independent and identically distributed with mean 0 and variance σ^2. A drift parameter d acts like a trend detector. A $d > 0$ denotes an upward trend. Forecasting random walk is simple since future values of k_{t+s} for $s > 0$ is k_t.

2.2.2 Self-exciting threshold autoregressive

Threshold autoregressive (TAR) is one of nonlinear model. It is a piecewise AR model where the regime switching variable will tell in which regime a process should follow. Consider a simple TAR model that consists of two regimes and one lag in each of regime. The model can be represented in Eq. (4):

$$k_t = \begin{cases} \emptyset^{(1)} k_{t-1} + u_t{}^{(1)} \text{if } k_{t-d} < r \\ \emptyset^{(2)} k_{t-1} + u_t{}^{(2)} \text{if } k_{t-d} \geq r \end{cases} \tag{4}$$

where k_t is the time varying parameter we are interested to model, ϕ are constants, $k_{t-1}$is the lag value in each regime, d is delay parameter,u_t is the error term, r is the threshold value and $k_{t-d}$is the regime switching variable. When the regime switching variable, $k_{t-d}$is less than threshold value r, the process shall follow first regime. If k_{t-d} is more or equal to r, the process shall follow second regime. The model is known to be a self-exciting threshold autoregressive (SETAR) model when the regime switching variable is the past realisation of the process. In this case, the regime switching variable would be

$$k_{t-d} = \beta_0 k_t + \beta_1 k_{t-1}$$

The model can also be more than just one regime and be written as $\text{SETAR}\left(Nr, p^{(1)}, \ldots, p^{(j)}\right)$. Nr is the number of regimes and p the number of lag in particular regime. The general form of SETAR can be expressed in Eq. (5):

$$k_t = \sum_{j=1}^{J} I_t^{(j)} \left(\phi_0^{(j)} + \sum_{i=1}^{P_j} \phi_i^{(j)} k_{t-i} + u_t^{(j)}\right), r_{j-1} \leq k_{t-d} \leq r_j \tag{5}$$

where $I_t^{(j)}$ is an indicator function for the jth regime. It will take the value 1 when in it is in regime j and 0 otherwise. k_{t-d} is the regime switching variable and u_t is the error term.

To estimate the parameters in the model, [p, clement] assuming normality, let $l_0(\hat{\phi}, \hat{\sigma}_a^2)$ be the log-likelihood function evaluated at the maximum estimates of $\phi = \left(\phi_0, \ldots, \phi_p\right)'$and σ_a^2. The likelihood function under the alternative can be determined if r_1 is given. Let $l_1\left(r_1; \hat{\phi}_1, \hat{\sigma}_1^2; \hat{\phi}_2, \hat{\sigma}_2^2\right)$be the log likelihood function evaluated at the maximum likelihood estimates of $\phi_i = \left(\phi_0^{(i)}, \ldots, \phi_p^{(i)}\right)'$and σ_i^2, conditioned on knowing the threshold r_1. The log likelihood ratio $l(r_1)$ defined as $l(r_1) = l_1(r_1; \hat{\phi}_1, \hat{\sigma}_1^2; \hat{\phi}_2, \hat{\sigma}_2^2) - l_0(\hat{\phi}, \hat{\sigma}_a^2)$ is then a function of the threshold r_1, which is unknown.

To determine the lag order, Akaike information criterion (AIC) shall be used. Tong (1990) suggested an AIC for threshold models as can be represented in Eq. (6):

$$\text{AIC}(g) = \sum_j \left\{n_j \ln \hat{\sigma}_{\varepsilon j}^2 + 2(p_j + 1)\right\} \tag{6}$$

where g is the lags, n_j is the number of observations in each regime, $\hat{\sigma}_{\varepsilon j}^2$is the residual variances of each regime, andp_j is the lag of each regime.

To estimate the parameter threshold, r and delay parameter, d in SETAR model, this study will follow Hansen and Seo's method by using a grid search procedure over two-dimensional space (Hansen & Seo, 2002; Cuthbertson, 2004).

The forecasting of nonlinear models has proven to be quite difficult. Some studies such as Daco and Satchell (1999) suggested nonlinear fit better however due to difficulty in determining the regime in which a process is at, forecast seems not to be as good. However, study by Clements and Smith (1997) proved that Monte Carlo method did quite well as compared to other forecasting method. So, in this study, the Monte Carlo method will be used to forecast our SETAR model. The Monte Carlo method for a one step forecast is given by Eq. (7):

$$\hat{k}_{m+1}^{MC} = \alpha_0 + \alpha_1 k_m + I_m(r)(\beta_0 + \beta_1 k_m) \tag{7}$$

where MC represents Monte Carlo, m is the starting point, $\alpha$and β are constants, $I(\bullet)$is the indicator function and $r$is the threshold parameter. Generally, the n step ahead Monte Carlo forecast is given by Eq. (8):

$$\hat{y}_{m+n}^{MC} = \frac{1}{N}\sum_{j=1}^{N}\hat{y}_{m+n}^{MC_j} \quad (8)$$

where j is the regime.

The models will be implemented with the help of StMoMo and tsDyn package in R studio developed by Dielman (1986) and Di Narzo et al. (2009).

2.3 Parameter estimation

This section assesses the estimated parameters for the Lee Carter, RWD and SETAR model. Figures 2, 3 and 4 describe on parameters of Lee Carter Model while Table 1 shows the estimation parameters for $\boldsymbol{k_t}$ data from SETAR model.

2.3.1 Lee Carter parameter estimates

Figures 2, 3 and 4 shows the estimated parameters of a_x, b_x and k_t for LC model. The parameter a_x depicts the shape of age specific death. Parameter $b_x$describes the changes in mortality with relation to k_t. In each specific age, mortality seems to be improving, judging by the positive values of b_x. Parameter k_t is the time varying factor in LC model. Figure 4 shows that over the years, it has a declining trend except for a few jumps in particular in 1998 and 1999. This parameter can also reflect the nature of mortality of certain population. The data used was Malaysian mortality data from 1980 to 2017. The resultant values of parameter $k_t$was then divided into two parts for modelling. The training data set was from 1980 to 2010 for in sample purpose while test data set used was from 2011 to 2017 for out of sample forecasting.

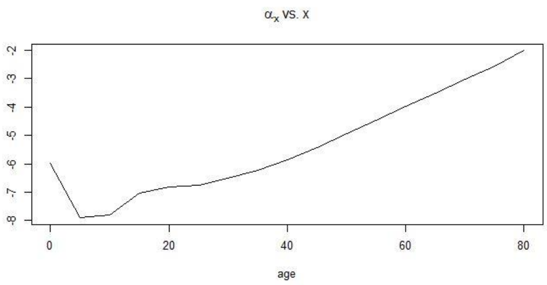


Figure 2. Estimated values of a_x over sample period of 1980-2017

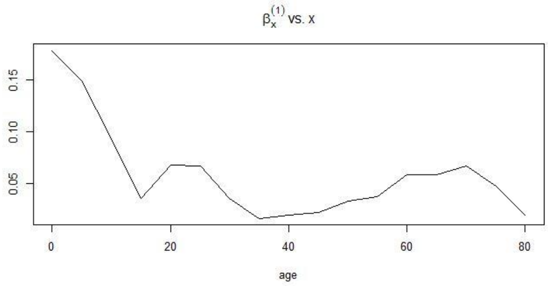


Figure 3. Estimated values of b_x over sample period of 1980-2017

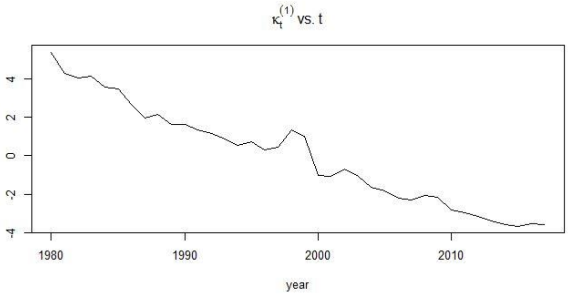


Figure 4. Estimated values of k_t over sample period of 1980-2017

2.3.2 RWD parameter estimate

In the RWD model, the only parameter needed to be estimated is drift, d. This parameter will show trend as well as the change in series. For data sample of $k_t$from 1980 to 2010, the estimate is

$d = -0.2715$. So, the model is given by Eq. (9):

$$k_t = k_{t-1} - 0.2715 + \varepsilon_t \tag{9}$$

where $\varepsilon_t \sim N(0,0.0918)$.

2.3.3 SETAR parameter estimate

There are a few steps to take in order to estimate parameters for SETAR. First, the number of regimes is determined and number of lags each regime should take by using grid search. Table 1 shows the grid search results with lowest AIC combination sits on top of the table. The number of regime, Nr is two regimes, with lag of first regime, $p^{(1)}$is one and lag of second regime, $p^{(2)}$ is one. The threshold value, r was estimated to be 1.0233.

Table 1. Estimated SETAR parameters for $k_t$data over sample period of 1980-2010. mL is the number of lag in lower regime and mH is the number of lag in upper regime

	mL	mH	Threshold	AIC
1	**1**	**1**	**1.0232530**	**-41.58236**
2	1	1	1.1568560	-41.44133
3	1	1	0.3126367	-41.01514
4	1	2	1.0232530	-40.15852
5	1	1	0.8542161	-40.13361
6	1	2	0.8542161	-40.12701
7	1	2	1.1568560	-39.98272
8	1	2	0.3126367	-39.82830
9	2	1	1.0232530	-39.76650
10	2	1	1.1568560	-39.54078

Following that, the coefficients,ϕ for each regime is estimated and regime switch variable, k_{t-d}. The k_{t-d} was determined to be

$$k_{t-d} = k_{t-1}$$

where $\beta_0 = 0$and $\beta_1 = 1$. And the SETAR(2,1,1) for k_t over sample period of 1980-2010 is given by Eq. (10):

$$k_t = \begin{cases} 1.0727k_{t-1} - 0.0234 + u_t \text{if } k_{t-1} < 1.023 \\ 1.0899k_{t-1} - 0.6308 + u_t \text{if } k_{t-1} \geq 1.023 \end{cases} \tag{10}$$

where $u_t \sim N(0{,}0.1894)$.

3. Results and Discussion

In this study, the goodness of fit for a model will be determine using several statistical measures. This will help in deciding which model is better in terms of in sample fit and out of sample forecast. The measures include Akaike information criterion (AIC), root mean squared error (RMSE) and mean absolute percentage error (MAPE). AIC is used to find parsimony model. It penalizes models that includes more parameters. The objective is to determine model with minimum AIC value as it represent model that fit data better. RMSE measures the standard deviation of estimated or forecasted value from the actual value. MAPE measures how far is the distance between estimated and the actual value. With the error measure, the model with lowest value will be deemed to be the most accurate. Should there be outliers in the actual values or data, Dielman (1986) suggest that MAPE to be used instead of RMSE. The out-of-sample results for RWD and SETAR models were then compared. Then, a long-term analysis up to 23 years ahead was conducted.

3.1 Scatter plot of residuals comparison

To get a good model fit, the scatter plot of residuals must be randomly distributed and show no clear pattern or clump. Figures 5 and 6 shows the deviance of residuals for both RWD and SETAR models to $k_t$data. The closer the residuals to 0 indicates a better fit between fitted value to original data. Both scatter plots show no real pattern apart from a few outliers.

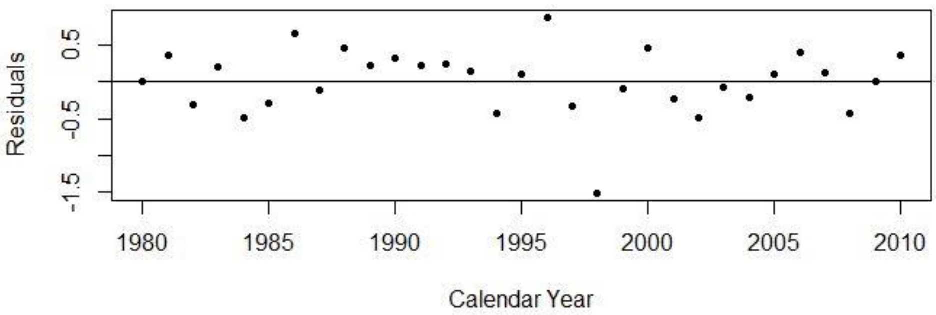


Figure 5. Scatter plot of residuals for RWD for period 1980 to 2010

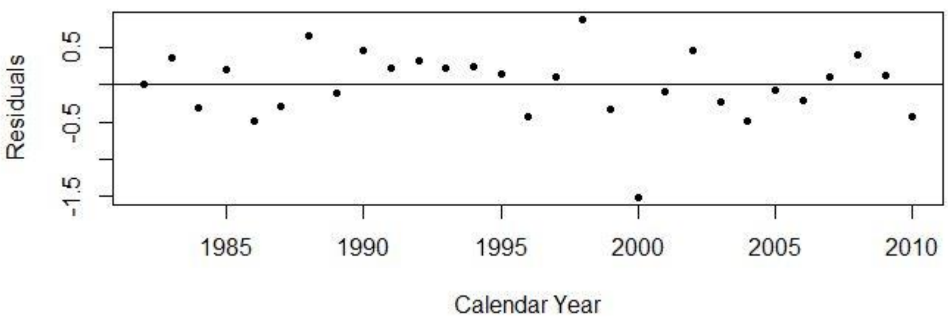


Figure 6. Scatter plot of residuals for SETAR for period 1980 to 2010

3.2 AIC and BIC comparison

Table 2 shows AIC and BIC values of both models. AIC and BIC can also rank models based on parsimony, that is model that can explain or fit data based on optimal number of parameters. A low parsimony model is a model that consist of more parameters to fit model. A high parsimony model is a model that uses less parameter to fit model. Ideally, this study prefers somewhere in the middle where it is optimal. A low AIC and BIC are preferred since the calculation impose a penalty on model that use more parameter. Here, since the number of parameters used in SETAR model is more, it is expected that it will fit the data better. However, since the log likelihood for SETAR is bigger, it resulted in lower AIC and BIC values. Therefore, SETAR model fits data better while being more parsimony.

Table 2. Number of parameters, log likelihood, AIC and BIC for RWD and SETAR model fitted to k_t for period 1980-2010

Model	Number of parameters	Log likelihood	AIC	BIC
RWD	2	-21.95	47.9	50.71
SETAR	6	25.885	**-39.77**	**-31.16**

3.3 MAPE and RMSE comparison

Table 3 presents the values of MAPE and RMSE for both models. The error representation of how far the fitted values to the original data values are. The lower value of MAPE and RMSE for SETAR model indicated the fitted values are much closer to the original data values.

Table 3. MAPE and RMSE values for RWD and SETAR model fitted to k_t for period 1980-2010

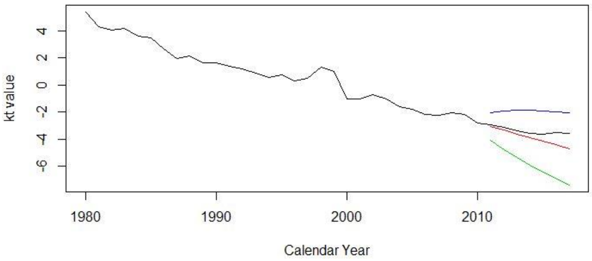
Model	MAPE	RMSE
RWD	0.715	0.5029
SETAR	**0.3932**	**0.4486**

3.4 Forecasting for out of sample comparison

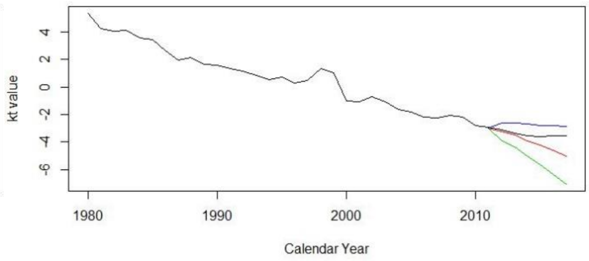
The values of k_t was forecasted via each model's forecast method and then compared against the original $k_t$data values from the LC model. MAPE and RMSE will measure the forecasting accuracy of both methods. Table 4 shows MAPE and RMSE values of both models. The results show that SETAR model have lower MAPE and RMSE values suggesting better forecasting accuracy than RWD in forecasting $k_t$over the sample period.

Table 4. RMSE and MAPE for RWD and SETAR 7 years' projection for k_t over sample period 2011-2017

Model	MAPE	RMSE
RWD	0.1370011	0.5984093
SETAR	**0.1152654**	**0.5689715**



RWD
(a)



SETAR
(b)

Figure 7(a) and Figure 7(b). Out of sample forecast of k_t based on RWD and SETAR models fitted to $\boldsymbol{k_t}$ over sample period 1980-2017. Black lines represent original k_t data values, blue lines represent upper 95% forecast interval, red line represents the mean forecast and green line represent lower 5% forecast interval

Out of sample forecast of k_t shows huge difference in both models. Figure 7(a) and Figure 7(b) shows the forecast along with upper 95% forecast interval and lower 5% forecast interval values. In RWD, the upper and lower interval does not begin at the same point of forecast origin. It opens up possibilities of wider mortality projections but it is also possible that the projection of a RWD will not follow historical data and start at a point further away from forecast origin. SETAR forecast's interval in the meantime does start from forecast origin indicating a more controlled forecast. Mean reversion directly starts at the origin. This means that the forecast value should return to the average line over time.

3.5 Forecasting overall k_t for 23 years ahead

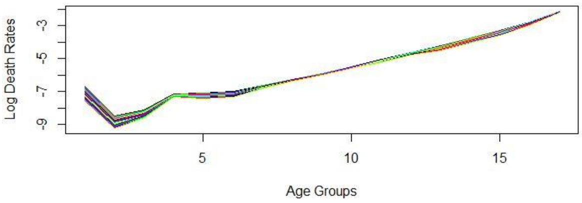
We then used the overall k_t values from LC model for sample period 1980 to 2017 and projected 23 years ahead to the year 2040 to see mortality improvements made using both models. For RWD, the model was given by

$$k_t = k_{t-1} - 0.2411 + \varepsilon_t$$

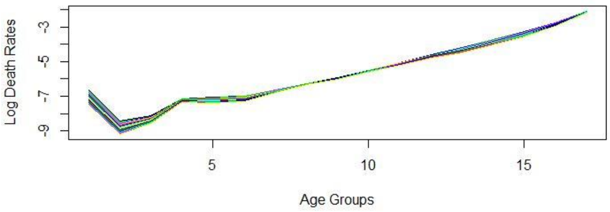
where $\varepsilon_t \sim N(0,0.0756)$. And the SETAR (2,1,1) for overall k_t data values were given by Eq. (11):

$$k_t = \begin{cases} 1.0330k_{t-1} - 0.0402 + u_t if\, k_{t-1} < 1.023 \\ 1.0899k_{t-1} - 0.6308 + u_t if\, k_{t-1} \geq 1.023 \end{cases} \quad (11)$$

where $u_t \sim N(0,0.1582)$



RWD
(a)



SETAR
(b)

Figure 8 (a) and Figure 8(b). 23 years ahead forecast for log death rate of Malaysian data using RWD and SETAR as k_t modeller

Figure 8 (a) and Figure 8(b) shows the forecast of log death rate for Malaysian mortality in the year 2018 up to 2040. Although it seems the same, but RWD did forecast mortality decline much quicker than SETAR model in all age group. However, this could be because of the wide forecast interval and poorer goodness fit to history data could lead to over-forecasting. Both models share the same foundation that is auto regression but the ability of SETAR to switch between regimes of auto regression through the regime switching variable makes it much better in model fit and hence better forecast.

4. Conclusion

This paper compares random walk with drift and self-exciting threshold auto regressive in modelling the time varying parameter, k_t in Lee Carter mortality model. Since it was noted that mortality in Malaysia is not relatively linear, we suggest the use of non-linear model in particular SETAR model to model k_t, improve the model fit and as a result produce better forecast. Each model was discussed in detail and the in sample fit and out of sample accuracy was analysed.

The result showed that SETAR model did fit k_t better than RWD. This was shown by the lower values of MAPE, RMSE, AIC and BIC. However, the residuals did not show much difference between the models.

The out of sample forecasting analysis showed that SETAR had lower errors in MAPE and RMSE than RWD. It is suggested that the forecast of SETAR is much closer to the original $k_t$data values. The forecast interval also indicates interesting finding in which RWD model forecast have a higher possibility to project wider mortality change. This could mean the forecast of RWD could deviate from forecast origin at any given time. In contrast, the forecast of SETAR is well within historical data. Lastly, the 23-year forecast into the future suggest that RWD will predict much faster mortality improvement across ages. However, since it has been shown that RWD have relatively larger error and wider forecast interval, the forecast could be due to underfit historical data and over forecast. This study indicates that nonlinear model results in better model fit and forecast than linear model in general and especially in modelling the time varying parameter, k_t in Lee Carter mortality model for Malaysian data.

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