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Characterization of Basin of Attraction for an Attractor in a Discrete Prey-Predator Sea Turtle-Human Interaction Model using Stability Index Approach

Mohd Tirmizi Mohd Lutfi[a], Ummu Atiqah Mohd Roslan[a,*], Fatemeh Helen Ghane[b], Ahmad Fadillah Embong[c]

[a] Special Interest Group on Modelling and Data Analytics, Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terengganu, 21030 Kuala Nerus, Terengganu, Malaysia; [b] Department of Mathematics, Ferdowsi University of Mashhad, Mashhad, P.O. Box 1159-91775, Iran; [c] Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 Skudai, Johor, Malaysia

***For correspondence:** ummuatiqah@umt.edu.my

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Abstract Drastic declines of the number of sea turtles have become a global issue since decades ago. This is due to overexploitation through human activities in various ways, such as poaching and trapping in the fishing nets, consuming their eggs and meat, other than being threatened by pollution, habitat destruction and many more. Therefore, it is our interest to investigate this issue through mathematical modelling. For that purpose, a discrete model of interaction between sea turtles and humans is proposed in this paper. The objective of this paper is to determine sustainability of sea turtle population in the future. In order to achieve this, an approach called the stability index, has been implemented on the basin of attraction of an attractor in the model proposed for different values of consumption rate α. The results show that the stability indices vary from $+\infty$ down to positive values as α increases. Biologically, $+\infty$ index means that the sea turtles will survive, while positive index means that riddled basin has occurred in which it is predicted that the sea turtles might extinct or not depending on the initial population existing in a habitat. Moreover, the time series have also been plotted for different values of α. The patterns show that for a low α, the sea turtles will survive in the next 10 years while when α is high, the sea turtle will extinct in less than two years. Thus, the results of this model could significantly be used to urge humans to stop exploiting these unique creatures and start appreciating their existence in the ecosystem.

Keywords: Dynamical systems, attractor, riddled basin, stability index, sea turtle-human model.

Introduction

Dynamical systems can be defined as systems that evolve with time according to definite rules. They can be useful to explain the dynamics behaviour for many real-world problems including population dynamics in ecological systems. This paper focuses on a discrete-time dynamical system in the form of:

$$x_{n+1} = g(x_n),$$

where $g: X \to X$, $x_n \in X$, g is a continuous map on metric space X with discrete-time $n \in \mathbb{Z}$. Imagine that we have an initial point $x_0 \in X$, and let g^n denote $n-$th iterate of g. While the sequence of the orbit of x_0 is considered as follows:

$$\{x_0, g(x_0), g^2(x_0), ..., g^n(x_0), ...\},$$

which are obtained by iterating the map.

The issue of human exploitation of a marine species namely sea turtles, can be studied by a dynamical system. The number of such species decreases abruptly each year due to human activities such as poaching and trapping in the fishing nets, consuming their eggs and meat, being threaten by pollution and many more. Hence, to discuss this issue, the first mathematical model on the interaction between sea turtles and humans was introduced by Mohd Roslan et al. [1], where they investigated the stability of equilibrium points. In the research, the parameter known as consumption rate of sea turtles by humans is varied.

In the article by [1], the stability of the system is studied by utilizing its solutions (equilibrium points), using the well-known linearization method. However, the stability of the system can also be investigated through the basin of attraction of the solution of the system. To achieve this, an approach called the 'stability index' was used to characterize the basin of attraction of the system in [1]. This stability index was firstly initiated by Podvigina and Ashwin [2] in 2011. They considered robust heteroclinic to study its local stability index in terms of eigenvalues of the linearization of the vector field. Years later, Lohse [3] studied the stability index of heteroclinic cycles and networks. Meanwhile, Keller [4] applied the stability index in the case of concave maps. In addition, Mohd Roslan and Ashwin [5] considered a piecewise expanding linear map to study the stability index both locally and globally for a point and attractor. Furthermore, in 2017, Mohd Roslan [6] computed the stability index for an attractor for electronic-circuit system. In 2019, Mohd Roslan and Mohd Lutfi [7] used to apply the stability index for an attractor in the competition model for flour beetle dynamic population. Recently, a remarkable work has been done by Karimi and Ghane [8] in which the stability index has been implemented to characterize the basin of an attractor in a two-species competition model. They observed that the attractor changed its stability from asymptotically stable attractor to riddled basin attractor to chaotic saddle whenever a parameter in the system varied. Moreover, the result showed that the value of stability index is positive for Lebesgue almost all points whenever the riddling occurred.

Motivated by the study of sea turtle-human interaction model in [1], in this research, the researchers intended to compute the stability index for this system. We began by introducing some basic definitions and notions in dynamical systems. We then define the stability index as proposed in [5] along with its algorithm. In the next section, we will introduce a discrete model of sea turtle-human. Finally, we will discuss the results of stability index for the discrete model in the last section. In addition, to support the results of the stability index, time series have been plotted.

Definitions and Preliminaries

In this section, some basic definitions of dynamical systems were defined. Generally, in dynamical systems, the researchers were keen on invariant sets. Some examples of invariant sets were the set of all fixed points, periodic orbits, limit cycles, and attractors. In this paper, the attractors seem engaging to be explored. Throughout this paper, X denoted a metric space. Glendinning [9] defines an invariant set as follows:

Definition 2.1. *A compact set $A \subset X$ is said to be a forward invariant set (or backward invariant set) if $x \in A$ implies that $g(x) \in A$ (or $g^{-1}(x) \in A$). Thus, A is invariant if it is both forward and backward invariant.*

The above definition suggested that if an initial point starts in the invariant set A, then its entire orbit will also stay on this invariant set. For an attractor, there exists its own basin of attraction, defined by the set of initial points whose orbits approaching to the invariant set. Thus, the mathematical definition for a basin of attraction as defined by Milnor [10] is presented here as below:

Definition 2.2. *The basin of attraction $\mathcal{B}(A)$ is the set of points $x \in X$ whose $\omega-$limit set is* *contained in* A *that is* $\mathcal{B}(A) = \{x \in X \mid \omega(x) \subset A\}$.

For non-empty A, the basin of attraction $\mathcal{B}(A)$ is always non-empty because of the compactness of A. This means $\mathcal{B}(A)$ is required to be large in a suitable sense for A to be an attractor [11]. To define the attractor, we refer the definition as suggested by [10].

Definition 2.3. *A compact invariant subset* $A \subset X$ *is called a Milnor attractor if it satisfies two conditions:*

(i) The basin of attraction $\mathcal{B}(A)$ *has strictly positive Lebesgue measure (i.e.* $\ell(\mathcal{B}(A)) > 0$*), where all points converge to* A*; and*

(ii) For any closed proper subset $A' \subset A$, $\ell(\mathcal{B}(A) \setminus \mathcal{B}(A')) > 0$.

However, Melbourne [12] has defined a stronger version of Milnor attractor as:

Definition 2.4. *Let* $B_\delta(A)$ *is a* δ*-neighbourhood of* A *in* X. A *is an essential attractor if*

$$\lim_{\delta \to 0} \frac{\ell\left(B_\delta(A) \cap \mathcal{B}(A)\right)}{\ell\left(B_\delta(A)\right)} = 1,$$

where $\ell(.)$ *is a Lebesgue measure on* X.

Consequently, according to [5], the attractor in Definition 2.4 is also referred as the asymptotically stable attractor (a.s.a.). The definition of Lyapunov stable is needed as the main key for a.s.a. [9]:

Definition 2.5. A *is said to be Lyapunov stable if for any neighbourhood* U *of* A*, there exists a neighbourhood* V *of* A *such that* $g^n(V) \subset U$ *for all* $n \in \bullet$.

Definition 2.6. A *is said to be an a.s.a. if it is Lyapunov stable and the basin* $\mathcal{B}(A)$ *of* A *contains a neighbourhood of* A*, i.e.* $\mathcal{B}(A) \supset B_\delta(A)$.

If there are two or more attractors in a system, therefore there will be more than one basin of attraction. There are two possible situations for the basins of attraction:

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(i) Such basins are separated by a basin boundary.
(ii) Basins can intercept with each other.

In this research, the latter case is considered when a phenomenon called as the 'riddled basin' is referred to. In 1992, Alexander [14] defined the following:

Definition 2.7. *The basin of attraction of an attractor is riddled if its complement intersects every neighbourhood in a set of positive Lebesgue measure.*

Ashwin [11] defines the measure of riddled basin as follows:

Definition 2.8. *A Milnor attractor* A *has a riddled basin if for every* $x \in \mathcal{B}(A)$ *and* $\delta > 0$*, one has*

$$\ell\left(B_\delta(x) \cap \mathcal{B}(A)\right) \ell\left(B_\delta(x) \cap \mathcal{B}(A)^c\right) > 0.$$

If the basins are riddled with each other, then they are said to be intermingled.

Definition 2.9. *Suppose that there are two basins of attraction, basin* $\mathcal{B}(C)$ *for attractor* C *and basin* $\mathcal{B}(D)$ *for attractor* D*. If the two basins are riddled with each other, i.e.* $\mathcal{B}(C)$ *is riddled with* $\mathcal{B}(D)$

and $\mathcal{B}(D)$ is riddled with $\mathcal{B}(C)$, then we say that the two basins are intermingled basins.

In this paper, the researchers were highlighted the changes of stability of attractor from being an asymptotically stable to riddled basin for the system considered using the approach of stability index which will be explained in the next section.

Methodology

In this section, the stability index is briefly explained. Based on previous works by [5, 6, 7], the stability index can be considered as a new approach that has great potential in the study of bifurcation. This stability index is particularly useful for discussing the stability of an attractor of a system that is by quantifying the local geometry of a basin of attraction of the attractor. This paper has been focusing on finding the stability index for an attractor. The authors in [5] has defined the stability index as:

Definition 2.10. *Let A be an attractor and let again $B_\delta(A)$ be the neighbourhood of A with radius $\delta > 0$. Denote*

$$\Sigma_\delta(A) := \frac{\ell\left(B_\delta(A) \cap \mathcal{B}(A)\right)}{\ell\left(B_\delta(A)\right)},$$

and

$$1-\Sigma_\delta(A) := \frac{\ell\left(B_\delta(A) \cap \mathcal{B}(A)^c\right)}{\ell\left(B_\delta(A)\right)},$$

where basin $\mathcal{B}(A)^c$ is a complement of basin $\mathcal{B}(A)$. Then the stability index of A is given by

$$\sigma(A) := \sigma_+(A) - \sigma_-(A),$$

which exists when the following limits converge:

$$\sigma_-(A) := \lim_{\delta \to 0} \frac{\ln\left(\Sigma_\delta(A)\right)}{\ln \delta}, \quad \sigma_+(A) := \lim_{\delta \to 0} \frac{\ln\left(1-\Sigma_\delta(A)\right)}{\ln \delta}.$$

Note that $0 \le \Sigma_\delta(A) \le 1$ and therefore $\sigma(A) \in (-\infty, \infty)$.

According to above definition, the value of stability index can be ranged from $-\infty$ to ∞ depending on the behaviour of a system as a chosen parameter is varied.

Algorithm of stability index

Mohd Roslan and Mohd Lutfi [7] stated the algorithm for stability index for the attractor in the system as follows:

Step 1: Choose a parameter in the system.
Step 2: Choose $\delta -$neighbourhood around A.
Step 3: Compute the proportion $\Sigma_\delta(A)$.
Step 4: Reduce the size of $\delta -$neighbourhood.
Step 5: Compute the proportion as $\delta \to 0$.
(i) If proportion $\Sigma_\delta(A) = 1$, then $\sigma(A) = \infty$.
(ii) If proportion $\Sigma_\delta(A) = 0$, then $\sigma(A) = -\infty$.
(iii) If $0 < \Sigma_\delta(A) < 1$, then $\sigma(A) > 0$ or $\sigma(A) < 0$.
Step 6: Repeat Steps 1 to 5 for different values of parameter chosen.

In this paper, the computations of stability index will be performed using MATLABR2019b software.

Numerical Example: Sea Turtle-Human Model

In this section, we have considered a discrete model of the prey-predator sea turtle-human in which the original model was proposed in [1]:

$$\begin{aligned} x_{n+1} &= rx_n(1-x_n) - \omega x_n - \alpha x_n y_n + x_n, \\ y_{n+1} &= sy_n(1-y_n) + \alpha x_n y_n - py_n + y_n, \end{aligned} \tag{1}$$

where there is an interaction between the number of sea turtles, x with the number of humans who consume the sea turtle, y at time, n. Parameters stated in (1) are the sea turtle population growth rate r, consumption rate by human α, human population growth rate s, sea turtle natural death rate ω, and human population death rate p, in which all parameters are positive constants. Equation (1) states that both sea turtle and humans obey the logistic growth rate since the first term for each line in the equation follows the usual logistic map equation.

Suppose that the dynamics of model (1) is written as mapping notation, G where $G(x_n, y_n) = (x_{n+1}, y_{n+1}) = \left(G_1(x_n, y_n), G_2(x_n, y_n)\right)$ is given by

$$G(x_n, y_n) = \left(rx_n(1-x_n) - \omega x_n - \alpha x_n - \alpha x_n y_n + x_n, sy_n(1-y_n) + \alpha x_n y_n - py_n + y_n\right),$$

and the iterates as

$$G^{(n)}(x_n, y_n) = \left(x_{n+1}^{(n)}, y_{n+1}^{(n)}\right).$$

From system (1), it is intrigued to investigate the dynamic behaviour of this system as the consumption rate α varies. We expect that the system will have different types of stability as α increases.

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Results and Discussion

In this section, the attractors and their basins of attraction for the map (1) were identified. Later, the numerical results of stability index for one of the attractors obtained were presented.

Attractors and basins of attraction

It can be observed that the map (1) has two invariant subspaces: $N_0 = \sim \times \{0\}$ and $N_1 = \{0\} \times \sim$. These subspaces are invariant by G since $G(\sim \times \{0\}) \subseteq \sim \times \{0\}$ and $G(\{0\} \times \sim) \subseteq \{0\} \times \sim$. Clearly, since $G_1|_{N_0}$ and $G_2|_{N_1}$ are conjugated to logistic map, therefore for some values of parameters known as α, r and s, have proven to admit chaotic attractors A_0 and A_1. In this case, the occurrence of riddled basin for A_0 and A_1 can be verified, by varying the parameter. In this paper, the parameter of interest is α, which represents the consumption rate of sea turtles by human. This shown that by varying α, riddled basin could occur for system (1).

The numerical approximation of the basins of attraction for model (1) were plotted in Figure 1 for different values of parameter α. The black region represented by $\mathcal{B}(A_0)$ is the basin such that the initial points attracted to $A_0 = \sim \times \{0\}$ while the white area represented by $\mathcal{B}(A_1)$ is the basin where initial points are attracted to attractor at $A_1 = \{0\} \times \sim$. From the figure, it is obvious that as α increases, the black region decreases. To characterize the stability of attractor based on this basin of attraction, the researchers have chosen the attractor A_0 as the main interest. It is intrigued to investigate how the stability of this attractor changes using the concept of stability index as proposed originally by Podivigina and Ashwin in 2011 [13]. If A_0 stable, then it is expected that the sea turtles will survive in the future. However, when A_0 is no longer stable, this means that the phenomenon of riddled basin might occurred which indicate that such population might extinct or not depending on the initial population in a certain habitat.

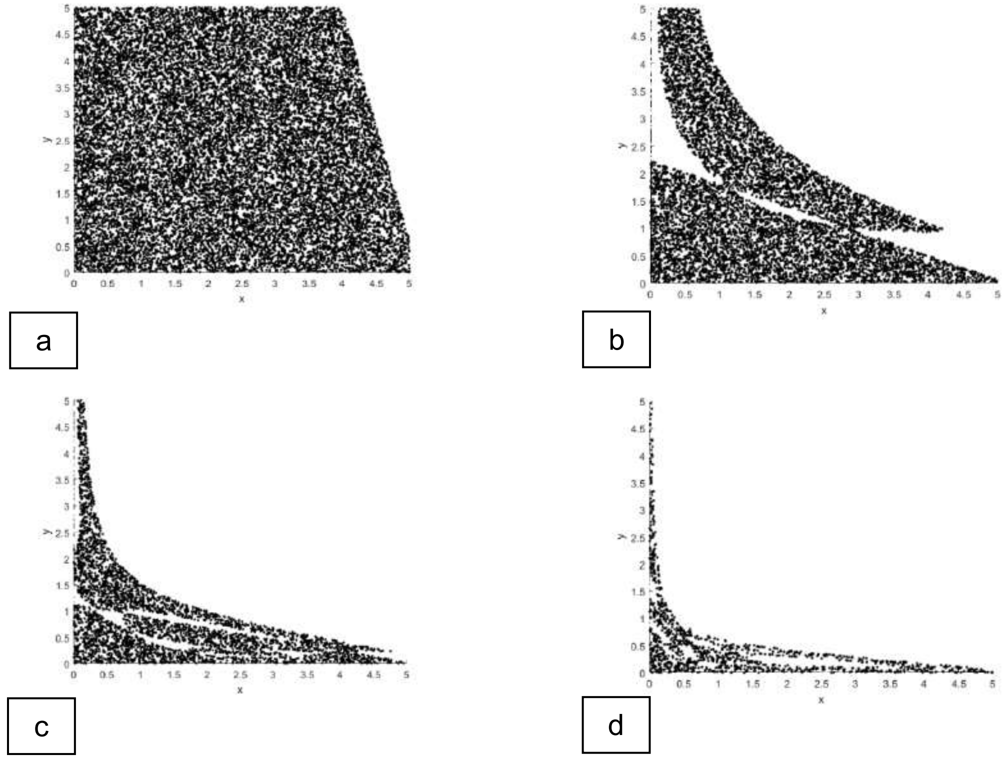

Figure 1. The approximate basin of attraction for system (1) with fixed parameters $r = 0.2174, s = 0.1, \omega = 0.1, p = 0.2$ and various α . (a) $\alpha = 0.05$, (b) $\alpha = 0.5$, (c) $\alpha = 1$, (d) $\alpha = 2$. The black region corresponds to basin of attraction $\mathcal{B}(A_0)$ for attractor A_0 while white region is the basin for the second attractor A_1.

Computations of stability index

In this section, the stability index for attractor A_0 was determined numerically. The algorithm in the previous section was applied to compute the stability index. First, the researchers plotted the proportion of points that converged to A_0 in Figure 2. It was observed that the proportion has gradually decreased as the values of parameter increased due to the reduction of black region as obtained in the previous figure, i.e. Figure 1.

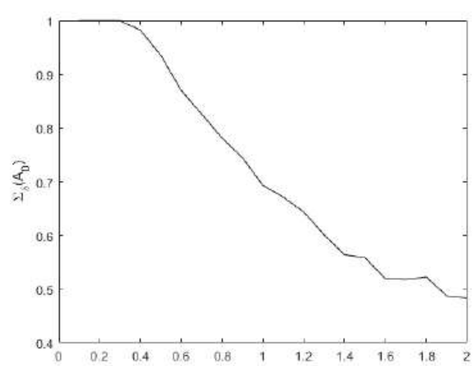

Figure 2. The proportion $\Sigma_\delta(A)$ versus $\alpha = 0, \ldots, 2$. The proportion remained at 1 until $\alpha_c = 0.4$ and then plummeted since the area of black region in Figure 1 becomes insignificant as α increases.

We then picked a δ-neighbourhood around A_0 in which $A_0 = \sim \times \{0\}$. The proportion of points that converged to A_0 within the δ-neighbourhood was computed thereafter. The proportion was evaluated by shrinking the size of the δ-neighbourhood. Figure 3 below illustrates how the value of δ-neighbourhood was changed to get the value of proportions that converge A_0 when $\alpha = 0.05$ and $\alpha = 5$.

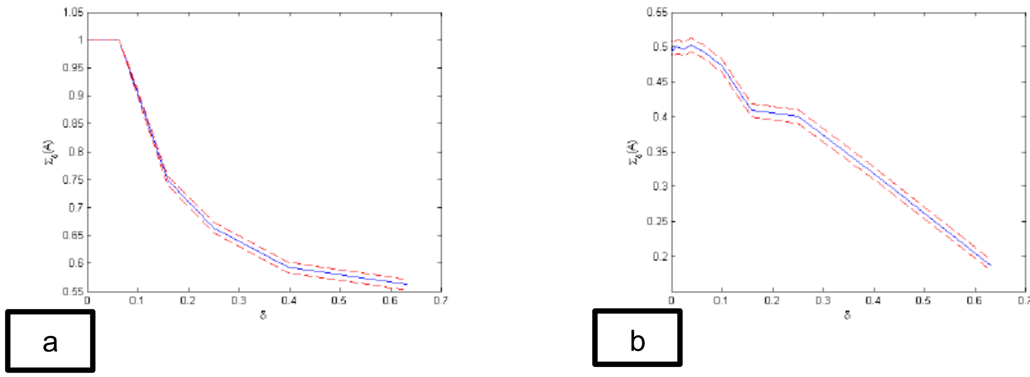

Figure 3. The numerical approximation for the pattern of proportions that converge A_0 for various size of δ-neighbourhood. (a) For $\alpha = 0.05$ and (b) for $\alpha = 5$.

Figure 3a indicates the proportion $\Sigma_\delta(A_0)$ increased and finally reached the value 1 as $\delta \to 0$. When $\Sigma_\delta(A_0) = 1$, this means that the δ-neighbourhood was only filled with the orbits of initial values that converged to A_0. Figure 3b illustrates the proportion $\Sigma_\delta(A_0)$ has increased, but $\Sigma_\delta(A_0)$ never reached value 1. When the value of δ-neighbourhood decreased, the proportion of points that converged to both attractor A_0 and A_1 were still there. This was the case where the riddled basin occurred in which there were always orbits of initial values which converge to another attractor, that is A_1 as $\delta \to 0$. The next stage, $\sigma_-(A_0)$ is determined from the slope of $\ln(\Sigma_\delta(A_0))$ versus $\ln(\delta)$. On the other hand, $\sigma_+(A_0)$ is determined from the slope of $\ln(1-\Sigma_\delta(A_0))$ versus $\ln(\delta)$. Finally, the stability index $\sigma(A_0)$ was evaluated by subtracting $\sigma_-(A_0)$ from $\sigma_+(A_0)$. The numerical result of stability index $\sigma(A_0)$ for the attractor A_0 for map (1) was shown in Figure 4 for $\alpha \in [0, 2]$.

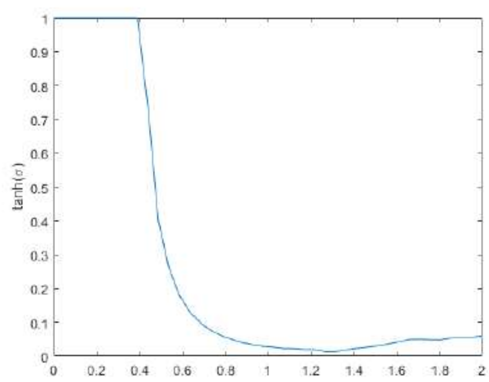

Figure 4. The numerical approximation of stability index for map (1) for the range of parameter $\alpha \in [0, 2]$. Note that the value of 1 on the vertical axis represents that the stability index has a value of $+\infty$.

We note that the value 1 on the vertical axis in the graph represents the index value, $\sigma(A_0) = +\infty$. When $0 \le \alpha \le 0.390244$, $\sigma(A_0) = +\infty$ since all the orbits of initial points within δ-neighbourhood of A_0 belong to $\mathcal{B}(A_0)$. When $0.390244 < \alpha \le 2$, the values of the indices decrease from positive value down to 0 since more points belong to $\mathcal{B}(A_1)$. This is where the riddled basin occurs since there are always trajectories that escape to $\mathcal{B}(A_1)$ no matter how small the δ-neighbourhood is. Thus, the results of stability index discussed above can be summarized as follows:

(i) When $0 \le \alpha \le 0.390244$, $\sigma(A_0) = +\infty$.

(ii) When $0.390244 < \alpha \le 2$, $\sigma(A_0) > 0$.

From the above results, it is clear that the stability indices only have values of $+\infty$ and positive values. We can then interpret the results as follows:

(i) When $\sigma(A_0) = +\infty$, this means that A_0 is a.s.a.

(ii) When $\sigma(A_0) > 0$, this corresponds to the case where A_0 now has a riddled basin.

Implications on ecological modelling

The main attractor A_0 was defined at $A_0 = \sim \times \{0\}$. This attractor is referred to the case where $y = 0$ in map (1), in which there is no human would consume the sea turtles' product. Meanwhile, $A_1 = \{0\} \times \sim$ is referred to the case where there is no sea turtle population in the future, i.e. $x = 0$ in the map (1). Ecologically, we can interpret as follows:

(i) For the low consumption rate α, i.e. when $0 \le \alpha \le 0.390244$, the sea turtles will survive in the future ($x > 0$).

(ii) For the high consumption rate α, i.e. when $0.390244 < \alpha \le 2$, the extinction possibility in the population of sea turtle depends on the species initial population in the ecosystem.

Prediction on the sustainability of sea turtle population

In this section, the researchers have investigated the sustainability of the population of sea turtles by performing time series plots for the map (1). Based on the plot of stability index in Figure 4, the low and the high consumption rates occurred when $\alpha \in [0, 0.390244]$ and $\alpha \in (0.390244, 2]$ respectively. To compare the survival rate of such population, two values of parameter were chosen within these ranges. Thus, we plotted the time series in Figure 5 for $\alpha = 0.05$ and $\alpha = 0.4$ for the time frame $t \in [0,10]$, where t was measured in years.

Based on Figure 5(a), for a low consumption rate, it was expected that the population of sea turtles could still be sustained in the next 10 years. However, the pattern shows that the number decreased dramatically as time increased. Meanwhile, in Figure 5(b), such a population was predicted to be extinct in less than 2 years due to the high consumption of the sea turtles. From the two figures, no matter how low or high the consumption rate is, it still affects the survival of sea turtles badly, which could lead to total extinction.

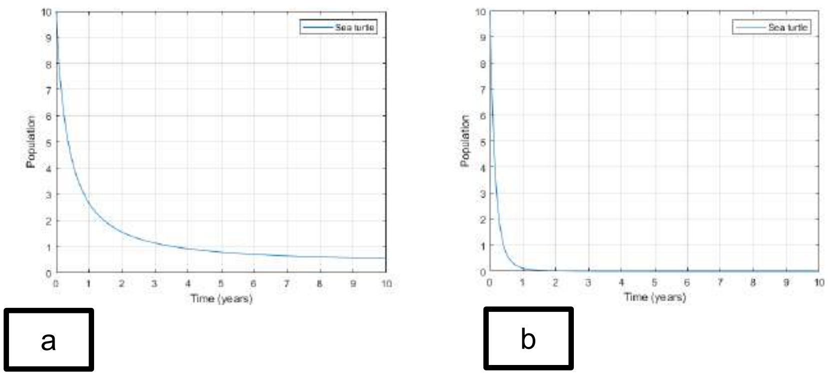

Figure 5. Plot of time series for map (1) when (a) $\alpha = 0.05$ and (b) $\alpha = 0.4.$

Conclusions

In conclusion, the dynamics behaviour of a discrete model of the sea turtle-human can be highly considered to plan for human reconciliation with the sea turtles. The solutions of the forms of attractors have been identified and clearly defined in this model. The study has discovered that there exist two attractors in the system (1): A_0 and A_1 which are defined at $\sim \times \{0\}$ and $\{0\} \times \sim$ respectively. A_0 corresponds to the case where sea turtles survive, while A_1 refers to the extinction of sea turtles. The basins of attraction of these attractors are then plotted using Matlab R2019b software.

To determine the stability of the system (1), A_0 was identified as our main interest. We then applied the stability index to characterize the basin of attraction of A_0. On the one hand, the results suggested that for a lower consumption rate α, A_0 was asymptotically stable which implied the survival of sea turtle population in the future. On the other hand, when α is high, A_0 was no longer stable whereby at this stage, riddled basin would occur. The concept of a riddled basin has involved the element of sensitive dependence on initial conditions, which implied that the probability of the sea turtle's extinction could depend on the initial population in a habitat. Moreover, these findings were supported by using the time series plot. From the plots, it was obvious that the sea turtles would still be extinct in the future no matter how low or high the consumption rate would be.

Therefore, it is an urge to increase public awareness to stop consuming sea turtles' product or harm the sea turtle population in any ways. When there is no exploitation on such unique creatures, it is hoped that the population can sustain and live in harmony in the ecosystem.

Conflicts of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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