ORDER PRODUCT PRIME GRAPH AND ITS VARIATIONS OF SOME FINITE GROUPS

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DEDICATION

To my father and my mother, who with love and effort have accompanied me in this process, without hesitating at any moment of seeing my dreams come true, which are also their dreams.

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ABSTRACT

The study of groups from geometric viewpoint has recently become one of the focus of researches in group theory, which started with the Cayley graph. Later, the study grew through the years, leading to the definition of many graphs of groups and investigation of graphical properties of finite groups. This development exists due to the fact that groups can be profitably studied as geometric objects in their own right, since the geometry exists both in the group itself and in the spaces it acts on. This study basically shows how groups and spaces interact together, which helps in understanding the symmetries of much more complicated objects. In this thesis, the order product prime graph of finite groups is defined as the graph whose vertices are the elements of the groups, and any two vertices are adjacent if and only if the product of their orders is a prime power. Moreover, another graph which is commuting order product prime graph of finite groups is defined as the graph whose vertices are the elements of finite groups, and any two vertices are adjacent if and only if they commute and the product of their order is a prime power. Furthermore, these definitions are extended to the order prime permutability graph of subgroups of finite groups as the graph whose vertices are the proper subgroups of finite groups, and any two vertices are adjacent if and only if they permute and the product of their order is a prime power. Also the order prime permutability graph of cyclic subgroups of finite groups is defined as the graph whose vertices are the proper subgroups of finite groups, and any two vertices are adjacent if and only if they are permuting cyclic subgroups and the product of their orders is a prime power. The order product prime graph is connected, complete and regular on all quasi-dihedral groups, cyclic groups of prime power order and generalized quaternion groups, Q_{4n}, where n is even prime power. On dihedral groups, the graph is connected only if the degree is prime power, but complete and regular if the degree is even prime power. The commuting order product prime graph is connected, complete and regular on cyclic groups of prime power order and connected on quasi-dihedral groups, dihedral groups of prime power degree and generalized quaternion groups, Q_{4n}, where n is even prime power. Next is the order prime permutability graph of subgroups, which is connected, complete and regular on cyclic groups of prime power order and connected on quasi-dihedral groups, dihedral groups, D_n and generalized quaternion groups, Q_{4m}, where m is even prime power or just prime. Finally, the order prime permutability graph of cyclic subgroups, is connected, complete and regular on cyclic groups of prime power order and connected on dihedral groups of prime degree and generalized quaternion groups, Q_{4p}. The properties of the graphs are used in obtaining their invariants on cyclic groups, dihedral groups, generalized quaternion groups and quasi-dihedral groups, which include the clique number, independence number, domination number, girth, diameter, vertex chromatic number, edge chromatic number and some other recently introduced chromatic numbers, which are the dominated chromatic number and locating chromatic number. Moreover, the general presentations of the graphs on the above groups are used in exploring the number of perfect codes of the graphs, which has also been recently introduced on graphs of groups.

ABSTRAK

Kajian kumpulan dari sudut pandangan geometri telah menjadi salah satu fokus penyelidikan dalam teori kumpulan, yang bermula dengan graf Cayley. Kemudian, kajian ini berkembang dari tahun ke tahun, yang membawa kepada banyak penakrifan graf kumpulan dan penyiasatan sifat geometri kumpulan terhingga. Perkembangan ini wujud disebabkan oleh hakikat bahawa kumpulan boleh dikaji sebagai objek geometri dalam hak mereka sendiri, kerana geometri wujud dalam kumpulan itu sendiri dan di ruang yang ia bertindak. Kajian ini pada dasarnya menunjukkan bagaimana kumpulan dan ruang berinteraksi bersama-sama, yang membantu dalam memahami simetri objek yang lebih rumit. Dalam tesis ini, graf peringkat hasil darab perdana bagi kumpulan terhingga ditakrifkan sebagai graf yang bucunya adalah unsur-unsur kumpulan tersebut, dan mana-mana dua bucu adalah bersebelahan jika dan hanya jika hasil darab peringkat mereka adalah kuasa bagi nombor perdana. Selain itu, satu lagi graf iaitu graf peringkat hasil darab perdana bertukar tertib kumpulan terhingga ditakrifkan sebagai graf yang bucunya adalah unsur-unsur kumpulan terhingga, dan mana-mana dua bucu adalah bersebelahan jika dan hanya jika mereka adalah kalis tukar tertib dan hasil darab peringkat mereka adalah kuasa bagi nombor perdana. Seterusnya, definisi ini diperluaskan kepada graf peringkat kebolehaturan perdana subkumpulan kumpulan terhingga sebagai graf yang bucunya adalah subkumpulan wajar kepada kumpulan terhingga, dan mana-mana dua bucu adalah bersebelahan jika dan hanya jika mereka adalah berpilih atur dan hasil darab peringkat mereka adalah kuasa bagi nombor perdana. Juga, graf peringkat kebolehaturan perdana subkumpulan kitaran kumpulan terhingga ditakrifkan sebagai graf yang bucunya adalah subkumpulan wajar kumpulan terhingga, dan mana-mana dua bucu adalah bersebelahan jika dan hanya jika mereka adalah subkumpulan kitaran berpilih atur dan hasil darab peringkat mereka adalah kuasa bagi nombor perdana. Graf peringkat hasil darab perdana adalah bersambung, lengkap dan sekata pada semua kumpulan kuasi-dihedral, kumpulan kitaran peringkat kuasa perdana dan kumpulan kuarternion umum, Q_{4n}, di mana n adalah kuasa perdana genap. Pada kumpulan dihedral, graf adalah bersambung hanya jika darjahnya adalah kuasa perdana, tetapi lengkap dan sekata jika darjahnya adalah kuasa perdana genap. Graf peringkat hasil darab perdana bertukar tertib adalah bersambung, lengkap dan sekata pada kumpulan kitaran peringkat kuasa perdana dan bersambung pada kumpulan kuasi- dihedral, kumpulan dihedral darjah kuasa bagi nombor perdana dan kumpulan kuarternion umum, Q_{4n}, di mana n kuasa perdana genap. Seterusnya adalah graf peringkat kebolehaturan perdana subkumpulan, yang mana bersambung, lengkap dan sekata pada kumpulan kitaran peringkat kuasa perdana dan bersambung pada kumpulan kuasi-dihedral, kumpulan dihedral, D_n dan kumpulan kuarternion umum, Q_{4n}, di mana n adalah kuasa perdana genap atau hanya perdana. Akhir sekali, graf peringkat kebolehaturan perdana subkumpulan kitaran adalah bersambung, lengkap dan sekata pada kumpulan kitaran peringkat kuasa perdana dan bersambung pada kumpulan dihedral darjah perdana dan kumpulan kuarternion umum, Q_{4n}. Sifat-sifat ini digunakan dalam mendapatkan invarian umum graf kepada kumpulan kitaran, kumpulan dihedral, kumpulan kuarternion umum dan kumpulan kuasi-dihedral, yang termasuk nombor klik, nombor ketakbersandaran, nombor dominasi, lilitan, diameter, nombor berkroma bucu, nombor berkroma tepi dan beberapa nombor berkroma lain yang baru-baru mi diperkenalkan, iaitu nombor berkroma yang dominasi dan nombor berkroma penentu. Selain itu, persembahan umum graf pada kumpulan graf di atas digunakan bagi meneroka bilangan kod sempurna kepada graf, yang juga baru-baru ini diperkenalkan pada graf kumpulan.

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$x \sim y$	-	Adjacency of vertices x and y
$\omega(\Gamma)$	-	Clique number of a graph Γ
$\overline{K}_n$	-	Complement of a complete graph on n vertices
K_n	-	Complete graph on n vertices
$\Gamma^{copp}(G)$	-	Commuting order product prime graph of a group G
$\mathbb{Z}_n$	-	Cyclic group of order n
$deg(x)$	-	Degree of vertex x
$diam(\Gamma)$	-	Diameter of the graph Γ
D_{2n}	-	Dihedral group of degree n
$d(x, y)$	-	Distance between two vertices x and y
$\chi_{dom}(\Gamma)$	-	Dominated chromatic number of the graph Γ
$\gamma(\Gamma)$	-	Domination number of the graph Γ
$\chi^{'}(\Gamma)$	-	Edge chromatic number of the graph Γ
$E(\Gamma)$	-	Edge set of a graph Γ
K_0	-	Empty graph with no vertex
$\emptyset$	-	Empty set
$\phi(n)$	-	Euler phi function of n
Q_{4n}	-	Generalized quarternion group of order $4n$
$girth(\Gamma)$	-	Girth of the graph Γ
Γ	-	Graph
$\Gamma_1 \cong \Gamma_2$	-	Graph Γ_1 is isomorphic to the graph Γ_2
(m, n)	-	Greatest common divisor of m and n
G	-	Group
$\langle a \rangle$	-	Group generated by the element a
$G \cong H$	-	Group G is isomorphic to the group H
$H \leq G$	-	H is a subgroup of G
$H \lhd G$	-	H is normal subgroup of G
e	-	Identity element
$\alpha(\Gamma)$	-	Independence number of a graph Γ

$[G : H]$	-	Index of the subgroup H in the group G
x^{-1}	-	Inverse of an element x
$\chi_L(\Gamma)$	-	Locating chromatic number of the graph Γ
$\Delta(\Gamma)$	-	Maximum degree of the graph Γ
$\delta(\Gamma)$	-	Minimum degree of the graph Γ
$\|V(\Gamma)\|$	-	Number of vertices of a graph Γ
$\|G\|, \|x\|$	-	Order of the group G, Order of the element x
$\Gamma^{op}(G)$	-	Order prime permutability graph of subgroups of a group G
$\Gamma^{opc}(G)$	-	Order prime permutability graph of cyclic-subgroups of a group G
$\Gamma^{opp}(G)$	-	Order product prime graph of a group G
QD_{2^n}	-	Quasi-dihedral group of order 2^n
$\Phi(n)$	-	Set of positive non-trivial proper divisors of n
$H \triangleleft G$	-	Subgroup H is normal in the group G
$\ni$	-	Such that
$\sum$	-	Summation
$K_m + K_n$	-	Sum of two complete graphs K_m and K_n
$K_m \cup K_n$	-	Union of two complete graphs K_m and K_n
$\chi(\Gamma)$	-	Vertex chromatic number of the graph Γ
$V(\Gamma)$	-	Vertex set of a graph Γ
$x \mid y$	-	x divides y

CHAPTER 1

INTRODUCTION

Various techniques have been used by researchers in investigating the properties of a group as well as classifying it according to its properties, which happen to be one of the great achievements of modern mathematics. One of the techniques found to be useful is by defining graphs to the groups and investigate their properties in terms of the corresponding geometric structures. The study creates a bridge to move from group theory to graph theory in order to obtain the properties of one in terms of the other. According to Devi [1], the properties of a group can be explored through the relationship among its elements or subgroups. This relationship can be considered as the vertex adjacency of the corresponding defined graph.

Another technique is by classifying the groups in terms of their graphical properties. The classification can be done based on the invariants or perfect codes of the graphs of groups. Graph invariants are the properties of graphs that preserve isomorphism, which can serve as analytical tools for investigating the abstract structures of graphs. These invariants include clique number, independence number, diameter, girth and many more. Also, the study of perfect code plays important role in the theory of error correction, since it classifies codes that achieve maximum possible error correction without ambiguity. The study of perfect codes has been recently extended to algebraic structures leading to the investigation of the perfect codes for graphs of groups and determining the subgroup perfect codes in finite groups.

In this research, some new graphs of finite groups are introduced, namely the order product prime graph, the commuting order product prime graph, the order prime permutability graph of subgroups, and the order prime permutability graph of cyclic subgroups of finite groups. The properties of these graphs are investigated for some categories of finite groups which are cyclic groups, dihedral groups, generalized quaternion groups, and quasi-dihedral groups. These groups are chosen because cyclic groups are normal subgroups of dihedral groups and also to generalized quaternion groups and quasi-dihedral groups. Thus, the relationship among these groups are further explored. Meanwhile, the general presentations for the graphs on these groups are investigated. Additionally, the cyclic groups, dihedral groups, generalized quaternion groups, and quasi-dihedral groups are classified in terms of their geometric structures as one of connected, complete, regular, and planar. Moreover, some invariants of the graphs on these groups which are the clique number, independence number, domination number, girth, diameter, vertex chromatic number, edge chromatic number, dominated chromatic number, and locating chromatic number are investigated. Furthermore, the number of the perfect codes of these graphs on the cyclic groups, dihedral groups, generalized quaternion groups, and quasi-dihedral groups are investigated.

1.1 Research Background

The study of groups in connection to graph theory was established by Cayley [2] in 1878, when he defined the graph that explains the abstract structure of a group generated by a set of generators. This study was reintroduced by Dehn in [3], where he called the Cayley graph a group diagram which led to the geometric group theory of today. More interestingly, the study attracted the attention of many researchers leading to the establishment for the study of the properties of algebraic structures such as groups, semi-groups, rings, modules, vector spaces, fields, and many more by defining graphs to the structures and study its properties using graphs theoretical tools. For example, after the introduction of the Cayley graph, Dehn in [3] used the graph to solve word problem for fundamental groups of surfaces with genuse greater than or equals to 2. Much later this graph has been used by Kelarev $et\ al.$ [4] to classify data which can be recorded as a set of strings or sequences of letters over a finite alphabet.

Beside Cayley graph, there is another graph that is called order prime graph of groups defined in [5] by Sattanathan and Kala as a graph with the elements of groups as vertices and two vertices a, b are adjacent if and only if $gcd\big(|a|, |b|\big) = 1$. This definition was generalized five years later by Rajendra and Reddy in [6], by defining general order prime graph of finite groups as a graph having the elements of groups as its vertices, and two vertices a and b are adjacent if and only if $gcd\big(|a|, |b|\big) = 1$ or p, where p is a prime number. Later on $\big(|a|, |b|\big)$ is used to represent $gcd\big(|a|, |b|\big)$.

In addition, non-commuting graph of groups defined by Newmann in [7], which is the complement of commuting graph. Later, in [8], Bertram used the combinatorial properties of the commuting graph to prove three fundamental and non-trivial theorems on finite groups. Growing body of literature shows that this concept was extended by defining graphs where the vertices are the subgroups of groups. For instance, Aschbacher in [9], defined the commuting graph on subgroups of groups, which led to defining some graphs whose vertices are the subgroups of groups. One of these graphs is the permutability graph of conjugacy classes of cyclic subgroups which was defined by Ballester $et\ al.$ in [10]. More recently, Rajkumar and Devi in [11], defined the permutability graph of subgroups of a given group, as a graph with vertex set consisting of all the proper subgroups of a group and two distinct vertices are adjacent if the corresponding subgroups permute. In the same year, Rajkumar and Devi in [12], introduced the permutability graph of cyclic subgroups as a graph having proper cyclic subgroups as its vertices and two vertices are adjacent if and only if they permute.

The vertex adjacencies of the previous graphs of groups are associated with single relationship, like the relationship among the generators of a group, coprimeness, relatively primeness or commutativity among the elements of a group. However, there are no graphs of groups, which the vertex adjacencies are associated with both primeness and commutativity within the groups elements or subgroups of a group. Accordingly, new graphs of groups called the order product prime graph of finite groups, commuting order product prime graph of finite groups, order prime permutability graph of p-subgroups of finite groups and order prime permutability graph of cyclic subgroups of finite groups are introduced in this thesis. Furthermore, those newly defined graphs are determined for some finite groups which are cyclic groups, dihedral groups, generalized quaternion groups, and quasi-dihedral groups.

1.2 Problem Statement

Group properties investigation and group classification require considerable amount of information that need reasonable time to complete. To tackle out this long process, the study of group is linked with the concept of graph. Accordingly, part of the aim of this research is to introduce some graphs of groups and use them to study the graphical properties of groups. Futhermore there are limited works involving finding the general presentations for graphs of groups, particularly on cyclic groups, dihedral groups, generalized quaternion groups and quasi-dihedral groups. Thus, this research involves investigating the general presentations of some types of graphs of groups. Finally, not much work are available in the literature on perfect codes on graphs of groups. Therefore, in this research, the perfect codes of the new graphs of groups are investigated.

1.3 Research Objectives

The objectives of this research are:

(a) To introduce some new graphs of finite groups and determine their general presentations on cyclic groups, dihedral groups, generalized quaternion groups, and quasi-dihedral groups.
(b) To investigate some basic properties of the new graphs on the groups in (a) such as connectivity, completeness, regularity, and planarity.
(c) To obtain graph invariants of the groups in (a) which consist of clique number, independence number, domination number, girth, diameter, vertex chromatic number, dominated chromatic number, locating chromatic number, and edge chromatic number.
(d) To investigate the number of perfect codes for the graphs on the groups in (a).

1.4 Scope of the Study

This research focuses on the study of the graphical properties of some finite groups. Some new graphs of groups which are the order product prime graph, commuting order product prime graph, order prime permutability graph of subgroups of groups, and order prime permutability graph of cyclic subgroups of groups are defined. Meanwhile, the properties of these graphs which are the general presentations of the graphs are investigated. Moreover, the general presentations are used in obtaining the invariants of the graphs. The groups under the scope of this research are cyclic groups, $\mathbb{Z}_n$, dihedral groups, D_{2n}, generalized quaternion groups, Q_{4n}, and quasi-dihedral groups, QD_{2^n}.

1.5 Significance of Findings

Even though there are many techniques for investigating the properties of a group, but one of those that found to be useful is by defining graph to the groups and investigate the graphical properties of the group.

One of the major contributions of this research is to introduce some new graphs of groups that will contribute in adding new dimension of finding graphical properties of some finite groups. Furthermore, the general presentations of the new graphs are obtained on cyclic groups, dihedral groups, generalized quaternion groups, and quasi-dihedral groups, which may inspire future works in geometric group theory.

1.6 Research Methodology

This thesis is divided into two parts. The first part focuses on introducing some new graphs of groups, while the second part focuses on investigating the properties of the new graphs on groups. Firstly, the fundamental and essential concepts of some existing graphs of groups are studied. In the first part, inspired by the previous researches on graphs of finite groups, some new graphs of finite groups which are order product prime graph, commuting order product prime graph, order prime permutability graph of p-subgroups, and order prime permutability graph of cyclic subgroups of finite groups are presented.

In the second part, the properties of these graphs are presented. At first, the general presentations of the graphs are presented using the properties of the elements and subgroups of the cyclic groups, $\mathbb{Z}_n$, dihedral groups, D_{2n}, generalized quaternion groups, Q_{4n}, and quasi-dihedral groups, QD_{2^n}. It is proven that the cyclic subgroups are normal subgroups of the dihedral groups, generalized quaternion groups, and quasi-dihedral groups. The computation of the general presentations of the graphs based on the definitions requires determining the order and centralizers of the elements and subgroups in the groups. Therefore, the order of the elements in the cyclic groups, are determined using the idea of a cyclic groups, $\mathbb{Z}_n$, since the order of an element $a^k \in \mathbb{Z}_n$, is $|a^k| = \frac{n}{gcd(n,k)}$. While the rest of the elements of the dihedral groups, generalized quaternion groups, and quasi-dihedral groups, their group presentation shows that the order of each of the element is of even prime power. Hence, by the concept of the order of the elements and subgroups with their centralizers, the general presentations of the graphs of groups are provided. Afterwards, the general presentations of the graphs are used in obtaining the invariants of the graphs, which are the clique number, indepedent number, domination number, girth, and diameter. Moreover, the independent sets are used in coloring the vertices of the graphs. In addition, the special case of Baranyai's Theorem and Vizing's Theorem were used in coloring the edges of the graphs. Finally, the independent sets and the concept of neighborhood of a vertex, which is defined as the set of all the vertices incident to it is used in obtaining the perfect codes of the graphs.

Figure 1.1 illustrates the research methodology of this thesis.

1.7 Thesis Organization

This thesis is composed of seven chapters which are the introduction, literature review, results on order product prime graph, results on commuting order product prime graph, results on order prime permutability graph of subgroups, results on order prime permutability graph of cyclic subgroups of finite groups, and the conclusion.

The first chapter, is the introduction chapter, which gives the brief overview of the thesis that includes the background of the research, problem statement, objectives of the research, scope of the research, significance of the study, research methodology, and thesis organization.

In Chapter 2, the review of the related literature is provided. This chapter is divided into three sections. The first section presents some preliminaries related to group theory. It also contains some definitions and basic concepts on group theory. In the second section, some terminologies on graph theory are presented. Finally, in the third section, some definitions, basic concepts, terminologies, and previous studies on graph of groups are presented.

Chapter 3 is divided into two sections. The first section is devoted to presenting the first new graph of groups, which is called the order product prime graph. Meanwhile, in the second section, the general presentations, the invariants and number of perfect codes of the graph on cyclic groups, dihedral groups, generalized quaternion groups, and quasi-dihedral groups are provided.

Chapter 4 is also divided into two sections. In the first section, the second new graph of groups is presented, which is called the commuting order product prime graph. Concurrently, in the second section, the general presentations, the invariants and number of perfect codes of the graph on cyclic groups, dihedral groups, generalized quaternion groups, and quasi-dihedral groups are given.

In Chapter 5, another graph of groups, which is called the order prime permutability graph of p-subgroups is presented. In the second section, the general presentations, the invariants and number of perfect codes of the graph on cyclic groups, dihedral groups, generalized quaternion groups, and quasi-dihedral groups are provided.

Similarly, Chapter 6 is also divided into two sections. The first section presents the order prime permutability graph of cyclic subgroups. In the second section, the general presentations, the invariants and number of perfect codes of the graph on cyclic groups, dihedral groups, generalized quaternion groups, and quasi-dihedral groups are presented.

Finally, Chapter 7 summarized and conclude the whole thesis which gives a brief summary of the findings. Moreover, areas of further research are also suggested in this chapter.

The outline of the thesis is illustrated in Figure 1.2.

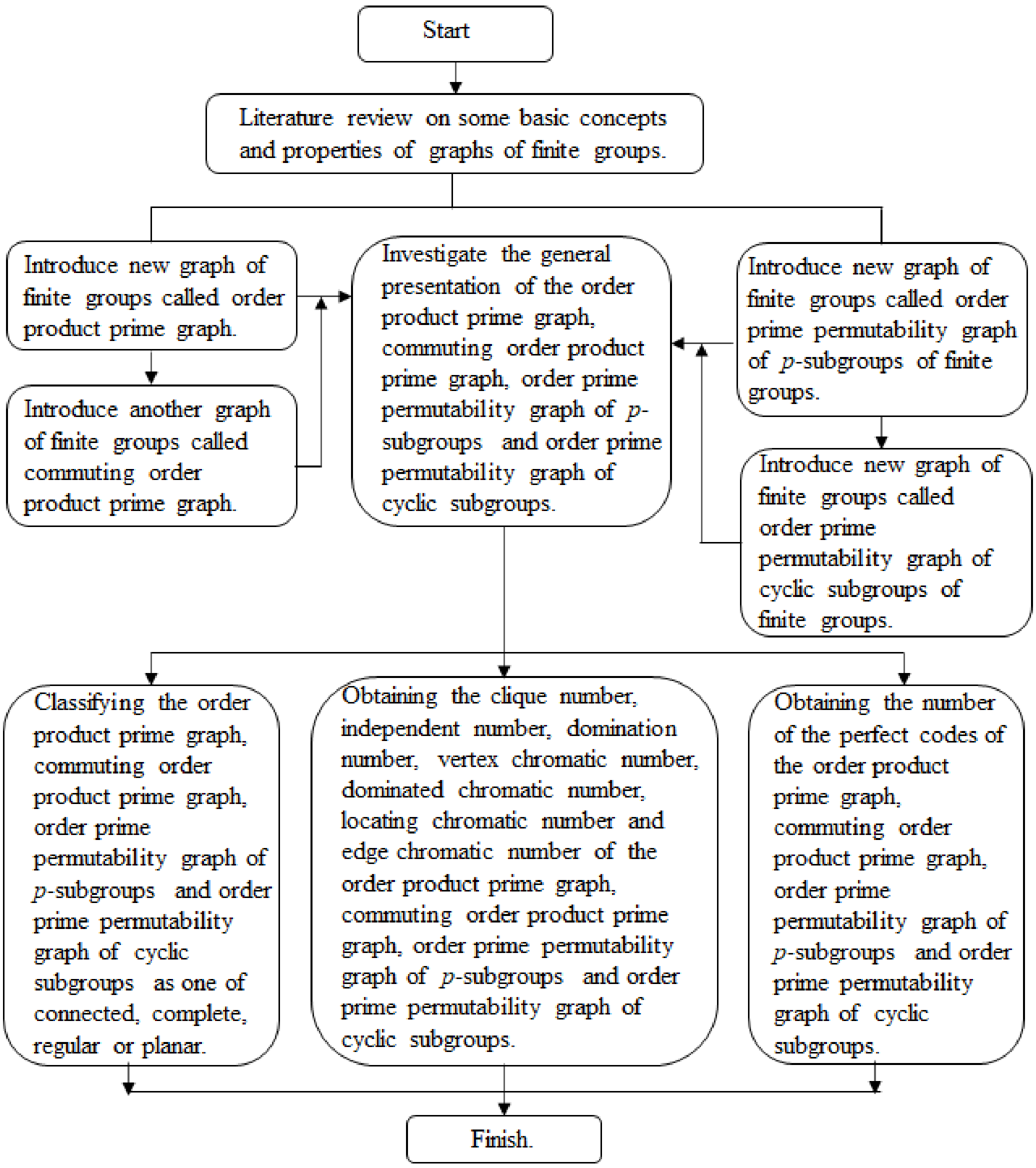

Figure 1.1 Flow chart of research methodology

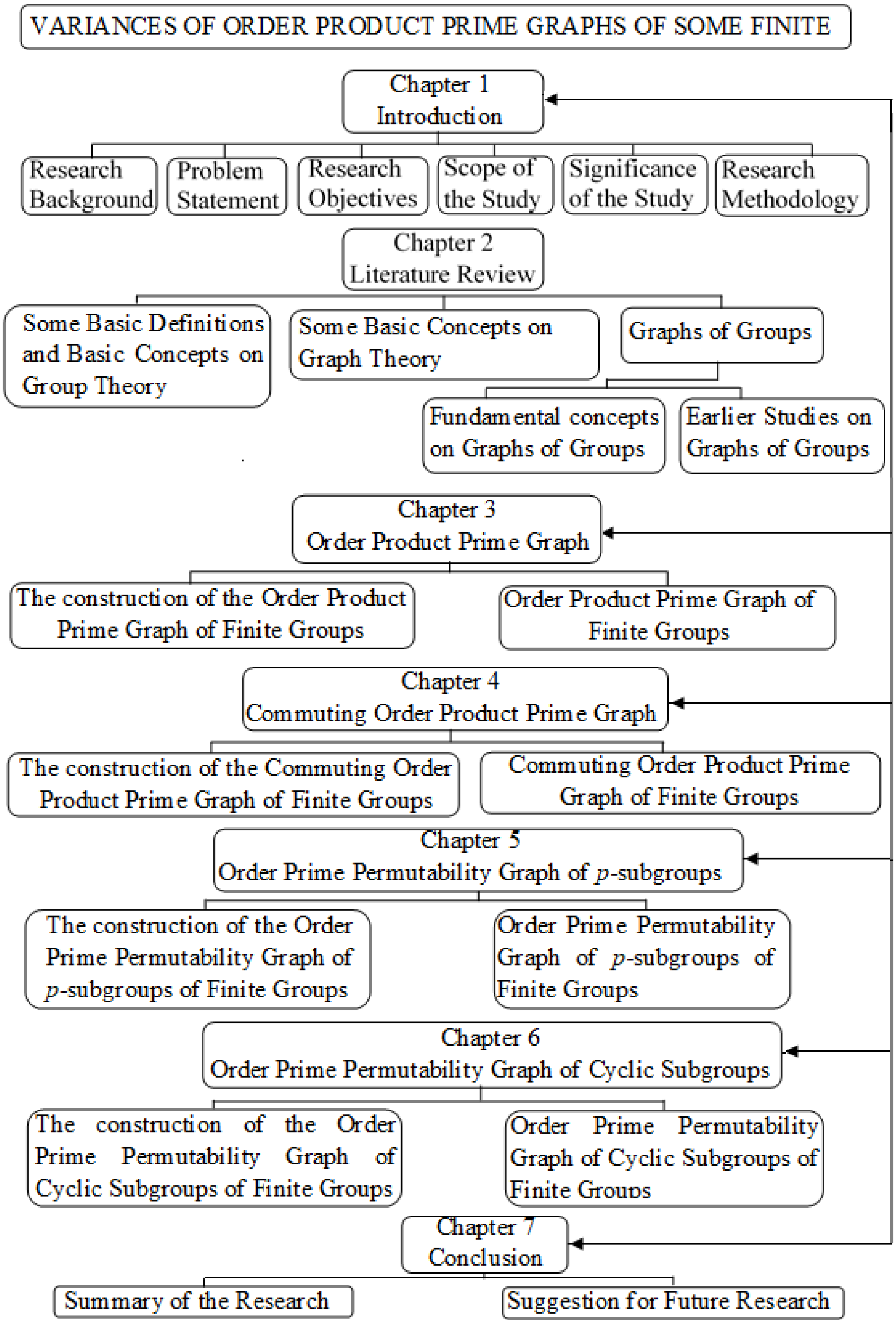

Figure 1.2 Flow chart of thesis organization

REFERENCES

1. Devi, P. *Studies on Graphs Associated with Subgroups of Groups*. Ph.D. Thesis. Gandhigram Rural Institut-Demmed University, Tamil Nadu; 2016.

2. Cayley, A. The Theory of groups: Graphical Representation. *American Journal of Mathematics*. 1878. 1(2): 174-176.

3. Dehn, M. *Papers on Group Theory and Topology*. United States of America: Springer-Verlag. 1987.

4. Kelarev, A., Ryan, J. and Yearwood J. Cayley graphs as classifiers for data mining. *Discrete Mathematics*. 2009. 309(17): 5360-5369.

5. Sattanathan, M. and Kala , R. An Introduction to Order Prime Graph. *Int. J. Contemp. Math. Sciences*. 2009. 4(10): 467-474.

6. Rajendra, R. and Reddy, K. S. P. On General Order Prime Graph of Finite Group. *Proceedings of Jangjeon Mathematical Society*. 2014. 17(4): 641-644.

7. Newmann, B. H. A Problem of Paul Erdos on Groups. *J. Austral. Math. Soc. 21 (Series A)*. 1976. 21(4): 467-472.

8. Bertram, E. A. Some Application of Graph Theory to Finite Groups. *Discrete Mathematics*. 1983. 44(1): 31-43.

9. Aschbacher, M. Simple Connectivity of p-Group Complexes. *Israel Journal of Mathematics*. 1993. 82: 1-43.

10. Ballester B. A., Cossey, J. and EstebanRomero, R. A Characterization via Graphs of the Soluble Groups in which Permutability is Transitive. *Algebra and Discrete Mathematics*. 2009. 4:10-17.

11. Rajkumar, R. and Devi, P. On Permutability Graphs of Subgroups of Groups. *Discrete Mathematics Algorithm and Applications* 2015. 7(2):1-11.

12. Rajkumar, R. and Devi, P. Permutability Graph of Cyclic Subgroups. *Malaya J. Mat.* 2015. 1:172-178.

13. Charles, C. P. *A Book of Abstract Algebra*. 2nd Edition. Mineola, NewYork. Dover Publications, Inc. 2010. ISBN 13:978-0-486-474 17-5.

14. Gallian, J. *Contemporary Abstract Algebra*. 9th Edition. Nelson Education, USA. Cengage Learning. 2015. ISBN: 978-1-305-65796-0.

15. Gallian, J. *Contemporary abstract algebra*. 8th. USA: Cengage Learning. 2012.

16. Rotman, J. J. *Advanced Modern Algebra*. 2nd Edition. USA: American Mathematical Society. 2010.

17. Jennifer, S. and Kathryn, S. On the Structure of Symmetric Spaces of Semi-dihedral Groups. *Involve*. 2017. 10(4): 665-676.

18. Cavior, S. R. *The subgroups of the dihedral group*. Mathematics Magazine. 48(2).1975. p107.

19. Bondy, J. A. and Murty, U. S. R. *Graph Theory with Applications*. Britain: Macmillan. 1976.

20. Sur, E. and Srimani, P. K. *Topological properties of star graphs*. Computers math. Applic. 1993. 25(12). 87-98.

21. Benjamin, A. Chartrand, G. and Zhang, P. *The Fascinating World of Graph Theory*. Princeton University Press. UK. 2015. ISBN: 978-0-691-16381-9

22. Balakrishnan, R. and Ranganathan, K. *A Text Book of Graph Theory*. 2nd Edition. New York: Springer Science+Business Media. 2012. Universitex: Second Edition ISBN 9781461445296(e-Book)

23. Balakrishnan, R. and Ranganathan, K. *A Text Book of Graph Theory*. New York: Springer Verlag. 2000. Universitex: ISBN 0387988599(e Book)

24. Diestel, R. *Graph Theory*. 5th Edition. Germany. Springer. 2017.

25. Chartrand, G. and Zhang, P. *A first course in graph theory*. Courier Corporation. 2013.

26. Chartrand, G. Erwin, D. Henning, M.A, Slater, P.J. and Zhang, P. The locating-chromatic number of a graph. *Bull. Inst. Combin. Appl.* 2002. 36(2002): 89-101.

27. Chartrand, G. Erwin, D. Henning, M.A, Slater, P.J. and Zhang, P. Graphs of order n with locating-chromatic number n-1. *Discrete Mathematics*. 2003. 269(1): 65-79.

28. Behtoei, A. and Omoomi, B. On the Locating Chromatic Number of Kneser Graphs. *Discrete Applied Mathematics*. 2011. 159(2011): 2214-2221.

29. Ghanem, M., Al-Ezeh, H. and Dabbour, A. Locating Chromatic Number of powers of paths and cycles. *Symmetric(MDPI)*. 2019. 11(389): 1-8.

30. Merouane, H. B., Haddad, M., Chellali, M. and Kheddouci, H. Dominated Colorings of Graphs. *Graphs and Combinatorics*. 2015. 31(3): 713-727.

31. Choopani, F., Jafarzadeh, A., Erfanian, A. and Mojdeh, D. A. On Dominated Coloring of Graphs and Some Nordhaus-Gaddum-type Relations. *Turkish Journal of Mathematics*. 42(2018): 2148-2156.

32. Alikhani, S. and Piri, M. R. Dominated Chromatic number of Some Operations on a Graph. *arXiv.1912.00016v1[math.CO]*. 2019: 1-9.

33. Ian, H. The NP-completeness of edge-coloring. *SIAM Journal on computing*. 1981. 10(3): 718-720.

34. Behzad, M., Chartrand, G. and Cooper, J. K. The color numbers of complete graphs. *Journal London Maths. Soc.*. 1967. 42: 226-228.

35. Scheinerman, E. R. and Ullman, D. H. Fractional Graph Theory: A Rational Approach to the Theory of Graphs. *Wiley-Interscience*. 2008.

36. Biggs, N. Perfect Codes in Graphs. *Journal of Combinatorial Theory(B)*. 1973.15. 289-296.

37. Kratochivil, J. Perfect codes over graphs. *Journal of Combinatorial Theory(B)*. 40. 1986. 224-228.

38. Feng, R., Huang, H. and Zhou, S. Perfect codes in circulant graphs. *Discrete Mathematics*. 340. 2017. 1522-1527.

39. Ma, X., Walls, G. L., Wang, K. and Zhou, S. Subgroup perfect codes in Cayley graphs. *arXiv:1904.01858v1[math. Co]*. 2019. 1-19.

40. Zhang, J. and Zhou, S. On subgroup perfect codes in Cayley graphs. *European Journal of Combinatorics*. 91. 2021. 1-10

41. Selvakumar K. and Subajini M. Classification of groups with toroidal coprime graphs. *Australasian Journal of Combinatorics*. 2017. 69(2): 174-183.

42. Williams, J. S. Prime Graph Components of Finite Groups. *Journal of Algebra*. 1981. 69: 487-513.

43. Iiyori, N. and Yamaki, H. Prime Graph Components of the Simple Groups of Lie Type over the Field of Even Characteristic. *Journal of Algebra*. 1991. 67(A).

44. Maria, S. L. The diameter of the prime graph of a finite group. *Journal of Group Theory*. 1999. 2: 157-172.

45. Gruber, A., Keller, T. M., Lewis, M. L., Naughton, K., Strasser, B. A Characterization of the Prime Graphs of Solvable Groups. *Journal of Algebra*. 2013. 442: 397-422.

46. Zhang, Q. L., Wang, J. H. and Liu , W. J. A Characterization of symmetric group of Prime Degree. *Bulletin of Iranian Mathematical Society*. 2009. 40(2): 473-480.

47. Burness, T. C. and Covato, E. On the Prime Graph of Simple groups. *Bulletin of the Australian Mathematical Society*. 2015. 91(2): 227-240.

48. Kelarev, A. V. and Quinn, S. J. A Combinatorial Property and Power Graphs of Groups. *Contributions to General Algebra*. 2000. 12(58): 3-6.

49. Imani, N., Sarbazi-Azad, H., Akl, S. G. and Moinzadeh P. Chromatic sets of power graphs and their application to resource placement in multicomputer networks. *Comput. Math. Appl. 58*. 2009. 58:403413.

50. Chakrabarty, I., Shamik, G. and Sen, M.K. Undirected power graphs of semigroups. *Semigroup Forum*. 2009. 78: 410426.

51. Cameron, P. J. and Gosh, S. The power graph of finite group. *Discrete Mathematics*. 2011. 311: 1220-1222.

52. Moghaddamfar, A. R., Shi, W. J., Zhou, W. and Zokayi, A. R. On Commuting Graph Associated with a Finite Group. *Sib. Math. J.*. 2005. 46(2): 325-332.

53. Ma, X., Wei, H. and Yang, L. Coprime Graph of a Group. *International Journal of Group Theory*. 2014. 3(3): 13-23.

54. Dorbidi, H. R. A Note on the Coprime Graph of a Group. *International Journal of Group Theory*. 2016. 5(4): 17-22.

55. Abd Rhani, N., Mohd Ali, N. M., Sarmin, N. H. and Erfanian, A. *The Relative Co-prime Graph of a Group*. Proceedings of the 4th Biennial International Group Theory Conference. 2017. (4BIGTC2017). 23-26. 155-158.

56. Abdollahi, A., Akbari, S. and Maimani, H. R. Noncommuting graph of a group. *Journal of Algebra*. 2006. 298: 468492.

57. Tamizh ,T. C., Selvakumar, K. and Raja, S. Commuting Graphs on Dihedral groups. *The Journal of Mathematics and Computer Science*. 2011. 2(2011), 402-406.

58. Vahidi, J. and Asghar, A. T. The Commuting Graphs on Groups D_{2n}, Q_n. *The Journal of Mathematics and Computer Science*. 2010. 2:123-127.

59. Mirzargar, M., Pach, P. P. and Ali, R. A. Remarks on Commuting Graphs of a Finite Groups. *Electronic Notes in Discrete Mathematics*. 2014. 45: 103-106.

60. Mahmoud R., Sarmin, N. H. and Erfanian, A. On the Energy of Non-commuting Graph of Dihedral Groups. *AIP Conference Proceedings of 2017*. American Institute of Physics. 2017. 1-5.

61. Kakeri, F., Erfanian, A. and Mansoori, F. Generalization of the Non-commuting graph of a group via a normal subgroup. *ScienceAsia*. 2016. 42: 231235.

62. Giudici, M. and Pope, A. The Diameters of Commuting Graph of Linear Groups and Matrix Rings Over The Integers Modulo n. *Australian Journal of Combinators*. 2010. 48: 221-230.

63. Bianchi, M., Gillio, A. and Verar di, L. Finite groups and subgroup-permutability. *Annali di Matematica Pura ed Applicata*. 1995. 169(1): 251-268.

64. Rajkumar, R. and Devi, P. Order Prime Graph of Subgroups of a Group. *arXiv:1510.00129v1)[math. GR]*. 2015.

LIST OF PUBLICATIONS

Journal Publications

1. **Bello, M.**, Mohd Ali, N. M., & Zulkifli, N. (2019). A Systematic Approach to Group Properties Using its Geometric Structure. *European Journal of Pure and Applied Mathematics*. 2020.13(1):84-95
2. **Bello, M.**, Mohd Ali, N. M. & Isha, S. I. (2020). Group perfect code in commuting order product prime graph. *Advances and Applications in Discrete Mathematics*. 2020.25(1):41-54
3. **Bello, M.**, Mohd Ali, N. M. & Isha, S. I. (2020). Graph coloring via commuting order product prime graph. *Journal of Mathematics and Computer Science*. 2020.23(2):155-169